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One-parameter Darboux-transformed quantum actions in Thermodynamics

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Abstract

We use nonrelativistic supersymmetry, mainly Darboux transformations of the general (one-parameter) type, for the quantum oscillator thermodynamic actions. Interesting Darboux generalizations of the fundamental Planck and pure vacuum cases are discussed in some detail with relevant plots. It is shown that the one-parameter Darboux-transformed Thermodynamics refers to superpositions of boson and fermion excitations of positive and negative absolute temperature, respectively. Recent results of Arnaud, Chusseau, and Philippe regarding a single mode oscillator Carnot cycle are extended in the same Darboux perspective. We also conjecture a Darboux generalization of the fluctuation-dissipation theorem.

I. INTRODUCTION

Recently, Arnaud, Chusseau, and Philippe (hereafter ACP) [1] studied the work done per Carnot cycle by a single mode oscillator of the ideal LC type operating between two baths of different temperatures. They used the old (1906) prescription of Einstein of discrete $\hbar\omega$ exchanges with the baths. Within this approach, they confirmed the Carnot theory of the efficiency of cyclic engines for this case. As known, the energy of the oscillator is given by Planck's distribution

$$U_P(\omega, \beta) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1} . \quad (1)$$

One can define an action function

$$f_P(x) = \frac{U_P}{\omega} = \frac{\hbar}{2} + \frac{\hbar}{\exp(\hbar x) - 1} = \frac{\hbar}{2} \coth\left(\frac{\hbar x}{2}\right) , \quad (2)$$

where $x = \beta\omega$. This action plays the role of a generalized force in the process of frequency change. In order to evaluate the efficiency, ACP introduced a 'two-variable entropy'

$$s(x, y) \equiv x f_P(y) - \int^x f_P(z) dz = \frac{\hbar x}{2} \coth\left(\frac{\hbar y}{2}\right) - \ln \sinh\left(\frac{\hbar x}{2}\right) + C , \quad (3)$$

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with the property $s(x, x) = s(x)$, the latter being the usual entropy. Employing standard thermodynamic formalism, ACP calculated the efficiency of both reversible and nonreversible Carnot cycles.

ACP made the interesting remark that the oscillator action $f_P(x)$ fulfills as particular solution the following Riccati equation

$$\frac{df}{dx} + f^2 = \left(\frac{\hbar}{2}\right)^2 . \quad (4)$$

As a matter of fact, the constant vacuum action $f_V = \frac{\hbar}{2}$ is also a solution of the same Riccati equation, whereas the pure thermal action $f_T = \frac{\hbar}{\exp(\hbar x) - 1}$ is a particular solution of the following equation

$$\frac{df}{dx} + \hbar f + f^2 = 0 , \quad (5)$$

which is a Bernoulli equation (or a particular type of Riccati equation). Here we would like to study the thermodynamic consequences of using methods belonging to supersymmetric quantum mechanics [2] focusing on one-parameter Darboux transformations of thermodynamic actions.

II. GENERAL SOLUTION OF THE SUPERSYMMETRIC PARTNER RICCATI EQUATION

In Witten's supersymmetric quantum mechanics [3], which is a simple application of Darboux transformations [2], one usually starts with a known particular solution of a Riccati equation without linear term

$$\frac{df_p}{dx} + f_p^2 = V_1(x), \quad (6)$$

that we call the bosonic Riccati equation. We are not interested here in a so-called factorization constant that can be placed at the right hand side. The function V_1 is an exactly solvable potential for the Schrödinger equation at zero energy

$$\left(\frac{d}{dx} + f_p\right) \left(\frac{d}{dx} - f_p\right) w_b = \frac{d^2 w_b}{dx^2} - V_1(x) w_b = 0 . \quad (7)$$

The particular solution w_b is usually called a bosonic zero mode. It is connected to the Riccati solution through $f_p = \frac{1}{w_b} dw_b/dx$. Next, changing the sign of the first derivative in Eq. (6) one calculates the outcome $V_2(x)$ using the same Riccati solution

$$-\frac{df_p}{dx} + f_p^2 = V_2(x) . \quad (8)$$

We call the latter equation the fermionic Riccati equation. The function $V_2(x)$ is known as the supersymmetric partner of the initial potential $V_1(x)$. The corresponding zero mode fulfills

$$\left(\frac{d}{dx} - f_p\right) \left(\frac{d}{dx} + f_p\right) w_f = \frac{d^2 w_f}{dx^2} - V_2(x) w_f = 0 . \quad (9)$$

On the other hand, already in 1984, Mielnik [4] studied the ambiguity of the factorization of the Schrödinger equation for the oscillator that led him to the general (one-parameter-dependent) solution of the companion Riccati equation for that case. In other words, one looks for the general solution [2]

$$-\frac{df_g}{dx} + f_g^2 = -\frac{df_p}{dx} + f_p^2 = V_2(x) . \quad (10)$$

The latter equation can be solved for f_g by employing the Bernoulli ansatz $f_g(x) = f_p(x) - \frac{1}{v(x)}$, where $v(x)$ is an unknown function [5]. One obtains for the function $v(x)$ the following Bernoulli equation

$$\frac{dv(x)}{dx} + 2v(x) f_p(x) = 1 , \quad (11)$$

that has the solution

$$v(x) = \frac{\mathcal{I}_{0b}(x) + \lambda}{w_b^2(x)} , \quad (12)$$

where $\mathcal{I}_{0b}(x) = \int_0^x w_b^2(y) dy$, and we consider λ as a positive integration constant that is employed as a free parameter.

Thus, the general fermionic Riccati solution is a one-parameter function $f_g(x; \lambda)$ of the following form

$$f_g(x; \lambda) = f_p(x) - \frac{d}{dx} \left[\ln(\mathcal{I}_{0b}(x) + \lambda) \right] = \frac{d}{dx} \left[\ln \left(\frac{w_b(x)}{[\mathcal{I}_{0b}(x) + \lambda]} \right) \right] = \frac{d}{dx} \ln w_b(x; \lambda) \quad (13)$$

where $w_b(x; \lambda) = \frac{w_b(x)}{\mathcal{I}_{0b}(x) + \lambda}$. The range of the λ parameter is conditioned by $\mathcal{I}_0(x) + \lambda \neq 0$ in order to avoid singularities. This is a well-known restriction [4]. According to the supersymmetric construction, one can use this general fermionic Riccati solution to calculate a one-parameter family of bosonic potentials as follows

$$\frac{df_g}{dx} + f_g^2 = V_{1,g} , \quad (14)$$

where

$$V_{1,g} = V_1 - 2 \frac{d^2}{dx^2} \ln(\mathcal{I}_{0b}(x) + \lambda) \quad (15)$$

enters the linear equation

$$\left(\frac{d}{dx} + f_g\right) \left(\frac{d}{dx} - f_g\right) w_b(x; \lambda) = \frac{d^2 w_b(x; \lambda)}{dx^2} - V_{1,g} w_b(x; \lambda) = 0 . \quad (16)$$

In the limit $\lambda \rightarrow \infty$, Eq. (14) goes into Eq. (6) because $f_g \rightarrow f_p$ and $V_{1,g} \rightarrow V_1$.

One can think of Eq. (14) as a generalization of the thermodynamic Riccati equation (4). Of interest are the one-parameter oscillator actions f_g rather than the ‘potentials’ $V_{1,g}$. Since in supersymmetric quantum mechanics $V_{1,g}$ are the general Darboux-transformed potentials, we shall call the f_g as Darboux-transformed actions.

A. The Planck case

Using $f = w'/w$ in Eq. (4), where the prime denotes the derivative with respect to x , leads to the linear equation

$$w'' - \left(\frac{\hbar}{2}\right)^2 w = 0, \quad (17)$$

having the particular zero-mode solution $w_a = W_a \sinh(\frac{\hbar}{2}x)$. Thus, the general Riccati solution is a one-parameter function $f_{gP}(x; \lambda)$ of the following form

$$\begin{aligned} f_{gP}(x; \lambda) &= f_P(x) - \frac{d}{dx} [\ln(\mathcal{I}_{0a}(x) + \lambda)] \\ &= \frac{d}{dx} \left[\ln \left(\frac{w_a(x)}{[\mathcal{I}_{0a}(x) + \lambda]} \right) \right]. \end{aligned} \quad (18)$$

Accordingly, the two-variable entropy will also become a parameter-dependent function

$$s_g(x, y; \lambda) \equiv x f_{gP}(y; \lambda) - \int^x f_{gP}(z; \lambda) dz, \quad (19)$$

where from all the basic calculations as performed by ACP can be easily repeated. For example, to calculate Carnot efficiencies one can use the generalized ACP formula

$$\eta_{C,g} = 1 - \frac{T_{cold} s_g(b, a; \lambda) - s_g(u; \lambda)}{T_{hot} s_g(a; \lambda) - s_g(v, u; \lambda)}, \quad (20)$$

and the same values of the parameters as in ACP, i.e., $T_{cold} = 1/4$, $T_{hot} = 1$, $a \equiv \beta_{hot}\omega_1 = 1$, $b \equiv \beta_{cold}\omega_2 = 4$, $u \equiv \beta_{cold}\omega_3 = 2c$, and $v \equiv \beta_{hot}\omega_4 = 2$, keeping c as a free parameter. The results of this subsection are illustrated in the plots of Figs. (1) - (4).

B. The vacuum case

For this case, the particular Riccati solution is the vacuum action $f_p = f_V = \frac{\hbar}{2}$. The corresponding zero mode is $w_V \propto e^{\hbar x/2}$. The usual entropy $s_V(x, x)$ of the vacuum fluctuations is zero as a result of a simple calculation, whereas the modified entropy has a kink-like behavior between the $\frac{\hbar}{2}$ (bosonic) solution and the $-\frac{\hbar}{2}$ (fermionic) solution. Plots of this case are displayed in Figs. (5) - (8).

C. The symmetric zero mode: Fermi-Dirac action at negative T

The cases in **A** and **B** could be considered particular cases of the general zero-mode $w_g = Ae^{\hbar x/2} + Be^{-\hbar x/2}$ that can be also used as solution in Eq. (7). The Planck action corresponds to $A = -B = \frac{1}{2}$ (antisymmetric zero-mode), while the vacuum case to $A =$ arbitrary and $B = 0$. One can use any other type of zero-modes. For example, the symmetric

zero-mode $w_s = W_s \cosh(\frac{\hbar}{2}x)$ (see Figs. (9) - (10)) is an interesting case since if we trace back to the action we get

$$f_s = -\frac{\hbar}{2} + \frac{\hbar}{\exp(-\hbar x) + 1} , \quad (21)$$

i.e., a Fermi-Dirac action for negative x . We have shown in a previous paper that the λ parameter is equivalent to the quotient A/B [6]. Thus, the general Riccati solution introduces effects of the second linear independent solution. The problem then turns into a subtle interpretation of the mathematical results. We have found at least one possible physical significance. For the Planck case, the second linear independent zero-mode is exactly the cosh function and the corresponding action is the aforementioned Fermi-Dirac action of negative x . We attach now the minus sign to the temperature parameter in $x = \omega/T$ and recall that the issue of negative absolute (spin) temperatures first appeared in Physics in 1951 when Purcell and Pound were able to produce sudden reversals of the direction of an external magnetic field applied to a crystal of LiF [7]. Since then many other experiments with negative temperatures have been devised in nuclear spin systems and the corresponding ‘violations’ of the second law of thermodynamics were a subject of discussion [8]. Thus, the one-parameter Darboux transformations of the Planck action are a way of introducing upon it the effect of a Fermi-Dirac action of negative absolute temperature. On the other hand, in the figures (9) - (10) we introduce effects of the Planck action of positive temperature on the Fermi-Dirac action of negative temperature.

III. THE DARBOUX GENERALIZATION OF THE FLUCTUATION-DISSIPATION THEOREM

Another interesting application refers to dissipative *RLC* systems where the oscillator action enters Nyquist-Johnson spectral power noise distributions of the type (fluctuation-dissipation theorem [9])

$$P(\omega, \beta) = \frac{\omega}{\pi} R(\omega, \beta) f_P . \quad (22)$$

One can think of the corresponding generalization

$$P(\omega, \beta; \lambda) = \frac{\omega}{\pi} R(\omega, \beta) f_g(\beta\omega; \lambda) \quad (23)$$

and hope to study even in simple experiments the significance of the present approach predicting a Darboux generalization of the fluctuation-dissipation theorem.

IV. CONCLUSION

In this work, Planck’s thermodynamic oscillator action is generalized to a one parameter Darboux family of actions. We also consider the bosonic vacuum case separately in the same way. The Planck action and the pure vacuum case correspond to the asymptotic limit of

the Darboux parameter $\lambda \rightarrow \infty$. In the Planck case, all the other λ cases correspond to a system made of bosons at temperature T interacting with an equal system of fermions at temperature $-T$. In the vacuum case, the $\lambda \neq \infty$ cases describe the interaction of the bosonic and fermionic vacua. The efficiencies of the ideal oscillator Carnot cycles based on the Darboux-modified Planck and vacuum entropies are calculated along the lines described by Arnaud, Chusseau and Philippe. Systems of negative Kelvin temperatures are hotter than those of positive T [8] and therefore they always represent the hot bath. In real, dissipative cases, the same type of generalization is suggested for the fluctuation-dissipation theorem. Finally, we mention that a multiple-parameter Darboux generalization is also possible [10].

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REFERENCES

- [1] J. Arnaud, L. Chusseau, and F. Philippe, *Carnot cycle for an oscillator*, physics/0105048 (to be published).
- [2] H.C. Rosu, *Short survey of Darboux transformations*, in Proc. "Symmetries in Quantum Mechanics and Quantum Optics", Burgos, Spain, Sept. 21-24, 1998, Eds. A. Ballesteros et al., (Serv. de Publ., Univ. Burgos, Burgos, Spain, 1999) pp. 301-315, quant-ph/9809056.
- [3] E. Witten, *Dynamical breaking of supersymmetry*, Nucl. Phys. B **188**, 513 (1981).
- [4] B. Mielnik, *Factorization method and new potentials with the oscillator spectrum*, J. Math. Phys. **25**, 3387 (1984); For a recent application to cosmology, see H.C. Rosu, *Darboux class of cosmological fluids with time-dependent adiabatic indices*, Mod. Phys. Lett. A **15**, 979 (2000), gr-qc/0003108.
- [5] D. Zwillinger, *Handbook of Differential Equations* (Academic, New York, 1989).
- [6] L.J. Boya, H. Rosu, A.J. Seguí-Santonja, F.J. Vila, *Strictly isospectral SUSY and Schrödinger general zero modes*, Nuovo Cimento B **113**, 409 (1998), quant-ph/9711059.
- [7] E.M. Purcell and R.V. Pound, *A nuclear spin system at negative temperature*, Phys. Rev. **81**, 279 (1951).
- [8] N.F. Ramsey, *Thermodynamics and Statistical Mechanics at negative absolute temperatures*, Phys. Rev. **103**, 20 (1956). See also, N. Bloembergen, *The concept of temperature in magnetism*, Am. J. Phys. **41**, 325 (1973).
- [9] H. Nyquist, *Thermal agitation of electric charge in conductors*, Phys. Rev. **32**, 110 (1928).
- [10] H.C. Rosu, *Multiple parameter structure of Mielnik's isospectrality in unbroken SUSYQM*, Int. J. Theor. Phys. **39**, 105 (2000), quant-ph/9904007.

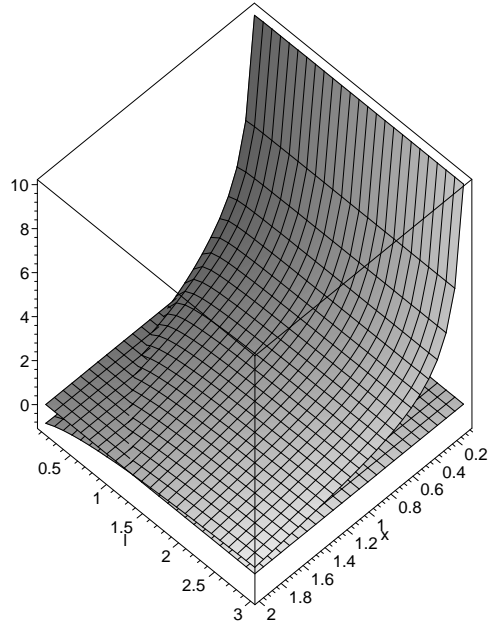


Fig. 1. The one-parameter Darboux-modified Planck action $f_g(x; \lambda)$ as a function of x and λ .

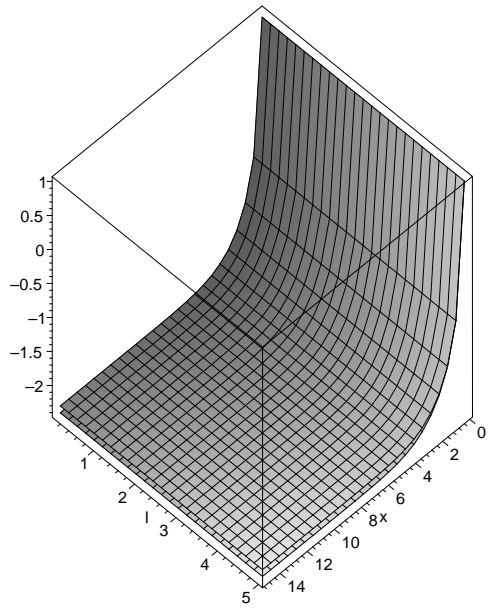


Fig. 2. The standard and the one-parameter entropy functions. There are only small differences between them. For more details see Fig. 3.

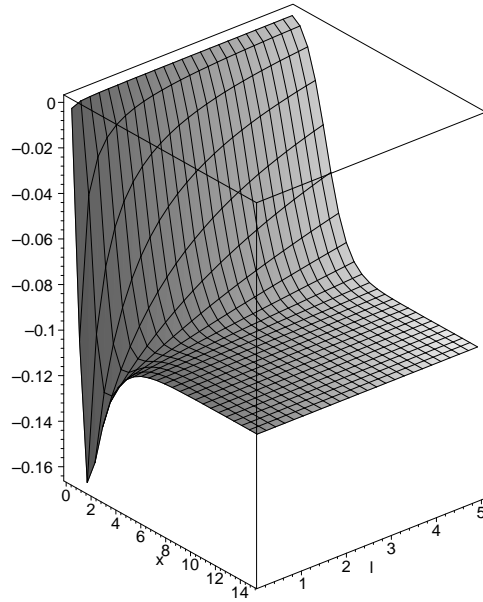


Fig. 3. The difference between entropies $\Delta s = s_g - s_p$.

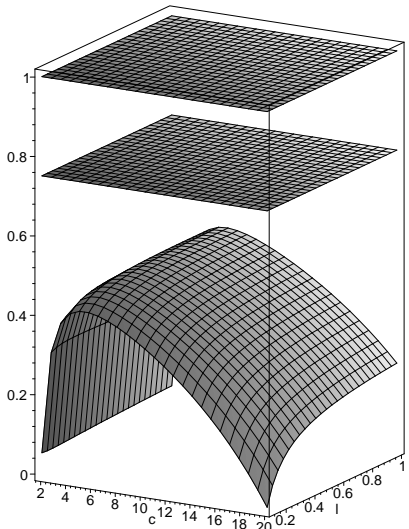


Fig. 4. Darboux-modified Carnot efficiencies for the Planck case. The plane at the height 0.75 corresponds to the maximum Carnot efficiency for the parameters used by ACP.

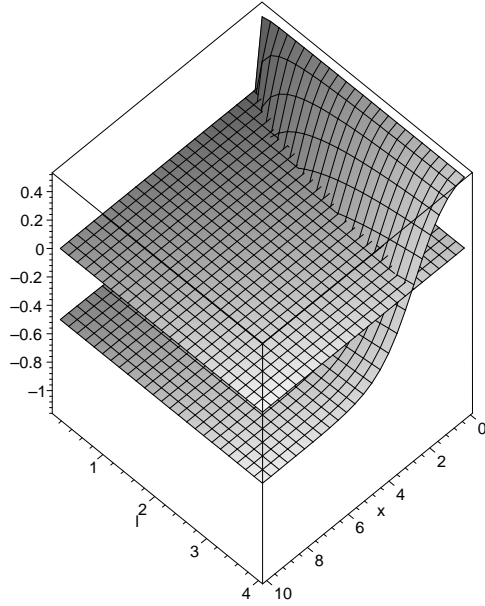


Fig. 5. The Darboux-modified zero-point action. It starts at the normal $\frac{\hbar}{2}$ value at $x = 0$ and goes to $-\frac{\hbar}{2}$ at large values of x . The shape is that of a usual kink (switching) function between $\pm\frac{\hbar}{2}$ for large values of the parameter. For small λ values see Fig. 6.

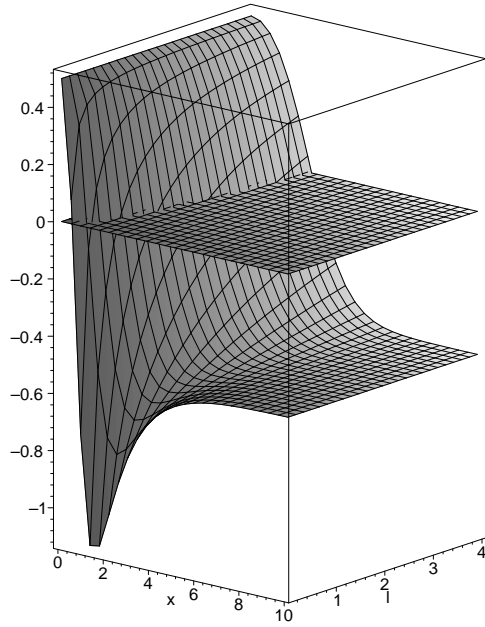


Fig. 6. The same as in the previous figure but for a different orientation to emphasize that at low values of λ the general Riccati thermodynamic vacuum kink deviates from the common definition of a kink and even turns singular for $\lambda_c = (x - \sinh x)/2$.

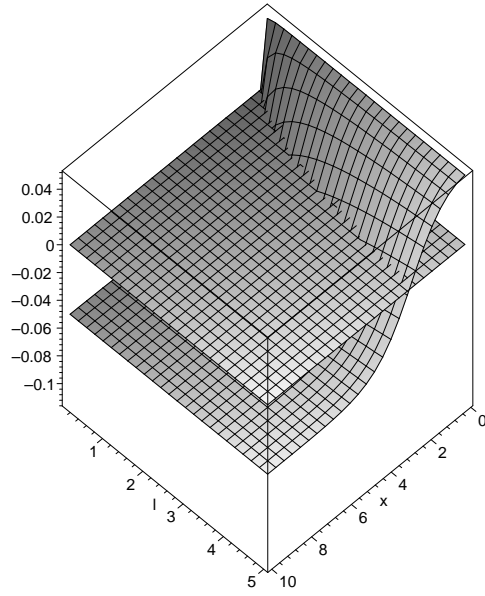


Fig. 7. The usual vacuum entropy and the Darboux-modified one.

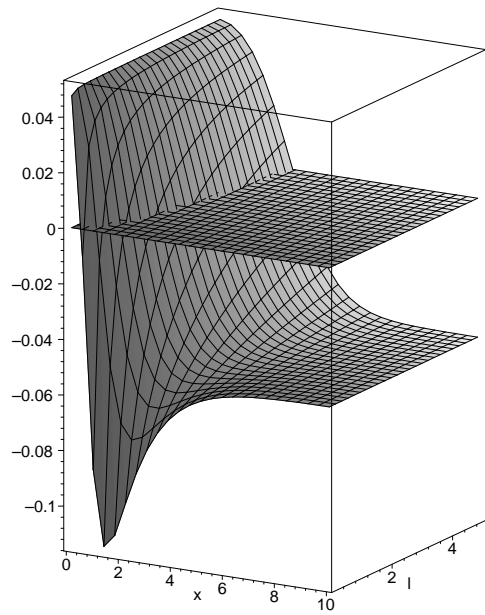


Fig. 8. The same as in the previous figure but for a different orientation.

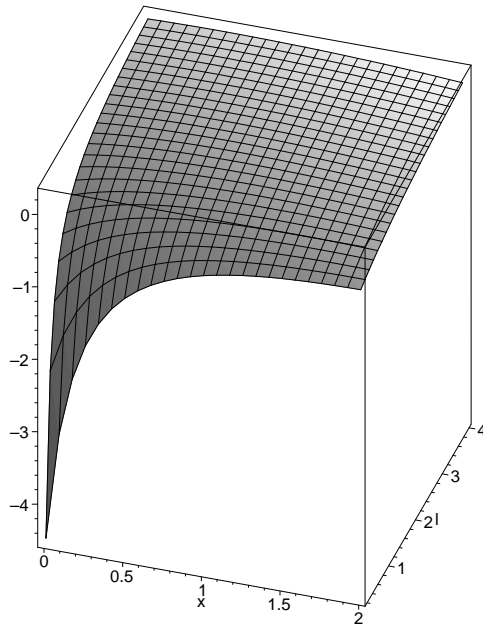


Fig. 9. The one-parameter Darboux-modified Fermi-Dirac action of negative T.

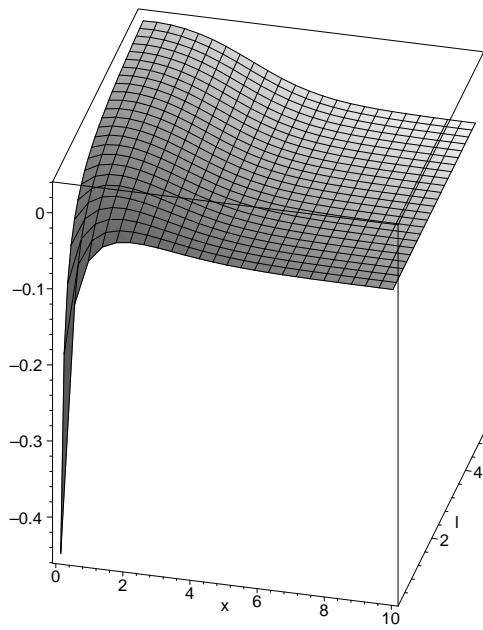


Fig. 10. The one-parameter Darboux-modified Fermi-Dirac entropy of negative T.

Plots not discussed in the text of the work.

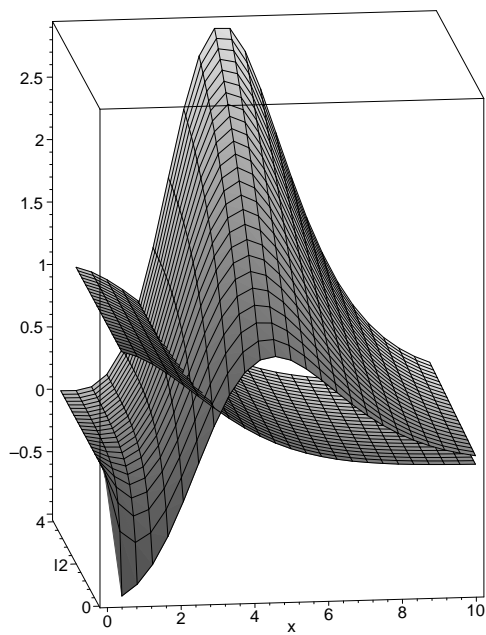


Fig. 11. Heat capacity for the one-parameter Planck case compared to the standard case.

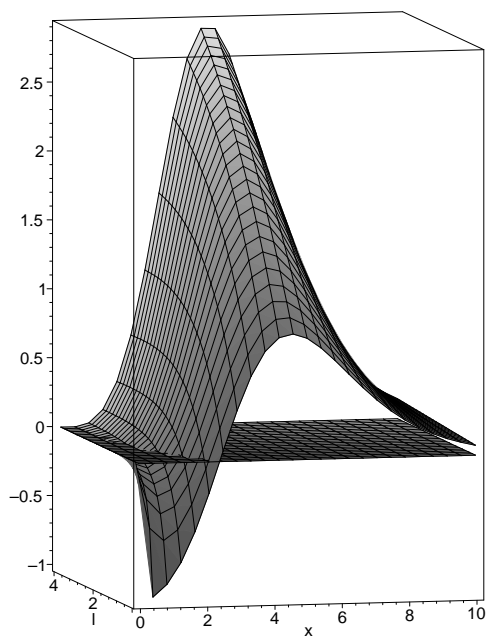


Fig. 12. Heat capacity for the one-parameter vacuum case compared to the standard case.

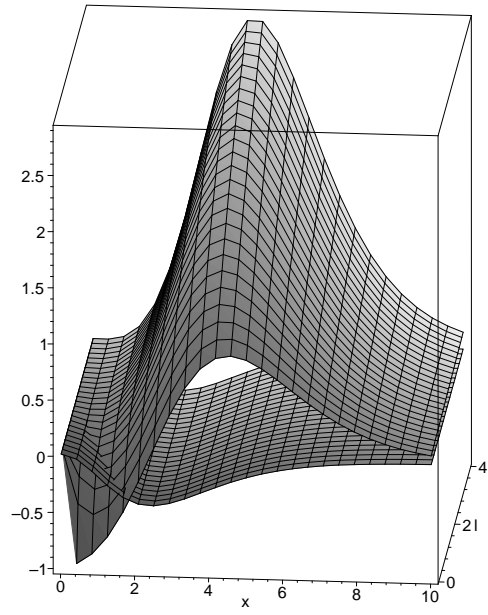


Fig. 13. Heat capacity for the one-parameter Fermi-Dirac case of negative T compared with the nonparametric case.