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NONLINEAR SECOND ORDER ODE'S: FACTORIZATIONS AND PARTICULAR SOLUTIONS

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We present particular solutions for the following important nonlinear second order differential equations: modified Emden, generalized Lienard, convective Fisher, and generalized Burgers-Huxley. For the latter two equations these solutions are obtained in the travelling frame. All these particular solutions are the result of extending a simple and efficient factorization method that we developed in Phys. Rev. E **71** (2005) 046607.

§1. Introduction

The purpose of this paper is to obtain, through the factorization technique, particular solutions of the following type of differential equations

$$\ddot{u} + g(u)\dot{u} + F(u) = 0 \ , \tag{1·1}$$

where the dot means the derivative $D = \frac{d}{d\tau}$, and $g(u)$ and $F(u)$ could in principle be arbitrary functions of u. This is a generalization of what we did in a recent paper for the simpler equations with $g(u) = \gamma$, where γ is a constant parameter.[1)] Factorizing Eq. (1·1) means to write it in the form

$$[D - \phi_2(u)]\,[D - \phi_1(u)]\,u = 0 \ . \tag{1·2}$$

Performing the product of differential operators leads to the equation

$$\ddot{u} - \frac{d\phi_1}{du}u\dot{u} - \phi_1\dot{u} - \phi_2\dot{u} + \phi_1\phi_2 u = 0\,, \tag{1·3}$$

for which one very effective way of grouping the terms is[1)]

$$\ddot{u} - \left(\phi_1 + \phi_2 + \frac{d\phi_1}{du}u\right)\dot{u} + \phi_1\phi_2 u = 0\,. \tag{1·4}$$

Identifying Eqs. (1·1) and (1·4) leads to the conditions

$$g(u) = -\left(\phi_1 + \phi_2 + \frac{d\phi_1}{du}u\right) \tag{1·5}$$

$$F(u) = \phi_1\phi_2 u \ . \tag{1·6}$$

If $F(u)$ is a polynomial function, then $g(u)$ will have the same order as the bigger of the factorizing functions $\phi_1(u)$ and $\phi_2(u)$, and will also be a function of the constant parameters that enter in the expression of $F(u)$.

In this research, we extend the method to the following cases: the modified Emden equation, the generalized Lienard equation, the convective Fisher equation, and the generalized Burgers-Huxley equation. All of them have significant applications in nonlinear physics and it is quite useful to know their explicit particular solutions. The present work is a detailed contribution to this issue.

§2. Modified Emden equation

We start with the modified Emden equation with cubic nonlinearity that has been most recently discussed by Chandrasekhar *et al*,[2)]

$$\ddot{u} + \alpha u \dot{u} + \beta u^3 = 0 \ . \tag{2·1}$$

1) $\phi_1(u) = a_1\sqrt{\beta}u$, $\phi_2(u) = a_1^{-1}\sqrt{\beta}u$, ($a_1 \neq 0$ is an arbitrary constant).
Then Eq. (1·5) leads to the following form of the function $g(u)$

$$g_1(u) = -\sqrt{\beta}\left(\frac{2a_1^2+1}{a_1}\right)u \ . \tag{2·2}$$

Thus we can identify $\alpha = -\sqrt{\beta}\left(\frac{2a_1^2+1}{a_1}\right)$, or $a_{1_\pm} = \frac{-\alpha\pm\sqrt{\alpha^2-8\beta}}{4\sqrt{\beta}}$, where we use a_1 as a fitting parameter providing that $a_1 < 0$ for $\alpha > 0$. Eq. (2·1) is now rewritten as

$$\ddot{u} - \sqrt{\beta}\left(2a_1 + a_1^{-1}\right)u\dot{u} + \beta u^3 \equiv \left(D - a_1^{-1}\sqrt{\beta}u\right)\left(D - a_1\sqrt{\beta}u\right)u = 0 \ . \tag{2·3}$$

Therefore, the compatible first order differential equation is $\dot{u} - a_1\sqrt{\beta}u^2 = 0$, whose integration gives the particular solution of Eq. (2·3)

$$u_1 = -\frac{1}{a_1\sqrt{\beta}(\tau-\tau_0)} \quad \text{or} \quad u_1 = \frac{4}{(\alpha\pm\sqrt{\alpha^2-8\beta})(\tau-\tau_0)} \ , \tag{2·4}$$

where τ_0 is an integration constant.

2) $\phi_1(u) = a_1\sqrt{\beta}u^2$, $\phi_2(u) = a_1^{-1}\sqrt{\beta}$. Then, one gets

$$g_2(u) = -\sqrt{\beta}\left(a_1^{-1} + 3a_1u^2\right) \ . \tag{2·5}$$

Therefore, g_2 is quadratic being higher in order than the linear g of the modified Emden equation. We thus get the particular case $GE = 3\beta$, $A = 0$ of the Duffing-van der Pol equation (see case **3** of the next section)

$$\ddot{u} - \sqrt{\beta}\left(a_1^{-1} + 3a_1u^2\right)\dot{u} + \beta u^3 \equiv \left(D - a_1^{-1}\sqrt{\beta}\right)\left(D - a_1\sqrt{\beta}u^2\right)u = 0 \ , \tag{2·6}$$

which leads to the compatible first order differential equation $\dot{u} - a_1\sqrt{\beta}u^3 = 0$ with the solution

$$u_2 = \frac{1}{[-2a_1\sqrt{\beta}(\tau-\tau_0)]^{1/2}} \ . \tag{2·7}$$

§3. Generalized Lienard equation

Let us consider now the following generalized Lienard equation

$$\ddot{u} + g(u)\dot{u} + F_3 = 0 \ , \tag{3·1}$$

where $F_3(u) = Au + Bu^2 + Cu^3$. We introduce the notation $\Delta = \sqrt{B^2 - 4AC}$, and assume that $\Delta^2 > 0$ holds. Then:

1) $\phi_1(u) = a_1 \left(\frac{(B+\Delta)}{2} + Cu\right)$, $\phi_2(u) = a_1^{-1}\left(\frac{(B-\Delta)}{2C} + u\right)$; $g(u)$ takes the form

$$g_1(u) = -\left[\frac{(B+\Delta)}{2}\, a_1 + \frac{(B-\Delta)}{2C}\, a_1^{-1} + \left(2Ca_1 + a_1^{-1}\right) u\right] \ . \tag{3·2}$$

For $g(u) = g_1(u)$, we can factorize Eq. (3·1) in the form

$$\left[D - a_1^{-1}\left(\frac{(B-\Delta)}{2C} + u\right)\right]\left[D - a_1\left(\frac{(B+\Delta)}{2} + Cu\right)\right] u = 0 \ . \tag{3·3}$$

Thus, from the compatible first order differential equation $\dot{u} - a_1(\frac{(B+\Delta)}{2} + Cu)u = 0$, the following solution is obtained

$$u_1 = \frac{(B+\Delta)}{2}\left(\exp\left[-a_1\left(\frac{(B+\Delta)}{2}\right)(\tau - \tau_0)\right] - C\right)^{-1} \ . \tag{3·4}$$

2) $\phi_1(u) = a_1(A + Bu + Cu^2)$, $\phi_2(u) = a_1^{-1}$; $g(u)$ is of the form

$$g_2(u) = -\left[(a_1 A + a_1^{-1}) + 2a_1 Bu + 3a_1 Cu^2\right] \ . \tag{3·5}$$

Thus, the factorized form of the Lienard equation will be

$$\left[D - a_1^{-1}\right]\left[D - a_1 \frac{F_3(u)}{u}\right] u = 0 \tag{3·6}$$

and therefore we have to solve the equation $\dot{u} - a_1 F_3(u) = 0$, whose solution can be found graphically from

$$a_1(\tau - \tau_0) = \ln\left(\frac{u^3}{F_3(u)}\right)^{\frac{1}{2A}} - \ln\left(\frac{2Cu + B - \Delta}{2Cu + B + \Delta}\right)^{\frac{1}{2A}\frac{B}{\Delta}} \ . \tag{3·7}$$

3) *The case $B = 0$ and $C = 1$: Duffing-van der Pol equation*

The $B = 0$, $C = 1$ reduction of terms in Eq. (3·1) allows an analytic calculation of particular solutions for the so-called autonomous Duffing-van der Pol oscillator equation[3)]

$$\ddot{u} + (G + Eu^2)\dot{u} + Au + u^3 = 0 \ , \tag{3·8}$$

where G and E are arbitrary constant parameters. Since we want to compare our solutions with those of Chandrasekar *et al*,[3)] we use the second Lienard pair of factorizing functions $\phi_1(u) = a_1(A + u^2)$ and $\phi_2(u) = a_1^{-1}$.Then

$$g_2(u) = -\left(Aa_1 + a_1^{-1} + 3a_1 u^2\right) \ . \tag{3·9}$$

Eq. (3·8) is now rewritten

$$\ddot{u} - \left(a_1 A + a_1^{-1} + 3a_1 u^2\right)\dot{u} + Au + u^3 \equiv \left[D - a_1^{-1}\right]\left[D - a_1(A+u^2)\right]u = 0 \ . \quad (3\cdot10)$$

Therefore, the compatible first order equation $\dot{u} - a_1(A+u^2)u = 0$ leads by integration to the particular solution of Eq. (3·10)

$$u = \pm\left(\frac{A\exp[2a_1A(\tau-\tau_0)]}{1-\exp[2a_1A(\tau-\tau_0)]}\right)^{1/2} = \pm\left(\frac{A\exp[-\frac{2}{3}AE(\tau-\tau_0)]}{1-\exp[-\frac{2}{3}AE(\tau-\tau_0)]}\right)^{1/2} , \quad (3\cdot11)$$

where the last expression is obtained from the comparison of Eqs. (3·8) and (3·10) that gives $a_1 = -\frac{E}{3}$ and $G = \frac{AE^2+9}{3E}$.

This is a more general result for the particular solution than that obtained through other means by Chandrasekar *et al*[3)] that corresponds to $E = \beta$ and $A = \frac{3}{\beta^2}$.

§4. Convective Fisher equation

Schönborn *et al*[4)] discussed the following convective Fisher equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u) - \mu u\frac{\partial u}{\partial x} \ , \quad \text{or} \quad \ddot{u} + 2(\nu - \mu u)\dot{u} + 2u(1-u) = 0 \ , \quad (4\cdot1)$$

where the transformation to the travelling variable $\tau = x - \nu t$ was performed in the latter form. The positive parameter μ serves to tune the relative strength of convection.

1) $\phi_1(u) = \sqrt{2}a_1(1-u)$, $\phi_2(u) = \sqrt{2}a_1^{-1}$. Then $g(u) = -\sqrt{2}\left([a_1 + a_1^{-1}] - 2a_1 u\right)$. Therefore, for this $g(u)$, we can rewrite the ordinary differential form in Eq. (4·1) as

$$\ddot{u} + 2\left(-\frac{1}{\sqrt{2}}(a_1 + a_1^{-1}) + \sqrt{2}a_1 u\right)\dot{u} + 2u(1-u) = 0 \ . \quad (4\cdot2)$$

If we set the fitting parameter $a_1 = -\frac{\mu}{\sqrt{2}}$, then we obtain $\nu = \frac{\mu}{2} + \mu^{-1}$. Eq. (4·2) is factorized in the following form

$$\left[D - \sqrt{2}a_1^{-1}\right]\left[D - \sqrt{2}a_1(1-u)\right]u = 0 \ , \quad (4\cdot3)$$

that provides the compatible first order equation $\dot{u} + \mu u(1-u) = 0$, whose integration gives

$$u_1 = \left(1 \pm \exp[\mu(\tau-\tau_0)]\right)^{-1} \ . \quad (4\cdot4)$$

2) Since we are in the case of a quadratic polynomial, a second factorization means exchanging $\phi_1(u)$ and $\phi_2(u)$ between themselves. This leads to a convective Fisher equation with compatibility equation $\dot{u} - \sqrt{2}a_1^{-1}u = 0$, where now $a_1 = -\sqrt{2}\mu$, having exponential solutions of the type

$$u_2 = \pm\exp[-\mu^{-1}(\tau-\tau_0)] \ . \quad (4\cdot5)$$

§5. Generalized Burgers-Huxley equation

In this section we obtain particular solutions for the generalized Burgers-Huxley equation discussed by Wang *et al*[5)]

$$\frac{\partial u}{\partial t} + \alpha u^{\delta}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^{\delta})(u^{\delta}-\gamma) \ , \tag{5·1}$$

or in the variable $\tau = x - \nu t$

$$\ddot{u} + (\nu - \alpha u^{\delta})\dot{u} + \beta u(1-u^{\delta})(u^{\delta}-\gamma) = 0 \ . \tag{5·2}$$

1) $\phi_1(u) = \sqrt{\beta}a_1(1-u^{\delta})$, $\quad \phi_2(u) = \sqrt{\beta}a_1^{-1}(u^{\delta}-\gamma)$. Then, one gets

$$g_1(u) = \sqrt{\beta}\left(\gamma a_1^{-1} - a_1 + [a_1(1+\delta) - a_1^{-1}]u^{\delta}\right) \tag{5·3}$$

and the following identifications of the constant parameters $\nu = -\sqrt{\beta}\left(a_1 - \gamma a_1^{-1}\right)$, $\alpha = -\sqrt{\beta}\left(a_1(1+\delta) - a_1^{-1}\right)$. Writing Eq. (5·2) in factorized form

$$\left[D - \sqrt{\beta}a_1^{-1}(u^{\delta}-\gamma)\right]\left[D - \sqrt{\beta}a_1(1-u^{\delta})\right]u = 0 \ , \tag{5·4}$$

the solution

$$u_1 = \left(1 \pm \exp[-a_1\sqrt{\beta}\delta(\tau-\tau_0)]\right)^{-1/\delta} \tag{5·5}$$

of the compatible first order equation $\dot{u} - \sqrt{\beta}a_1 u(1-u^{\delta}) = 0$ is also a particular kink solution of Eq. (5·2). It is easy to solve the second identification equation for $a_1 = a_1(\alpha,\beta,\delta)$ leading to

$$a_{1_{\pm}} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta(1+\delta)}}{2\sqrt{\beta}(1+\delta)} \ . \tag{5·6}$$

Then Eq. (5·5) becomes a function $u = u(\tau;\alpha,\beta,\delta)$, and $\nu = \nu(\alpha,\beta,\gamma,\delta)$.

2) $\phi_1(u) = \sqrt{\beta}e_1(u^{\delta}-\gamma)$, $\quad \phi_2(u) = \sqrt{\beta}e_1^{-1}(1-u^{\delta})$. This pair of factorizing functions lead to

$$g_2(u) = \sqrt{\beta}\left(\gamma e_1 - e_1^{-1} + [e_1^{-1} - e_1(1+\delta)]u^{\delta}\right) \tag{5·7}$$

and the ν and α identifications: $\nu = \sqrt{\beta}\left(e_1\gamma - e_1^{-1}\right)$, $\alpha = \sqrt{\beta}\left(e_1^{-1} - e_1(1+\delta)\right)$. Eq. (5·2) is then factorized in the different form

$$\left[D - \sqrt{\beta}e_1^{-1}(1-u^{\delta})\right]\left[D - \sqrt{\beta}e_1(u^{\delta}-\gamma)\right]u = 0 \ . \tag{5·8}$$

The corresponding compatible first order equation is now $\dot{u} - \sqrt{\beta}e_1 u(u^{\delta}-\gamma) = 0$, and its integration gives a different particular solution of Eq. (5·2) with respect to that obtained for the first choice of factorizing brackets:

$$u_2 = \left(\frac{\gamma}{1 \pm \exp[e_1\sqrt{\beta}\gamma\delta(\tau-\tau_0)]}\right)^{1/\delta} \ . \tag{5·9}$$

u_2 is different of u_1 because the parameter α has changed for the second factorization. Solving the α identification for $e_1 = e_1(\alpha, \beta, \delta)$ allows to express the solution given by Eq. (5·9) in terms of the parameters of the equation, $u = u(\tau; \alpha, \beta, \gamma, \delta)$, and also one gets $\nu = \nu(\alpha, \beta, \gamma, \delta)$. If we set $\delta = 1$ in Eq. (5·9), then from $\alpha = \sqrt{\beta}(e_1^{-1} - 2e_1)$ one can get $e_{1\pm} = \frac{\alpha \pm \sqrt{\alpha^2 + 8\beta}}{4\sqrt{\beta}}$ that can be used to obtain $\nu_\pm = \nu(\alpha, \beta, \gamma)$. The solutions given by Eqs. (5·5) and (5·6) and in (5·9) have been obtained previously by Wang *et al*[5] by a different procedure.

§6. Conclusion

In this paper, the efficient factorization scheme that we proposed in a previous study[1] has been applied to more complicated second order nonlinear differential equations. Exact particular solutions have been obtained for a number of important nonlinear differential equations with applications in physics and biology: the modified Emden equation, the generalized Lienard equation, the Duffing-van der Pol equation, the convective Fisher equation, and the generalized Burgers-Huxley equation.

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