A new theory on tidal currents rotation

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[1] Applying the linearized equations of motion, projection on the complex plane and a representation of the velocity vector and the forcing sea surface elevation gradient in function of the eigenvectors of the homogeneous system, we developed a theory that explains essential properties of tidal currents. We found the existence of a fundamental vector which we call sense of rotation **s**. The ellipticity, until now defined as a scalar quantity, results from the theory as a vector ε, normalized from **s**. ε is a measure of the eccentricity and sense of rotation of tidal ellipses and has some properties that are similar to those of the angular velocity. We derived an expression for the major and minor semi-axis in function of physical properties and characteristics of amphidromic systems. These and other results of the theory allow the analysis of important aspects of tidal currents. **Citation:** Carbajal, N., and J. H. Gaviño (2007), A new theory on tidal currents rotation, *Geophys. Res. Lett.*, *34*, L01609, doi:10.1029/2006GL027670.

1. Introduction

[2] A lot of work has been done to investigate the generation of tides and to explain the rich spectrum of their manifestations in the world. All this research has been summarized, among others, in known books on tides [*Defant*, 1961; *Melchior*, 1966; *Godin*, 1972; *Pugh*, 1987; *Cartwright*, 1999]. The observed tides in the oceans consist fundamentally of two parts; oscillations induced by the tide-generating forces associated to the gravitational potentials of sun and moon and the free oscillations induced by the interaction of the primary tides with the ocean basins. Loading effects should also be considered in more precise studies of tides. Numerical simulations and data assimilation played an important role in reproducing observed patterns and in understanding relevant phenomena associated to the propagation of tides [*Gotlib and Kagan*, 1982; *Zahel*, 1991, 1997; *Zahel and Müller*, 2005]. Another important research target has been the estimation of the resonance modes of the world's oceans [*Platzman et al.*, 1981; *Gaviño*, 1984]. The modes explain the magnification of the oscillations induced by the gravitational potentials in all regions of the oceans. The modes of resonance are free oscillations described by the homogeneous equations of motion. The importance of the free modes of resonance is that they allow the reproduction of the so called synthetic tide.

[3] The principal manifestations of tides in the world's oceans, i.e., the up and down movement of the sea surface elevation and the behavior of currents are summarized by the amphidromic systems of the different tidal constituents and their respective distribution of tidal ellipses. The amphidromic system describes the patterns of amplitudes and phases of the oscillating sea surface elevation. The tidal ellipses contain information on amplitude, phase and sense of rotation of the tidal currents. There have been several attempts to find a direct mathematical link between features of amphidromic systems and properties of tidal currents [*Prandle*, 1982; *Carbajal*, 2000, 2004]. In this research work, a new theory has been developed by treating the equations in a more fundamental form. It gives a new interpretation of important parameters associated to tidal currents. The calculation of tidal currents from satellite altimetry is an important issue [*Ray*, 2001]. We suggest how the theory can be applied to calculate tidal currents from gradients of the sea surface elevation.

2. Theory

[4] Consider the vertically averaged and linearized equations of motion, $\partial_t \mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{b}$, where the vectors

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} -g\dfrac{\partial\zeta}{\partial x} \\ -g\dfrac{\partial\zeta}{\partial y} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} u \\ v \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} -r & f \\ -f & -r \end{bmatrix} \quad (1)$$

have been introduced. t is the time, (x, y) are space variables, (u, v) are the components of the current velocity in the x and y directions, respectively, ζ is the sea surface elevation, $f = 2\Omega\sin\varphi$ is the Coriolis parameter, Ω is the angular velocity of the Earth, φ is the latitude, g is the acceleration due to gravity and r is the linear friction coefficient. We characterize column vectors as $\mathbf{v}$ and row vectors as its transposed $\mathbf{v}^t$. The variable $\mathbf{b}$ can be considered as a forcing vector. The free modes contained in the equation $\partial_t \mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{b}$ are obtained with $\mathbf{b} = \mathbf{0}$ and a harmonic oscillation of the form: $\mathbf{v} = \mathbf{v}_0 e^{\sigma t}$, where σ is the eigenfrequency. $\mathbf{v}_0$ is the space depending part of the velocity vector. We obtain the well known eigenvalues of the matrix $\mathbf{A}$; to the first eigenvalue, $\sigma_1 = -r + i\,f$, corresponds the eigenvector $\mathbf{v}_1^t = [1 \;\; i]$ where $i = \sqrt{-1}$. From the second eigenvalue $\sigma_2 = -r - i\,f$, the eigenvector $\mathbf{v}_2^t = [1 \; -i]$ is obtained. $\mathbf{v}_1$ and $\mathbf{v}_2$ have the absolute value $\sqrt{2}$. In order to develop the theory, we consider the velocity vector to have in the complex plane the general form: $\mathbf{v}^t = [u_r + iu_i \;\; v_r + iv_i] = [u_r \;\; v_r] + i[u_i \;\; v_i] = \mathbf{v}_\mathbf{r}^t + i\mathbf{v}_\mathbf{i}^t$. The sense of rotation of this vector is given by the cross product of the real by the imaginary part, i.e. $[u_r \;\; v_r] \times [u_i \;\; v_i] = \mathbf{v}_\mathbf{r}^t \times \mathbf{v}_\mathbf{i}^t = (u_r v_i - v_r u_i)\mathbf{k}^t$, where $\mathbf{k}$ is a unit vector parallel to $\mathbf{v}_\mathbf{r} \times \mathbf{v}_\mathbf{i}$. We now apply this concept to the eigenvectors $\mathbf{v}_1^t = [1 + 0i$

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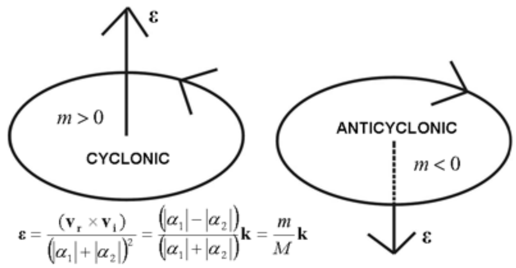

Figure 1. Rotation of tidal currents. Conceptual representation of tidal ellipses with the vector $\mathbf{s} = \mathbf{v_r} \times \mathbf{v_i}$, indicating the sense of rotation, cyclonic or anticyclonic, of tidal currents and the ellipticity vector ε, specifying both, sense of rotation of tidal currents and the eccentricity of the ellipses.

$0 + i]$ and $\mathbf{v}_2^t = [1 + 0i \quad 0 - i]$. The cross product of the real by the imaginary part of the $\mathbf{v}_1^t$ is $[1\ 0] \times [0\ 1] = 1\mathbf{k}^t$. The result is positive and therefore this eigenvector rotates cyclonically (C). We obtain for the second eigenvector $\mathbf{v}_2^t$, $[1\ 0] \times [0\ -1] = -1\mathbf{k}^t$, i.e. the sense of rotation of this vector is anticyclonic (AC). Any velocity vector $\mathbf{v}_0$ can be written as a linear combination of the eigenvectors $\mathbf{v}_1$ and v_2, i.e. $\mathbf{v}_0 = \mathbf{v_1}\alpha_1 + \mathbf{v_2}\alpha_2$. This mathematical form is equivalent to representing the vector as a function of its C (α_1) and AC (α_2) components. To calculate the α_1 and α_2, we carry out the below indicated interior products: $2\alpha_1 = \mathbf{v}_1^{t*}\mathbf{v}_0 = (u_r + v_i) + i(u_i - v_r)$ and $2\alpha_2 = \mathbf{v}_2^{t*}\mathbf{v}_0 = (u_r - v_i) + i(u_i + v_r)$, where the symbol * means complex conjugation. By taking absolute values in the last two equations, we have

$$|2\alpha_1|^2 = |u|^2 + |v|^2 + 2(u_r v_i - u_i v_r) = |\mathbf{v}_0|^2 + 2\mathbf{k}^t(\mathbf{v_r} \times \mathbf{v_i}) \tag{2}$$

$$|2\alpha_2|^2 = |u|^2 + |v|^2 - 2(u_r v_i - u_i v_r) = |\mathbf{v}_0|^2 - 2\mathbf{k}^t(\mathbf{v_r} \times \mathbf{v_i}) \tag{3}$$

[5] Subtracting (3) from (2) and considering that $|\alpha_1|$ and $|\alpha_2|$ are the C and AC components of the velocity vector, we conclude that

$$\mathbf{s} = (\mathbf{v_r} \times \mathbf{v_i}) = (|\alpha_1| + |\alpha_2|)(|\alpha_1| - |\alpha_2|)\mathbf{k} = Mm\mathbf{k} \tag{4}$$

[6] Where we have introduced the sense of rotation vector $\mathbf{s}$. $M = |\alpha_1| + |\alpha_2|$ and $m = |\alpha_1| - |\alpha_2|$ are the major and minor semi-axes of the tidal ellipses, respectively. $\mathbf{s}$ is a vector pointing normal to the plane formed by $\mathbf{v_r}$ and $\mathbf{v_i}$. When $m > 0$, $\mathbf{s}$ points upwards and the sense of rotation is cyclonic (C) and when $m < 0$, $\mathbf{s}$ points downwards and the sense of rotation is anticyclonic (AC). The scalar parameter ellipticity ε, defined as the ratio $\varepsilon = m/M$, can be seen as a normalization of the vector $\mathbf{s} = (\mathbf{v_r} \times \mathbf{v_i})$. In this way, we introduced the ellipticity vector

$$\varepsilon = \frac{(\mathbf{v_r} \times \mathbf{v_i})}{(|\boldsymbol{\alpha}_1| + |\boldsymbol{\alpha}_2|)^2} = \frac{(|\boldsymbol{\alpha}_1| - |\boldsymbol{\alpha}_2|)}{(|\boldsymbol{\alpha}_1| + |\boldsymbol{\alpha}_2|)}\mathbf{k} = \frac{m}{M}\mathbf{k} \tag{5}$$

[7] The ellipticity vector ε has some similar features to those of the angular velocity (Figure 1).

[8] This is an important result since the derivation of the sense of rotation of tidal currents as a vector results directly from fundamental properties of the equations of motion and not as a scalar quantity with a purely kinematic character [*Godin*, 1972; *Pugh*, 1987].

[9] In the particular case that the forcing term is the barotropic pressure gradient, we proceed as follows:

$$\mathbf{b}^t = -g\nabla^t\zeta = -g\left[\partial_x \quad \partial_y\right](\zeta_r + i\zeta_i) = -g\left[\zeta_{rx} + i\zeta_{ix} \quad \zeta_{ry} + i\zeta_{iy}\right]. \tag{6}$$

[10] Where ζ_r and ζ_i are the real and the imaginary components of the pressure gradient, respectively. Further, considering $\mathbf{b} = [\mathbf{v_1}\beta_1 + \mathbf{v_2}\beta_2]e^{-i\omega t}$ results

$$-2\beta_1 = -\mathbf{v}_1^{t*}\mathbf{b} = g\left[\left(\zeta_{rx} + \zeta_{iy}\right) + i\left(\zeta_{ix} - \zeta_{ry}\right)\right] \tag{7}$$

$$-2\beta_2 = -\mathbf{v}_2^{t*}\mathbf{b} = g\left[\left(\zeta_{rx} - \zeta_{iy}\right) + i\left(\zeta_{ix} + \zeta_{ry}\right)\right] \tag{8}$$

[11] Taking absolute values in the previous equations and after some algebraic manipulations, we obtain expressions for β_1 and β_2 as a function of the amplitude $|\zeta| = (\varsigma_r^2 + \varsigma_i^2)^{1/2}$ and of the phase $\phi = \arctan(\varsigma_i/\varsigma_r)$, i.e.

$$\begin{aligned} |\beta_1|^2 &= \frac{g^2}{4}\left(|\nabla\zeta|^2 + \mathbf{e}^t\left(\nabla|\zeta|^2 \times \nabla\phi\right)\right) \\ |\beta_2|^2 &= \frac{g^2}{4}\left(|\nabla\zeta|^2 - \mathbf{e}^t\left(\nabla|\zeta|^2 \times \nabla\phi\right)\right) \end{aligned} \tag{9}$$

[12] Where $\mathbf{e}$ is a unit vector parallel to $\nabla|\zeta|^2 \times \nabla\phi$. The term $\nabla\zeta_r \times \nabla\zeta_i = (1/2)\nabla|\zeta|^2 \times \nabla\phi$, which appears in the algebraic calculations, is of fundamental importance since it determines the influence of the gradients of phases and amplitudes of the sea surface elevation on the sense of rotation of tidal currents associated to a particular constituent.

[13] The equation $\partial_t\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{b}$ can be solved with $\mathbf{v} = (\mathbf{v_1}\alpha_1 + \mathbf{v_2}\alpha_2)e^{-i\omega t}$ and the forcing term $\mathbf{b} = [\mathbf{v_1}\beta_1 + v_2\beta_2]e^{-i\omega t}$. From these equations, we obtain the relations

$$\alpha_1 = \frac{\beta_1}{r - i(\omega + f)}, \quad \alpha_2 = \frac{\beta_2}{r - i(\omega - f)} \tag{10}$$

[14] Combining equations (9) and (10) and developing the product $\nabla|\zeta|^2 \times \nabla\phi$, it results

$$\begin{aligned} |\alpha_1|^2 &= \frac{g^2\left[|\nabla\zeta|^2 + \left|\nabla|\zeta|^2\right||\nabla\phi|\sin\gamma\right]}{4\left[r^2 + (\omega + f)^2\right]} \\ |\alpha_2|^2 &= \frac{g^2\left[|\nabla\zeta|^2 - \left|\nabla|\zeta|^2\right||\nabla\phi|\sin\gamma\right]}{4\left[r^2 + (\omega - f)^2\right]} \end{aligned} \tag{11}$$

[15] These last two equations relate the C and AC components of the velocity with physical parameter and features of the amphidromic systems. γ is the angle between $\nabla|\zeta|^2$ and $\nabla\phi$. Since $|\alpha_1|$ and $|\alpha_2|$ are the cyclonic and

anticyclonic components of the velocity vector, we also get expression for the major (*M*) and minor (*m*) semi-axes of the ellipses. These are given by

$$|\alpha_1| + |\alpha_2| = M = \frac{g}{2}\left[\sqrt{\frac{|\nabla\zeta|^2 + |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{r^2 + (\omega + f)^2}} + \sqrt{\frac{|\nabla\zeta|^2 - |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{r^2 + (\omega - f)^2}}\right] \quad (12)$$

$$|\alpha_1| - |\alpha_2| = m = \frac{g}{2}\left[\sqrt{\frac{|\nabla\zeta|^2 + |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{r^2 + (\omega + f)^2}} - \sqrt{\frac{|\nabla\zeta|^2 - |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{r^2 + (\omega - f)^2}}\right] \quad (13)$$

[16] If one substitutes equations (12) and (13) in equation (5), the sense of rotation and eccentricity of the ellipses can be calculated. Although we have derived expressions for all properties of tidal currents, we also gain information on the sense of rotation by defining the parameter

$$F = \frac{|\alpha_1|}{|\alpha_2|} = \sqrt{\frac{r^2 + (\omega - f)^2}{r^2 + (\omega + f)^2}}\sqrt{\frac{|\nabla\zeta|^2 + |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{|\nabla\zeta|^2 - |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}} = \alpha\beta \quad (14)$$

with

$$\alpha = \sqrt{\frac{r^2 + (\omega - f)^2}{r^2 + (\omega + f)^2}}, \quad \beta = \sqrt{\frac{|\nabla\zeta|^2 + |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}{|\nabla\zeta|^2 - |\nabla|\zeta|^2||\nabla\phi|\sin\gamma}}. \quad (15)$$

α contains information on the physical properties and β includes only features of the amphidromic system. In fact, starting from equation (14), a similar formula obtained by Carbajal [*Carbajal*, 2004] can be derived. For $F = 1$, rectilinear motion occurs. For $F > 1$ the cyclonically sense of rotation takes place in the northern hemisphere and for $F < 1$ it is anticyclonic.

[17] A possible and interesting application of this theory can be further derived. With the sea surface elevation gradient $\mathbf{b}_0 e^{-i\omega t}$ measured, for example, by satellite altimetry and using the equation $\partial_t \mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{b}_0 e^{-i\omega t}$, the velocity vector can be calculated with a solution of the form $\mathbf{v} = (\mathbf{v}_1\alpha_1 + \mathbf{v}_2\alpha_2)e^{-i\omega t}$. $\mathbf{b}_0$ is the time independent part of $\mathbf{b}$. With the α's given by the equations (10), we have

$$2\alpha_1 = \frac{2\beta_1}{r - i(\omega + f)} = \frac{\mathbf{v}_1^{t*}\mathbf{b}}{r - i(\omega + f)} = -\frac{(r + i(\omega + f))\{(\zeta_{rx} + \zeta_{iy}) + i(\zeta_{ix} - \zeta_{ry})\}}{r^2 + (\omega + f)^2} \quad (16)$$

$$2\alpha_2 = \frac{2\beta_2}{r - i(\omega - f)} = \frac{\mathbf{v}_2^{t*}\mathbf{b}}{r - i(\omega - f)} = -\frac{(r + i(\omega - f))\{(\zeta_{rx} - \zeta_{iy}) + i(\zeta_{ix} + \zeta_{ry})\}}{r^2 + (\omega - f)^2} \quad (17)$$

[18] Once α_1 and α_2 are known, the velocity vector can be calculated using $\mathbf{v} = (\mathbf{v_1}\alpha_1 + \mathbf{v_2}\alpha_2)e^{-i\omega t}$. In a future work we will present the application of this theory.

3. Conclusions

[19] The fact that the sense of rotation **s** and the ellipticity ε are vectors should have consequences in the analysis of tidal currents. Since in the theory the major and minor semi-axes of the ellipses, the sense of rotation and the ellipticity depend on physical parameters like frequency, Coriolis parameter and friction coefficient and on properties of amphidromic systems, it will allow investigating more explicitly the influence of all these parameters on the behavior of tidal currents. Further, we have derived new expressions for calculating tidal currents from known sea surface elevation gradients, for example from measured satellite altimetry in ocean regions. It would be of interest extending this theory to include eddy viscosity terms and direct gravitational forcing. Finally, although the continuity equation is not necessary for the analysis of properties of tidal currents, we have obtained, of course, expressions for it in function of fundamental quantities. It will be the subject of future research work.

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