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# One-parameter nonrelativistic supersymmetry for microtubules

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**Abstract.** The one-parameter nonrelativistic supersymmetry of Mielnik [J. Math. Phys. 25, 3387 (1984)] is applied to the simple supersymmetric model of Caticha [Phys. Rev. A 51, 4264 (1995)] in the form used by Rosu [Phys. Rev. E 55, 2038 (1997)] for microtubules. By this means, we introduce Montroll double-well potentials with singularities that move along the positive or negative traveling direction depending on the sign of the free parameter of Mielnik's method. Possible interpretations of the singularity are either microtubule associated proteins (motors) or structural discontinuities in the arrangement of the tubulin molecules.

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Microtubules (MTs) are hollow cylinder tubes, 25 nm in outer diameter and 14 nm inner diameter, made of two types of 4 nm-long dimers of a polar protein known as tubulin that can self-assemble both *in vivo* and *in vitro* to lengths from several nm up to mm in some neurons. They form the main filamentary component of the cytoskeleton of all eukaryotic cells. Along their walls the tubulin dimers are distributed onto 13 (the seventh Fibonacci number) so-called protofilaments laterally associated. Brain tissues are especially enriched in MTs. Many interesting speculations on MTs have been advanced in recent years [1].

Based on well-established results of Collins, Blumen, Currie and Ross [2] regarding the dynamics of domain walls in ferrodistorive materials, Tuszyński and collaborators [3, 4] considered MTs to be ferrodistorive and studied kinks of the Montroll type [5] as excitations responsible for the energy transfer within this highly interesting biological context.

The Euler-Lagrange dimensionless equation of motion of ferrodistorive domain walls as derived in [2] from a Ginzburg-Landau free energy with driven field and dissipation included is of the traveling reaction-diffusion type

$$\psi'' + \rho\psi' - \psi^3 + \psi + \sigma = 0, \quad (1)$$

where the primes are derivatives with respect to a traveling coordinate  $\xi = x - vt$ ,  $\rho$  is a friction coefficient and  $\sigma$  is related to the driven field [2].

There may be ferrodistorive domain walls that can be identified with the Montroll kink solution of Eq. (1)

$$M(\xi) = \alpha_1 + \frac{\sqrt{2}\beta}{1 + \exp(\beta\xi)}, \quad (2)$$

where  $\beta = (\alpha_2 - \alpha_1)/\sqrt{2}$  and the parameters  $\alpha_1$  and  $\alpha_2$  are two nonequal solutions of the cubic equation

$$(\psi - \alpha_1)(\psi - \alpha_2)(\psi - \alpha_3) = \psi^3 - \psi - \sigma. \quad (3)$$

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In a previous paper [6], one of the authors noted that Montroll's kink can be written as a typical tanh kink

$$M(\xi) = \gamma - \tanh\left(\frac{\beta\xi}{2}\right), \quad (4)$$

where  $\gamma \equiv \alpha_1 + \alpha_2 = 1 + \frac{\alpha_1\sqrt{2}}{\beta}$ . The latter relationship allows one to use a simple construction method of exactly soluble double-well potentials in the Schrödinger equation proposed by Caticha [7]. The scheme is a non-standard application of Witten's supersymmetric quantum mechanics [8] having as the essential assumption the idea of considering the traveling  $M$  kink as the switching function between the two lowest eigenstates of the Schrödinger equation with a double-well potential. Thus

$$\phi_1 = M\phi_0, \quad (5)$$

where  $\phi_{0,1}$  are solutions of  $\phi''_{0,1} + [\epsilon_{0,1} - u(\xi)]\phi_{0,1} = 0$ , and  $u(\xi)$  is the double-well potential to be found. Substituting Eq. (5) into the Schrödinger equation for the subscript 1 and subtracting the same equation multiplied by the switching function for the subscript 0, one obtains

$$\phi'_0 + R_M\phi_0 = 0, \quad (6)$$

where  $R_M$  is given by

$$R_M = \frac{M'' + \epsilon M}{2M'}, \quad (7)$$

and  $\epsilon = \epsilon_1 - \epsilon_0$  is the lowest energy splitting in the double-well Schrödinger equation. In addition, notice that Eq. (6) is the basic equation introducing the superpotential  $R$  in Witten's supersymmetric quantum mechanics, i.e., the Riccati solution. For Montroll's kink the corresponding Riccati solution reads

$$R_M(\xi) = -\frac{\beta}{2}\tanh\left(\frac{\beta\xi}{2}\right) + \frac{\epsilon}{2\beta}\left[\sinh(\beta\xi) + 2\gamma\cosh^2\left(\frac{\beta\xi}{2}\right)\right] \quad (8)$$

and the ground-state Schrödinger function is found by means of Eq. (6)

$$\begin{aligned} \phi_{0,M}(\xi) &= \phi_0(0) \cosh\left(\frac{\beta\xi}{2}\right) \exp\left(\frac{\epsilon}{2\beta^2}\right) \\ &\exp\left(-\frac{\epsilon}{2\beta^2}\left[\cosh(\beta\xi) - \gamma\beta\xi - \gamma\sinh(\beta\xi)\right]\right), \end{aligned} \quad (9)$$

while  $\phi_1$  is obtained by switching the ground-state wave function by means of  $M$ . This ground-state wave function is of supersymmetric type

$$\phi_{0,M}(\xi) = \phi_{0,M}(0) \exp\left[-\int_0^\xi R_M(y)dy\right], \quad (10)$$

where  $\phi_{0,M}(0)$  is a normalization constant.

The Montroll double well potential is determined up to the additive constant  $\epsilon_0$  by the 'bosonic' Riccati equation

$$\begin{aligned} u_M(\xi) &= R_M^2 - R'_M + \epsilon_0 = \frac{\beta^2}{4} + \frac{(\gamma^2 - 1)\epsilon^2}{4\beta^2} + \frac{\epsilon}{2} + \epsilon_0 + \\ &+ \frac{\epsilon}{8\beta^2}\left[(4\gamma^2\epsilon + (2\gamma^2 + 1)\epsilon\cosh(\beta\xi) - 8\beta^2)\cosh(\beta\xi) - \right. \end{aligned}$$

$$- 4\gamma \left( \epsilon + \epsilon \cosh(\beta\xi) - 2\beta^2 \right) \sinh(\beta\xi) \Big]. \quad (11)$$

Plots of the asymmetric Montroll potential and ground state wavefunction are given in Figs. (1) and (2) for a particular set of the parameters. If, as suggested by Caticha, one chooses the ground state energy to be

$$\epsilon_0 = -\frac{\beta^2}{4} - \frac{\epsilon}{2} + \frac{\epsilon^2}{4\beta^2} (1 - \gamma^2), \quad (12)$$

then  $u_M(\xi)$  turns into a traveling, asymmetric Morse double-well potential of depths depending on the Montroll parameters  $\beta$  and  $\gamma$  and the splitting  $\epsilon$

$$U_{0,m}^{L,R} = \beta^2 \left[ 1 \pm \frac{2\epsilon\gamma}{(2\beta)^2} \right], \quad (13)$$

where the subscript  $m$  stands for Morse and the superscripts  $L$  and  $R$  for left and right well, respectively. The difference in depth, the bias, is  $\Delta_m \equiv U_0^L - U_0^R = 2\epsilon\gamma$ , while the location of the potential minima on the traveling axis is at

$$\xi_m^{L,R} = \mp \frac{1}{\beta} \ln \left[ \frac{(2\beta)^2 \pm 2\epsilon\gamma}{\epsilon(\gamma \mp 1)} \right], \quad (14)$$

that shows that  $\gamma \neq \pm 1$ .

A one-parameter supersymmetric extension of the previous results is possible. It is quite known in the literature on supersymmetric quantum mechanics where it has been introduced by Mielnik, Fernandez and Nieto [9] and is based on the Darboux covariant isospectrality of Schroedinger equations. In the biological context it has been applied to the DNA molecule by Drigo-Filho and Ruggiero [10]. The point is that  $R_M$  as given in Eq. (8) is only the particular solution of the Riccati equation occurring in Eq. (11). A more general, parameter-dependent Riccati equation of the form  $u_M(\xi; \lambda) = R_M^2(\xi; \lambda) - R_M'(\xi; \lambda) + \epsilon_0$  can be constructed whose solution is a one-parameter function of the form

$$R_M(\xi; \lambda) = R_M(\xi) + \frac{d}{d\xi} \left[ \ln(I_M(\xi) + \lambda) \right] \quad (15)$$

and the corresponding one-parameter Montroll potential is given by

$$u_M(\xi; \lambda) = u_M(\xi) - 2 \frac{d^2}{d\xi^2} \left[ \ln(I_M(\xi) + \lambda) \right]. \quad (16)$$

In these formulas,  $I_M(\xi) = \int^\xi \phi_{0,M}^2(\xi) d\xi$  and  $\lambda$  is an integration constant that is used as a deforming parameter of the potential and is related to the irregular zero mode. The one-parameter Darboux-deformed ground state wavefunction can be shown to be

$$\phi_{0,M}(\xi; \lambda) = \sqrt{\lambda(\lambda + 1)} \frac{\phi_{0,M}}{I_M(\xi) + \lambda}, \quad (17)$$

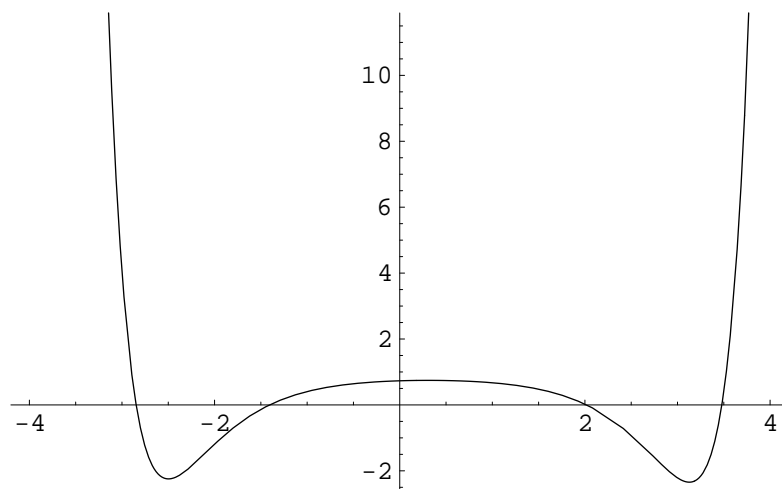
where  $\sqrt{\lambda(\lambda + 1)}$  is the normalization factor implying that  $\lambda \notin [0, -1]$ . Plots of  $u_M(\xi; \lambda)$  and  $\phi_{0,M}(\xi; \lambda)$  for  $\lambda = 10$  are presented in Figs. (3) - (4). See also Fig. (5) for a plot of the function  $I_M(\xi)$  producing the parametric Darboux deformation. A singularity at  $I_M(\xi) + \lambda = 0$  is introduced in both potential and wavefunction. If the parameter  $\lambda$  is positive the singularity is to be found on the negative  $\xi$  axis, while for negative  $\lambda$  it is on the positive side. For large values of  $\pm\lambda$  the singularity moves towards  $\mp\infty$  and the potential and ground state wave function recover the shapes of the non-parametric potential and wavefunction. The one-parameter Morse case corresponds formally to the change of subscript  $M \rightarrow m$  in Eqs. (15) and (16). For the single well Morse potential the one-parameter procedure has been studied

by Filho [11] and Bentaiba *et al* [12]. Potentials and wavefunctions with singularities are not so strange as it seems [13]. Similar to the case of the  $\delta$  potential in condensed matter physics, we interpret the singularity as representing the effect of an impurity moving along the microtubule in one direction or the other depending on the sign of the parameter  $\lambda$ . In the case of microtubules, The impurity may represent a protein attached to the microtubule or a structural discontinuity in the arrangement of the tubulin molecules. This interpretation of impurities has been given by Trpišová and Tuszyński in non-supersymmetric models of nonlinear microtubule excitations [14].

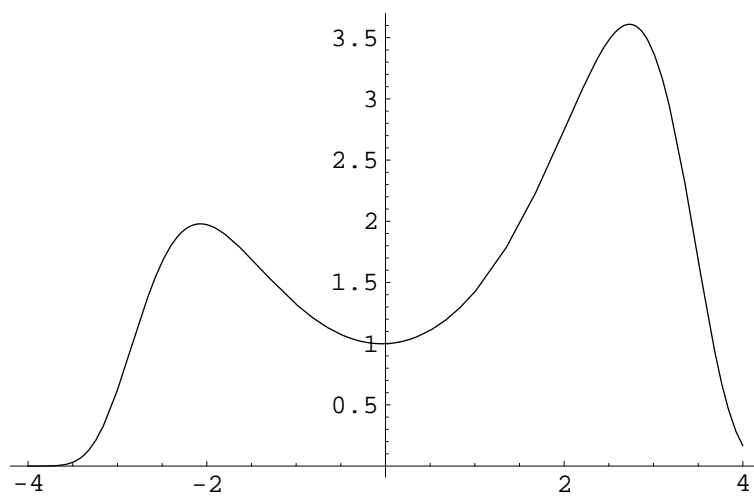
In conclusion, the supersymmetric approaches allow for a number of interesting *exact* results and point to a direct connection between Schroedinger double-well potentials and nonlinear kinks encountered in nonequilibrium chemical processes. MTs are an important application but the procedures described here can be used in many other applications. Moreover, the supersymmetric constructions can be used as a background for clarifying further details of the exact models. Although it is not so clear why one should take a certain type of kink as switching function between the Schroedinger split modes, it is interesting that proceeding in this way one will be led to some familiar double-well potential in chemical physics.

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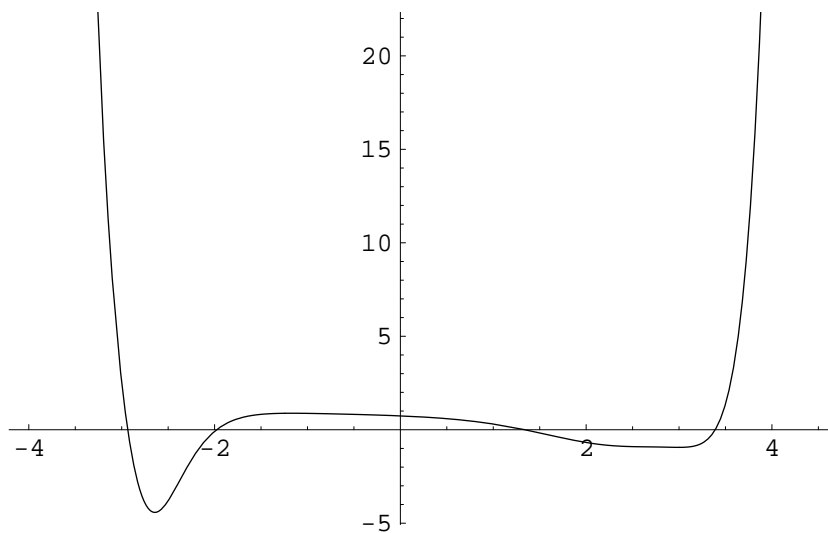
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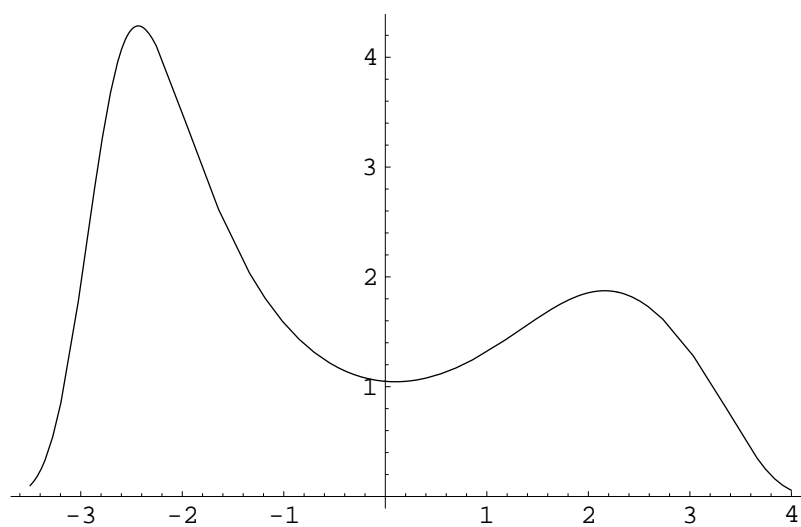
**Figure 1.** The Montroll asymmetric double-well potential (MDWP) calculated using Eq. (11) for  $\epsilon_0 = 0$ . In all figures  $\alpha_1 = 1, \alpha_2 = -1.5, \beta = -2.5/\sqrt{2}, \gamma = -0.5, \epsilon = 0.1$ .



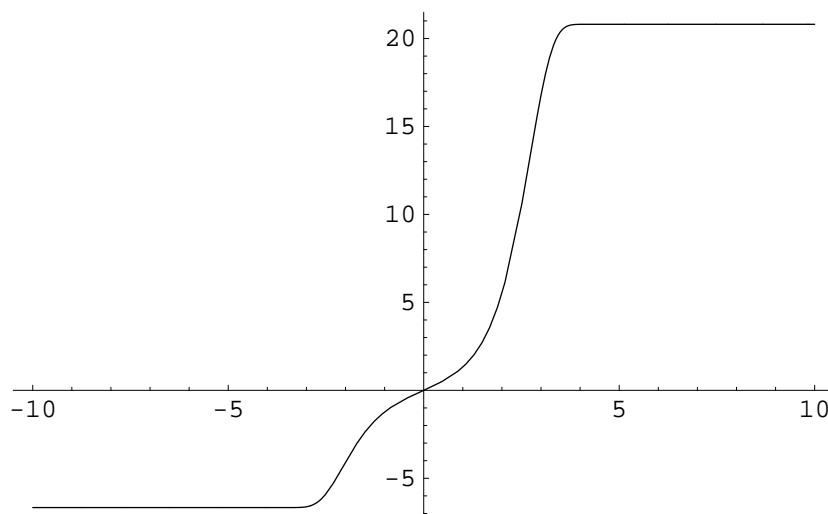
**Figure 2.** The Montroll ground state wave function cf. Eq. (9) for  $\phi_0(0) = 1$ .



**Figure 3.** The one parameter Darboux-modified MDWP for  $\lambda = 10$



**Figure 4.** The ground state wave function corresponding to  $\lambda = 10$ .



**Figure 5.** Plot of the integral  $I_M(\xi)$  that produces the deformation of the potential and wavefunctions.