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Control of mechanical systems on Lie groups based on Vertically Transverse Functions

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Abstract

The transverse function approach to control provides a unified setting to deal with practical stabilization and tracking of arbitrary trajectories for controllable driftless systems. Controllers derived from that approach offer advantages over those based on more classical techniques for control of nonholonomic systems. Nevertheless, its extension to more general classes, such as critical underactuated mechanical systems, is not immediate. The present paper explores a possible extension by developing a framework that allows one to cast point stabilization problems for (left-invariant) second-order systems on Lie groups, including simple mechanical systems. The approach is based on "vertical transversality," a property exhibited by the derivative of transverse functions. In this paper we lay out the theoretical foundations of our approach and present an example to illustrate some of its features.

1 Introduction

The transverse function approach (TFA) is a recent approach to solve stabilization and tracking problems for a wide class of control systems, including driftless ("nonholonomic") systems which are typically critical in that they do not satisfy Brockett's stabilizability condition [13, 14, 15]. More recently, the approach has been partially extended to an even larger class, including critical underactuated mechanical systems [16, 11, 17]. As originally introduced, the approach addresses the control of systems

$$\dot{x} = X_0(t, x) + \sum_{i=1}^{m} u^i X_i(x), \tag{1}$$

where $X_0(t, \cdot), X_1, \ldots, X_m$ denote smooth vector fields on an n-dimensional, smooth manifold M, under the assumption that the Lie algebra $\mathrm{Lie}(\boldsymbol{X})$, generated by the set of *control vector fields* $\boldsymbol{X} = \{X_1, \ldots, X_m\}$, spans T_pM at some point p in M (i.e., $\boldsymbol{X}$ satisfies the Lie algebra rank condition, LARC, at p). Systems of the form (1) comprise controllable *driftless* systems with (possibly null) time-varying disturbances, which may represent model uncertainties or terms that typically arise in trajectory tracking problems. Among other virtues, the feedback laws obtained via the TFA allow one to tackle difficulties linked to the control of "critical" systems, that is, systems that do not satisfy Brockett's stabilizability conditions, or generalizations thereof. In particular, those feedback laws address the problems of stabilization of equilibria and of more general trajectories by trading *convergence* to the desired value for convergence to a given *neighborhood* of that value, settling for practical, instead of asymptotic stability. This trade-off is in accordance with results in [9], where it is pointed out that constructing "universal" controllers that asymptotically stabilize arbitrary system trajectories is a hopeless goal for some classes of systems subject to nonholonomic constraints. Under appropriate conditions, the feedback laws derived using the TFA also ensure practical stabilization of *non-feasible* trajectories and, moreover, they

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are *smooth*, so they do not exhibit some of the nonrobustness issues alluded to in [10]. Along with its applications, the original formulation of the TFA has been generalized in several directions. For example, in [15], the approach was enhanced to yield controllers that guarantee *asymptotic* stabilization to a point whenever the drift vector field X_0 allows it. In, [14], the TFA and the constructive algorithm to define the transverse functions was refined for the particular case of driftless systems defined on Lie groups. More recently, in order to incorporate a larger class of systems, including critical underactuated mechanical systems (in particular those that are not kinematic in the sense of [7]), two approaches have been independently developed. The first one, initially introduced in [16] and further advanced in [17], cleverly uses transverse functions in a construction which guarantees practical stabilization of the configuration variables ("positions and orientations"), and even convergence of the velocities to zero provided the drift vector field so allows it. The second approach, introduced in [11], makes use of transverse functions in a significantly different way as compared to the first one. Although, for the time being, the approach developed in [16, 17] seems to be more promising to yield practical stabilization and tracking results, the one in [11] provides an alternative which we explore in more detail in the present paper. More specifically, we introduce a framework that allows one to formulate practical point-stabilization problems for second-order systems, in particular those defined on (tangent) Lie groups. Some examples of these include underactuated manipulators, blimp-like systems and underwater vehicles. Our methodology builds upon two main ingredients, the first of which is the observation that the tangent mapping Tf of every transverse function f as defined in [13], satisfies what we call *vertical transversality*. The second ingredient is the adjunction, to the system being controlled, of an auxiliary control system—a dynamic extension—which evolves on the domain of Tf. An error signal is defined to measure the difference between the state of the controlled system and the image by Tf of the state of the auxiliary system. The two ingredients are related by the fact that the vertical transversality of Tf endows one with full control over the *second-order* time-derivatives of the error signal, thus allowing one to enforce a dynamics for which the error vanishes asymptotically under appropriate conditions. The latter, however, are not necessarily satisfied in a number of practical situations; instead, for every given application, a zero-dynamics analysis is required to establish the long-term behavior of the controlled system. This issue motivates the search for conditions to guarantee that the asymptotic behavior of the solutions is acceptable. In its full generality, this remains an open problem; however, in the case of systems underactuated by one control, necessary and sufficient conditions for the existence of solutions and for the boundedness of their velocity coordinates are stated below.

The paper is organized as follows. In Section 2 we recall basic concepts needed to establish our results, while in Section 3 we recall, from [13], the construction of transverse functions for driftless systems. In Section 4 we single out the notion of vertically transverse function. Our approach to cast point-stabilization problems for second-order systems is outlined in Section 5, where additional results are stated regarding the nature of the zero-dynamics and some stability properties of the resulting system. In Section 6 we work out an example and in Section 7 we include concluding remarks. Finally, the Appendix in Section 8 contains some lemmas and the more technical proofs.

2 Preliminary notions

2.1 Basic concepts

We recall standard notions from differential geometry mainly to fix notations (cf. e.g. [1] or [19]). Given a smooth (Hausdorff, paracompact) manifold Q, and its first and second tangent bundles TQ and TTQ, we let $\pi_Q : TQ \longrightarrow Q$ and $\pi_{TQ} : TTQ \longrightarrow TQ$ denote their respective tangent bundle projections. The tangent space to Q at a point $q \in Q$ is denoted by T_qQ. Given smooth manifolds Q, P and a mapping $f : Q \longrightarrow P$, we write $T_qf : T_qQ \longrightarrow T_{f(q)}P$ for the tangent mapping of f at q and Tf for the respective bundle map covering f. If base point q is clear from the context, we usually write $Tf(v)$ instead of $T_qf(v)$. The sets of *smooth* vector fields on Q and on TQ will be denoted by $\Gamma(TQ)$ and $\Gamma(TTQ)$, respectively, and $C^\infty(Q)$ denotes the $\mathbb{R}$-algebra of smooth, real-valued functions on Q. For simplicity we frequently write X_q instead of $X(q)$ for the value of a vector field X at a point q. A coordinate chart on Q is a pair $(U, (q^1, \ldots, q^n))$, also written (U, q) for conciseness, with U open in Q and $q = (q^1, \ldots, q^n) : U \longrightarrow \mathbb{R}^n$ a homeomorphism. Any chart (U, q) on Q determines, in a natural way (cf. e.g. [19, §1.25]), charts on TQ

and, in turn, on TTQ. The coordinates corresponding to those charts will usually be denoted by $(q,\dot{q})$ and $(q,\dot{q},\alpha_L,\alpha_H)$, respectively, and referred to as *natural coordinates* (induced by (U,q)) on TQ and TTQ, respectively. We shall write $r=(r^1,\dots,r^n)$ for the canonical coordinates on $\mathbb{R}^n$. *In what follows it is assumed, unless otherwise specified, that manifolds (including Lie groups) are finite-dimensional, connected and smooth (i.e., of class C^∞), and that mappings on manifolds are smooth.*

2.2 Second-order, vertical and related constructs

The concepts recalled in this section occur less frequently in the literature; for more details on those notions the reader may wish to consult e.g. [1, 12, 2]. A vector field $X\in\Gamma(TTQ)$ is said to be a ***second-order vector field*** (one also says that "X defines a second-order equation on Q" or simply that "X is second order") if $T\pi_Q\circ X=\mathrm{id}_{TQ}$. In natural coordinates, $X\in\Gamma(TTQ)$ is second-order if, and only if, it is of the form $X(q,\dot{q})=(q,\dot{q},\dot{q},X_H(q,\dot{q}))$. This definition extends naturally to vector fields along a curve in TQ, namely, if X is defined on the image of a curve $\gamma:(t_0,t_1)\longrightarrow TQ$ (for example, if X is the vector field *tangent* to γ so that $X_{\gamma(t)}=T_t\gamma\left(\partial/\partial r|_t\right)$) then X is said to be ***second-order along*** γ if for every $t\in(t_0,t_1)$, $T\pi_Q(X_{\gamma(t)})=\gamma(t)$. Associated with any such curve γ is the corresponding ***base curve*** $\pi_Q\circ\gamma:(t_0,t_1)\longrightarrow Q$. Given $v\in TQ$, the ***vertical space over*** v is $\ker(T_v\pi_Q)$, a subspace of T_vTQ which we denote by T_vTQ^{vert} and whose elements are said to be ***vertical (tangent) vectors***. In natural coordinates, α is vertical in $T_{(q,\dot{q})}TQ$ if, and only if, it is of the form $\alpha=(q,\dot{q},0,\alpha_H)$. The disjoint union of vertical spaces over points in TQ inherits a natural structure that makes it a vector subbundle of TTQ, called the vertical subbundle TTQ^{vert}. A section $X\in\Gamma(TTQ^{\mathrm{vert}})$ of this subbundle is called a ***vertical vector field***. Given tangent vectors $v,w\in TQ$ such that $\pi_Q(v)=\pi_Q(w)$, one defines the ***vertical lift of*** w ***by*** v as the vector in T_vTQ given by $\mathrm{lift}(v,w)=T_0\gamma_{v,w}\left(\partial/\partial r|_0\right)$, where $\gamma_{v,w}:\mathbb{R}\longrightarrow TQ$ is the curve determined by $\gamma_{v,w}(t)=v+tw$. (This choice of notation should not cause any confusion, since the related notion of *horizontal lift* is not used in this paper.) Given a vector field $X\in\Gamma(TQ)$, the ***vertical lift of*** X is the vector field $X^{\mathrm{lift}}\in\Gamma(TTQ)$ defined by $X^{\mathrm{lift}}_v=\mathrm{lift}(v,X_{\pi_Q(v)})$. In natural coordinates, if $X(q)=(q,\hat{X}(q))$ then $X^{\mathrm{lift}}(q,\dot{q})=(q,\dot{q},0,\hat{X}(q))$. System (1), under the assumption that $X_0(t,\,\cdot\,),X_1,\dots,X_m\in\Gamma(TTQ)$ for all $t\in\mathbb{R}$, is said to be a ***second-order (control-affine) system on*** TQ if $X_0(t,\,\cdot\,)+\sum_{i=1}^m u^iX_i$ is a second-order vector field for every $u=(u^1,\dots,u^m)\in\mathbb{R}^m$ and every $t\in\mathbb{R}$. One easily checks that if (1) is second order, then $X_0(t,\,\cdot\,)$ is itself second order for every $t\in\mathbb{R}$, and the vector fields $X_1,\dots,X_m$ are vertical. The ***Liouville vector field associated with*** Q is the vector field $C\in\Gamma(TTQ)$ given by $C_v=\mathrm{lift}(v,v)$ (we also write C^Q to emphasize the associated manifold Q). A vector field $S\in\Gamma(TTQ)$ is said to be a ***spray*** if S is second order and $[C,S]=S$. In natural coordinates, $C(q,\dot{q})=(q,\dot{q},0,\dot{q})$, and $X\in\Gamma(TTQ)$ is a spray if, and only if, X is of the form $X(q,\dot{q})=(q,\dot{q},\dot{q},X_H(q,\dot{q}))$ and the components of X_H are quadratic on the coordinates $\dot{q}$.

2.3 Tangent Lie groups

Given an n-dimensional Lie group G, we consider its associated *tangent group* TG (cf. [12, Chap. 9]) with Lie group structure determined by the multiplication $\mu:TG\times TG\longrightarrow TG$ given by

$$\mu(v,w)=T\widehat{L}_{\pi_G(v)}(w)+T\widehat{R}_{\pi_G(w)}(v),\tag{2}$$

where $\widehat{L}_g:h\mapsto gh$ and $\widehat{R}_g:h\mapsto hg$ denote left and right translations on G, respectively. In the sequel, we also write vw instead of $\mu(v,w)$. Endowed with this structure, TG admits 0_e (the zero vector in T_eG) as identity element. In the sequel, when both a Lie group G and its tangent group TG are involved in a discussion, we systematically use $\widehat{R},\widehat{L}$ to denote translations on G, and R,L to represent translations on TG. If $g:(t_0,t_1)\longrightarrow G$ is a curve on a Lie group G, we denote by $g^{-1}:(t_0,t_1)\longrightarrow G$ the curve defined by $t\mapsto(g(t))^{-1}$.

3 Transverse functions for driftless systems

In this section we recall the results of [13] on the existence and construction of transverse functions for driftless systems. Consider a set of vector fields $\boldsymbol{X}=\{X_1,\dots,X_m\}\subset\Gamma(TQ)$ and a point $p\in Q$ such that $\{Y_p : Y \in \mathrm{Lie}(\boldsymbol{X})\} = T_pQ$. It follows from [13, Thm. 1] that, given a neighborhood U of p, there exist an integer $\kappa \geq n-m$ and a *transverse function* $f : \mathbb{T}^\kappa \longrightarrow Q$ that satisfies $f(\mathbb{T}^\kappa) \subset U$ and, for every $\theta \in \mathbb{T}^\kappa$,

$$T_{f(\theta)}Q = Tf(T_\theta \mathbb{T}^\kappa) + \mathrm{span}_{\mathbb{R}}\{X_{1,f(\theta)}, \dots, X_{m,f(\theta)}\}, \tag{3}$$

where $\mathbb{T}^\kappa$ denotes the κ-torus $(\mathbb{R}/2\pi\mathbb{Z})^\kappa$. In the sequel we shall refer to any such mapping as ***Morin–Samson transverse function for $\boldsymbol{X}$ (near p)***. Note that, while in general the sum in (3) is not direct, i.e., κ need not equal $n-m$, in some cases f can be chosen so that it is, for instance when $Q = G$ is an n-dimensional Lie group and the vector fields $X_1, \dots, X_m$ are left-invariant. In the latter case, the explicit construction in [14] of a transverse function can be detailed as follows. Let $\xi_1, \dots, \xi_m$ be elements of $\mathfrak{g}$, the Lie algebra of G, such that $\mathrm{Lie}(\{\xi_1, \dots, \xi_m\}) = \mathfrak{g}$ and assume that X_i is the vector field associated to ξ_i by setting $X_{i,g} = T_e\widehat{L}_g(\xi_i)$ for $i = 1, \dots, m$ and $g \in G$. Next, define inductively a family $(G_k)_{k\in\mathbb{N}}$ of subspaces of $\mathfrak{g}$ by setting $G_0 = \mathrm{span}_{\mathbb{R}}\{\xi_1, \dots, \xi_m\}$ and $G_k = G_{k-1} + [G_0, G_{k-1}]$ for $k \geq 1$. Then consider mappings $\lambda, \rho : \{m+1, \dots, n\} \longrightarrow \{1, \dots, n\}$ and an ordered basis $\{\zeta_1, \dots, \zeta_n\}$ of $\mathfrak{g}$ such that

1. $G_k = \mathrm{span}_{\mathbb{R}}\{\zeta_1, \dots, \zeta_{\dim(G_k)}\}$ for $k = 1, \dots, \min\{k : G_k = \mathfrak{g}\}$.
2. Whenever $k \geq 2$ and $\dim(G_{k-1}) \leq i \leq \dim(G_k)$, one has $\zeta_i = [\zeta_{\lambda(i)}, \zeta_{\rho(i)}]$, with $\zeta_{\lambda(i)} \in G_a$, $\zeta_{\rho(i)} \in G_b$ and $a + b = k$.

The set $\{\zeta_1, \dots, \zeta_n\}$, together with the mappings λ and ρ, constitute a ***graded basis*** of $\mathfrak{g}$. With such basis one associates an n-tuple $(r_1, \dots, r_n)$, referred to as a ***weight vector***, by requiring that $r_i = k$ if, and only if, $\zeta_i \in G_k \setminus G_{k-1}$. Given a graded basis, the construction of the transverse function proceeds by selecting strictly positive reals $\varepsilon_{m+1}, \dots, \varepsilon_n$ and by defining mappings $f_i : \mathbb{T} \longrightarrow G$ $(i = m+1, \dots, n)$ as follows:

$$f_i(\theta) = \exp\left(\varepsilon_i^{r_{\lambda(i)}} \sin(\theta)\zeta_{\lambda(i)} + \varepsilon_i^{r_{\rho(i)}} \cos(\theta)\zeta_{\rho(i)}\right).$$

With these mappings at hand, a transverse function $f : \mathbb{T}^{n-m} \longrightarrow G$ is then obtained by setting

$$f(\theta_{m+1}, \dots, \theta_n) = f_n(\theta_n) f_{n-1}(\theta_{n-1}) \cdots f_{m+1}(\theta_{m+1}).$$

4 Vertically transverse functions for second-order systems

Here we show how tangent mappings of transverse functions for driftless systems define "vertically transverse functions," which bears relevance for second-order systems. Let Q be a manifold (the *configuration manifold*, by analogy with the case of mechanical systems) and let $p \in Q$. Starting with a set of vector fields $\{X_1, \dots, X_m\} \subset \Gamma(TQ)$ that satisfies the Lie algebra rank condition at p, we define a "lifted" system on TQ by considering a second-order vector field $Z \in \Gamma(TTQ)$ and the system

$$\dot{v} = Z_v + \sum_{i=1}^{m} u^i X_{i,v}^{\mathrm{lift}}. \tag{4}$$

As can be anticipated, the choice Z will have an impact on the solutions to control problems addressed below. The approach proceeds by assuming that the *target system* (also called the *controlled* system) is of the form (4). This form encompasses a class of second-order and simple mechanical systems (fully actuated or underactuated, possibly subject to constraints). The goal is to provide control laws for a class of target systems (4), building upon the properties of any transverse function f for $\{X_1, \dots, X_m\}$, a function whose existence is guaranteed by the assumptions. Our first result in this vein states that Tf satisfies a condition that somehow extends (3), namely, that along the image of Tf, the image of the vertical subbundle $(TT\mathbb{T}^\kappa)^{\mathrm{vert}}$ by TTf, together with the distribution spanned by the lifted control vector fields $\{X_1^{\mathrm{lift}}, \dots, X_m^{\mathrm{lift}}\}$, generate the vertical subbundle of TTQ over $Tf(T\mathbb{T}^\kappa)$. This is made precise in the following proposition.

Proposition 1 *Let $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ satisfy the Lie algebra rank condition at a point $p \in Q$, and let $f : \mathbb{T}^\kappa \longrightarrow Q$ be a Morin–Samson transverse function for $\boldsymbol{X}$ near p. Then, for every $\omega \in T\mathbb{T}^\kappa$,*

$$T_{Tf(\omega)}TQ^{\mathrm{vert}} \quad = \quad TTf((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}) + \mathrm{span}_{\mathbb{R}}\{X^{lift}_{1,Tf(\omega)}, \ldots, X^{lift}_{m,Tf(\omega)}\}. \tag{5}$$

Moreover, if f is such that the sum in (3) is direct, then so is the sum in (5).

Proof. Let $\theta \in \mathbb{T}^\kappa$, let $\omega \in T_\theta\mathbb{T}^\kappa$ and assume that $v \in T_{Tf(\omega)}TQ^{\mathrm{vert}}$. The verticality of v and Lemma 1-(iii) imply the existence of $\widetilde{v} \in T_{f(\theta)}Q$ such that $v = \mathrm{lift}(Tf(\omega), \widetilde{v})$. In addition, from (3) we deduce the existence of a vector $\overline{\omega} \in T_\theta\mathbb{T}^\kappa$ and real numbers $a_1, \ldots, a_m$ such that $\widetilde{v} = Tf(\overline{\omega}) + \sum_{i=1}^m a^i X_{i,f(\theta)}$. Applying the linear mapping $\mathrm{lift}(Tf(\omega), \cdot\,)$ to both members of this equation we get

$$\begin{aligned} v &= \mathrm{lift}(Tf(\omega), Tf(\overline{\omega})) + \mathrm{lift}\left(Tf(\omega), \textstyle\sum_{i=1}^m a^i X_{i,f(\theta)}\right) \\ &= TTf(\mathrm{lift}(\omega, \overline{\omega})) + \textstyle\sum_{i=1}^m a^i X^{\mathrm{lift}}_{i,Tf(\omega)} \qquad \text{(by Lemma 1-(ii))}. \end{aligned}$$

Since $\mathrm{lift}(\omega, \overline{\omega}) \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$, this proves that (5) holds. Now suppose that the sum in (3) is direct, let θ and ω be as above, and assume that

$$v \in TTf((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}) \cap \mathrm{span}_{\mathbb{R}}\{X^{\mathrm{lift}}_{1,Tf(\omega)}, \ldots, X^{\mathrm{lift}}_{m,Tf(\omega)}\}.$$

We shall show that $v = 0$. By assumption, there exist $\alpha \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$ and real numbers $a^1, \ldots, a^m$ such that $v = TTf(\alpha) = \sum_{i=1}^m a^i X^{\mathrm{lift}}_{i,Tf(\omega)}$. Since α is a vertical tangent vector, Lemma 1-(iii) implies that it is the lift by ω of a tangent vector $\overline{\omega} \in T_\theta\mathbb{T}^\kappa$, i.e., $\alpha = \mathrm{lift}(\omega, \overline{\omega})$. The mapping $\mathrm{lift}(Tf(\omega), \cdot)$ is linear, hence

$$\begin{aligned} \mathrm{lift}\left(Tf(\omega), \textstyle\sum_{i=1}^m a^i X_{i,f(\theta)}\right) &= \textstyle\sum_{i=1}^m a^i \mathrm{lift}\left(Tf(\omega), X_{i,f(\theta)}\right) \\ &= \textstyle\sum_{i=1}^m a^i X^{\mathrm{lift}}_{i,Tf(\omega)} \\ &= TTf(\mathrm{lift}(\omega, \overline{\omega})) \\ &= \mathrm{lift}(Tf(\omega), Tf(\overline{\omega})) \qquad \text{(by Lemma 1-(ii))}. \end{aligned}$$

Since $\mathrm{lift}(Tf(\omega), \cdot)$ is injective as well, a consequence of Lemma 1-(iii), one infers that $\sum_{i=1}^m a^i X_{i,f(\theta)} = Tf(\overline{\omega})$. However, the sum in (3) is direct, by assumption, hence $\sum_{i=1}^m a^i X_{i,f(\theta)} = Tf(\overline{\omega}) = 0$. Using again the linearity of $\mathrm{lift}(Tf(\omega), \cdot\,)$, we conclude that $v = 0$, which completes the proof. ∎

By extension of the nomenclature used in [13], a mapping $F : T\mathbb{T}^\kappa \longrightarrow TQ$ of class C^1 such that

$$\forall \omega \in T\mathbb{T}^\kappa : \quad T_{F(\omega)}TQ^{\mathrm{vert}} = TF((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}) + \mathrm{span}_{\mathbb{R}}\{X^{\mathrm{lift}}_{1,F(\omega)}, \ldots, X^{\mathrm{lift}}_{m,F(\omega)}\},$$

is said to be ***vertically transverse*** to the vector fields in $\{X^{\mathrm{lift}}_1, \ldots, X^{\mathrm{lift}}_m\}$. Let us remark that if $f : \mathbb{T}^\kappa \longrightarrow Q$ is transverse to $\boldsymbol{X}$ near p, the previous proposition ensures that Tf is vertically transverse to $\{X^{\mathrm{lift}}_1, \ldots, X^{\mathrm{lift}}_m\}$. In such a case, which arises as a natural way to define vertically transverse functions, $F = Tf$ is a bundle mapping over f, thus, in a coordinate chart $(U, (\theta^1, \ldots, \theta^\kappa))$ for $\mathbb{T}^\kappa$, both conditions (3) and (5) essentially boil down to

$$\mathbb{R}^n = \mathrm{span}_{\mathbb{R}}\left\{\partial\hat{f}/\partial\theta^1(\theta), \ldots, \partial\hat{f}/\partial\theta^\kappa(\theta)\right\} + \mathrm{span}_{\mathbb{R}}\left\{\hat{X}_{1,f(\theta)}, \ldots, \hat{X}_{m,f(\theta)}\right\},$$

where $\hat{f}$ and $\hat{X}$ are the representatives of f and X, respectively. These two conditions are not equivalent, however, since in general F need not be a bundle mapping.

5 Applications of vertically transverse functions to control

5.1 Framework for practical point-stabilization

As a first example of how vertically transverse functions may be applied for control, suppose that the configuration manifold G is a Lie group and that the vector fields $X_i \in \Gamma(TG)$ $(i = 1, \ldots, m)$ are obtained by left-translating vectors $\xi_i \in \mathfrak{g}$ that satisfy $\mathrm{Lie}(\{\xi_1, \ldots, \xi_m\}) = \mathfrak{g}$. In this case, using the procedure recalled in Section 3, for any open set $U \subset G$ one can define a transverse function $f : \mathbb{T}^{n-m} \longrightarrow U$ and, from Proposition 1, we see that Tf satisfies, for every $\omega \in T\mathbb{T}^{n-m}$,

$$T_{Tf(\omega)}TG^{\text{vert}} = TTf((T_\omega T\mathbb{T}^{n-m})^{\text{vert}}) \oplus \text{span}_{\mathbb{R}}\{X^{\text{lift}}_{1,Tf(\omega)}, \ldots, X^{\text{lift}}_{m,Tf(\omega)}\}. \tag{6}$$

Mimicking the procedure described in [14], we extend system (4) dynamically by selecting a global frame for $(TT\mathbb{T}^{n-m})^{\text{vert}}$, that is, a set $\{\Omega_1, \ldots, \Omega_{n-m}\} \subset \Gamma(TT\mathbb{T}^{n-m})$ such that $\text{span}_{\mathbb{R}}\{\Omega_{1,\omega}, \ldots, \Omega_{n-m,\omega}\} = (T_\omega T\mathbb{T}^{n-m})^{\text{vert}}$ for all $\omega \in T\mathbb{T}^{n-m}$. The existence of a global frame is guaranteed by the triviality of $TT\mathbb{T}^{n-m}$ as a vector bundle. We then select a second-order vector field $\Delta \in \Gamma(TT\mathbb{T}^{n-m})$, typically the spray associated with a flat metric on $\mathbb{T}^{n-m}$, and define the *auxiliary system*

$$\dot{\omega} = \Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega}. \tag{7}$$

If $\omega : (t_0, t_1) \longrightarrow T\mathbb{T}^{n-m}$ and $w : (t_0, t_1) \longrightarrow \mathbb{R}^{n-m}$ are functions of classes C^1 and C^0, respectively, which satisfy the differential equation (7), we shall refer to the couple (ω, w) as an *auxiliary trajectory.* At this point we define an *error signal* whose intent, intuitively speaking, is to quantify the deviation of the state v of (4) from the image by Tf of the state ω of (7). The definition profits from the Lie group structure on TG defined by taking μ as in (2); we set $z = \mu(v, (Tf(\omega))^{-1})$, which, for the sake of simplicity, we write as $z = v \cdot Tf(\omega)^{-1}$.[1] The motivation for this definition is the following. Much as in the original TFA, in our approach described below, the auxiliary system (7) behaves as a (variable-frequency) oscillator whose state $\omega(t)$ is mapped by Tf into TG. Under appropriate conditions, $\omega(t)$ ultimately enters a compact neighborhood K of the zero section of $T\mathbb{T}^{n-m}$, which implies that if the feedback is designed so that $z(t)$ tends to zero, then $v(t)$ ultimately approaches $Tf(K)$. Thus the base (or "configuration") trajectory $q(t) = \pi_G(v(t))$ approaches the set $f(\mathbb{T}^{n-m})$ and, since the latter can be constructed to lie in an arbitrary neighborhood of the target configuration, $q(t)$ ultimately enters that neighborhood, as required by practical stability of the configuration trajectories.

Now, if (ω, w) is any arbitrary auxiliary trajectory and B is defined along the curve $Tf \circ \omega : (t_0, t_1) \longrightarrow TG$ by $B_{Tf\circ\omega(t)} = (Tf \circ \omega)'(t) = T_t(Tf \circ \omega)\,(\partial/\partial r|_t)$, then, by Lemma 1-(v), B satisfies a second-order equation. The following result is the key to writing an explicit expression for the error dynamics.

Proposition 2 *Let TG be a tangent Lie group, $A \in \Gamma(TTG)$ a second-order vector field (not necessarily left-invariant) and let B be a second-order vector field defined along a smooth curve $b : (t_0, t_1) \longrightarrow TG$ by $\dot{b}(t) = B_{b(t)}$. Then **(i)** if $a : (t_0, t_1) \longrightarrow TG$ is an integral curve of A, the curve $c = ab^{-1}$ satisfies, for $t \in (t_0, t_1)$,*

$$\dot{c}(t) = TR_{b^{-1}(t)}(A_{a(t)} - TL_{c(t)}(B_{b(t)})); \tag{8}$$

*and **(ii)** (8) defines a (non-autonomous) second-order differential equation on TG.*

Proof. (i) Let $\widetilde{B}$ be the vector field along the curve b^{-1} defined by $\widetilde{B}_{b^{-1}(t)} = (b^{-1})'(t)$. Differentiating $c(t) = \mu(a(t), b^{-1}(t))$, with μ given by (2), one finds that $\dot{c}(t) = TL_{a(t)}(\widetilde{B}_{b^{-1}(t)}) + TR_{b^{-1}(t)}(A_{a(t)})$. Also, by differentiating $e = \mu(b^{-1}(t), b(t))$ one easily concludes that $\widetilde{B}_{b^{-1}(t)} = -TR_{b^{-1}(t)} \circ TL_{b^{-1}(t)}(B_{b(t)})$. Therefore, given that $L_{a(t)} \circ R_{b^{-1}(t)} \circ L_{b^{-1}(t)} = L_{c(t)} \circ R_{b^{-1}(t)} = R_{b^{-1}(t)} \circ L_{c(t)}$, we get

$$\begin{aligned} \dot{c}(t) &= TL_{a(t)}(-TR_{b^{-1}(t)} \circ TL_{b^{-1}(t)}(B_{b(t)})) + TR_{b^{-1}(t)}(A_{a(t)}) \\ &= TR_{b^{-1}(t)}(A_{a(t)} - TL_{c(t)}(B_{b(t)})). \end{aligned}$$

(ii) For each β in the image of b, define $C_\beta : \gamma \mapsto TR_{\beta^{-1}}(A_{\gamma\beta} - TL_\gamma(B_\beta))$. Clearly, for $t \in \mathbb{R}$ and $\gamma \in TG$ one has $C_{b(t)}(\gamma) \in T_\gamma TG$, i.e., $C_{b(t)}$ is a section of TTG and thus a (time-varying) vector field on TG. It remains to show that $T\pi_G \circ C_\beta = \text{id}_{TG}$ for every $\beta \in b((t_0, t_1)) \subset TG$. Pick any such β and observe that $T(\pi_G \circ R_{\beta^{-1}}) = T(\widehat{R}_{\pi_G(\beta^{-1})} \circ \pi_G)$ and $T(\pi_G \circ L_\gamma) = T(\widehat{L}_{\pi_G(\gamma)} \circ \pi_G)$, which, for $\gamma \in TG$, entails that

$$\begin{aligned} (T\pi_G \circ C_\beta)(\gamma) &= T\pi_G(TR_{\beta^{-1}}(A_{\gamma\beta} - TL_\gamma(B_\beta))) \\ &= T\widehat{R}_{\pi_G(\beta^{-1})} \circ T\pi_G(A_{\gamma\beta} - TL_\gamma(B_\beta)) \\ &= T\widehat{R}_{\pi_G(\beta^{-1})}(T\pi_G(A_{\gamma\beta}) - T\widehat{L}_{\pi_G(\gamma)} \circ T\pi_G(B_\beta)). \end{aligned}$$

[1] We write $Tf(\omega)$ for $Tf \circ \omega$ and $Tf(\omega)^{-1}$ for $(Tf(\omega))^{-1}$ to simplify the exposition.

However, both A and B are second order, thus $T\pi_G(A_{\gamma\beta}) = \gamma\beta$ and $T\pi_G(B_\beta) = \beta$. Using these equations, along with $\gamma\beta - T\widehat{L}_{\pi_G(\gamma)}(\beta) = T\widehat{R}_{\pi_G(\beta)}(\gamma)$, we obtain $(T\pi_G \circ C_\beta)(\gamma) = T\widehat{R}_{\pi_G(\beta^{-1})}(\gamma\beta - T\widehat{L}_{\pi_G(\gamma)}(\beta)) = \gamma$, as required. ∎

In order to apply Proposition 2, we define curves $a = v$ and $b = Tf \circ \omega$, as well as the corresponding vector fields

$$A_v = Z_v + \sum_{i=1}^m u^i X_{i,v}^{\text{lift}} \quad \text{and} \quad B_{Tf(\omega)} = TTf\left(\Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega}\right),$$

thus yielding

$$\dot{z} \;=\; TR_{Tf(\omega)^{-1}}\left(Z_v + \sum_{i=1}^m u^i X_{i,v}^{\text{lift}} - TL_z \circ TTf\left(\Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega}\right)\right).$$

Grouping the drift and controlled vector fields, and using $v = z \cdot Tf(\omega)$ as well as the left-invariance of $X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}$, we obtain the following *error dynamics*, which, by virtue of Proposition 2-**(ii)**, is a second-order equation:

$$\begin{aligned} \dot{z} \;=\; & TR_{Tf(\omega)^{-1}}(Z_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega)) + \\ & TR_{Tf(\omega)^{-1}} \circ TL_z\left(\sum_{i=1}^m u^i X_{i,Tf(\omega)}^{\text{lift}} - \sum_{i=1}^{n-m} w^i TTf(\Omega_{i,\omega})\right). \end{aligned} \tag{9}$$

We now address how vertical transversality may be used for control purposes. The main idea is that, for second-order systems, the control inputs can *only* shape the second-order time derivatives of the base trajectories, which amounts to assigning values for those derivatives in the vertical subbundle. The latter, according to (5), is spanned by the control distribution and the image of the vertical subbundle $(T_\omega T\mathbb{T}^\kappa)^{\text{vert}}$. This fact provides one with full control over the error system and, therefore, it enables one to impose any desired (smooth) second-order error dynamics Y. The following result makes this statement precise.

Theorem 1 *Given a second-order vector field $Y \in \Gamma(TTG)$, there exists a smooth feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ such that the error $z = v \cdot Tf(\omega)^{-1}$ satisfies $\dot{z} = Y_z$ along the trajectories of the compound system*

$$(\dot{v}, \dot{\omega}) = \left(Z_v + \sum_{i=1}^m \alpha^i(v \cdot Tf(\omega)^{-1}, \omega) X_{i,v}^{\textit{lift}},\; \Delta_\omega + \sum_{i=1}^{n-m} \alpha^{i+m}(v \cdot Tf(\omega)^{-1}, \omega)\Omega_{i,\omega}\right). \tag{10}$$

Proof. Finding the required feedback amounts to setting the right-hand-side of (9) equal to Y_z, solving the resulting equation for the u^i and w^j in terms of (z, ω), and then checking that the solutions define a smooth mapping $\alpha : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$. The first step leads to

$$\begin{aligned} \sum_{i=1}^m u^i \; & X_{i,Tf(\omega)}^{\text{lift}} - \sum_{i=1}^{n-m} w^i TTf(\Omega_{i,\omega}) \\ = \; & (TR_{Tf(\omega)^{-1}} \circ TL_z)^{-1}(Y_z - TR_{Tf(\omega)^{-1}}(Z_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega))) \\ = \; & TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - TR_{Tf(\omega)^{-1}}(Z_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega))) \\ = \; & TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z), \end{aligned}$$

where, for each $\omega \in T\mathbb{T}^{n-m}$, we have defined the vector field $D_\omega \in \Gamma(TTG)$ by setting

$$D_\omega : z \mapsto TR_{Tf(\omega)^{-1}}(Z_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega)).$$

Since the right-hand-side of (9) is second order, so is D_ω for every $\omega \in T\mathbb{T}^{n-m}$ and, by linearity of $T\pi_G$ on restriction to the fibers, $T\pi_G \circ (Y_z - (D_\omega)_z) = T\pi_G(Y_z) - T\pi_G((D_\omega)_z) = z - z = 0$, which shows that $Y - D_\omega$ is vertical. On the other hand, $T\pi_G \circ TL_\xi = T\widehat{L}_{\pi_G(\xi)} \circ T\pi_G$ and $T\pi_G \circ TR_\xi = T\widehat{R}_{\pi_G(\xi)} \circ T\pi_G$ for every $\xi \in TTG$, both of which imply that $TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z)$ is vertical as well. By virtue of (6) and the assumption that $\{\Omega_1, \ldots, \Omega_m\}$ is a (global) frame for $(TT\mathbb{T}^{n-m})^{\text{vert}}$, there exists a mapping $\alpha : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ such that, for every $(z, \omega) \in TG \times T\mathbb{T}^{n-m}$,

$$\begin{aligned} \sum_{i=1}^m \alpha^i(z, \omega) \; & X_{i,Tf(\omega)}^{\text{lift}} \; - \sum_{i=1}^{n-m} \alpha^{i+m}(z, \omega) TTf(\Omega_{i,\omega}) \\ = \; & TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z). \end{aligned} \tag{11}$$

One easily checks that α so defined is smooth. ∎

5.2 Zero-dynamics of the closed-loop system

We have seen how to formulate a point-stabilization problem in a setup based on vertically transverse functions. Nevertheless, while Theorem 1 states that the error dynamics can be arbitrarily assigned as a second-order equation, further investigation is needed to assess the nature of the trajectories of the compound system (10). The latter may even fail to be positively complete, i.e., the maximum intervals of existence of some of its solutions may be bounded in $\mathbb{R}$. This is a consequence of how the problem was formulated, namely as an *output regulation* problem, where the "output" is the mapping that defines the error: $(v,\omega)\mapsto z = v\cdot Tf(\omega)^{-1}$. As it is well known, imposing a given dynamics on the output signal may induce undesirable effects on the system's solutions, thus compelling one to the study the associated *zero-dynamics*, the nature of which may determine the asymptotic behavior of the compound system. Throughout this section we assume that the feedback law $\alpha = (\alpha^1,\dots,\alpha^n) : TG\times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ has been determined, according to Theorem 1, from the given of a vector field $Y\in\Gamma(TTG)$ for which $0_e\in TG$ is an equilibrium point, so that (9), controlled by $\alpha(z,\omega)$, writes as $\dot z = Y_z$. Under this assumption, the zero-dynamics is determined simply by setting, in Equation (9), $z=0_e$, $u^i=\alpha^i(0_e,\omega)$ and $w^j=\alpha^{j+m}(0_e,\omega)$, $i=1,\dots,m$, $j=1,\dots,n-m$. After simplification, the substitution yields, for $\omega\in T\mathbb{T}^{n-m}$,

$$Z_{Tf(\omega)} + \textstyle\sum_{i=1}^m \alpha^i(0_e,\omega)X^{\mathrm{lift}}_{i,Tf(\omega)} = TTf(\Delta_\omega) + \sum_{j=1}^{n-m}\alpha^{j+m}(0_e,\omega)TTf(\Omega_{j,\omega}). \tag{12}$$

Since $T_\omega Tf$ is linear for every $\omega\in T\mathbb{T}^{n-m}$, (12) is equivalent to the equality

$$Z\circ Tf + \textstyle\sum_{i=1}^m \nu^i\cdot(X_i^{\mathrm{lift}}\circ Tf) = TTf\circ\left(\Delta + \sum_{j=1}^{n-m}\nu^{j+m}\Omega_j\right), \tag{13}$$

where the objects on both members are regarded as mappings $T\mathbb{T}^{n-m}\longrightarrow TTG$, and $\nu : T\mathbb{T}^{n-m}\longrightarrow\mathbb{R}^n$ is defined by $\nu(\omega)=\alpha(0_e,\omega)$. Equation (13) shows that, on one hand, *the zero-dynamics is independent of the particular choice of desired error dynamics* Y, and, on the other, the zero-dynamics for the auxiliary and controlled systems are Tf-related, thus it suffices to study trajectories of the former in order to characterize the trajectories of the target system. In this respect, an interesting fact is that *the auxiliary zero-dynamics inherits the structure of the target system*, that is, if the latter is an affine connection control system (in the sense of [8]), possibly with nonzero potential term, then so is the zero-dynamics of the auxiliary system. To state this claim more accurately, first let us define a projection operator $\mathcal{P}$ which maps vectors in TTG^{vert} lying over points of $N := Tf(T\mathbb{T}^{n-m})$, to vectors in $TTf((TT\mathbb{T}^{n-m})^{\mathrm{vert}})$. If $\{\Lambda_1,\dots,\Lambda_{n-m}\}\subset\Gamma(T\mathbb{T}^{n-m})$ is a set of vector fields whose lifts are a global frame for the vertical subbundle $(TT\mathbb{T}^{n-m})^{\mathrm{vert}}$, and $P\in\Gamma(TTG^{\mathrm{vert}})$ is a vertical vector field, the vertical transversality of Tf implies the existence of a unique mapping $a=(a^1,\dots,a^n) : T\mathbb{T}^{n-m}\longrightarrow\mathbb{R}^n$ such that $P\circ Tf = \sum_{i=1}^m a^i(X_i^{\mathrm{lift}}\circ Tf) + \sum_{i=1}^{n-m} TTf\circ(a^{i+m}\Lambda_i^{\mathrm{lift}})$. The projector $\mathcal{P}$ is then defined so that $\mathcal{P}\circ P\circ Tf = \sum_{i=1}^{n-m} TTf\circ(a^{i+m}\Lambda_i^{\mathrm{lift}})$. Correspondingly, there exists a unique vertical vector field $\Pi = \sum_{i=1}^{n-m} a^{i+m}\Lambda_i^{\mathrm{lift}}$ such that $\mathcal{P}\circ P\circ Tf = TTf\circ\Pi$.[2] The claim can now be restated as follows.

Theorem 2 *Let $\nu : T\mathbb{T}^{n-m}\longrightarrow\mathbb{R}^n$ be such that (13) holds and assume that $Z = S+P$, where $S\in\Gamma(TTG)$ is a spray and $P\in\Gamma(TTG^{\mathrm{vert}})$. Let $\Pi\in\Gamma((TT\mathbb{T}^{n-m})^{\mathrm{vert}})$ be the vector field that satisfies $\mathcal{P}\circ P\circ Tf = TTf\circ\Pi$. Then there exists a spray $\Sigma\in\Gamma(TT\mathbb{T}^{n-m})$ such that*

$$\Delta + \sum_{j=1}^{n-m}\nu^{j+m}\Omega_j = \Sigma + \Pi.$$

Proof in the Appendix.

Remark 1 If in the statement of Theorem 2 one thinks of P and Π as force-like terms arising from potential functions (for instance, P might be the lift of minus the gradient of the potential energy in a simple mechanical system), then *only the projection* $\mathcal{P}\circ P\circ Tf$ has an effect on the potential term Π of

[2] Formally, $\mathcal{P}$ is a vector bundle mapping $\mathcal{P} : Tf^*(TTG^{\mathrm{vert}})\longrightarrow TTf((TT\mathbb{T}^{n-m})^{\mathrm{vert}})$ covering Tf, where $Tf^*(TTG)$ is the pullback bundle induced by Tf.

the zero-dynamics. The complementary component $(P - \mathcal{P} \circ P) \circ Tf$, however, gets incorporated into the spray Σ. The decomposition of Z as a sum $S + P$ is not unique and different choices of S and P lead to different vector fields Σ and Π (the sum $\Sigma + \Pi$ being, of course, equal in every case).

Remark 2 Suppose that $Z = S$, the geodesic spray of a metric g. The effect of the control inputs $\nu(\omega)$ on the controlled and auxiliary systems, which yield the zero-dynamics (13), may be interpreted as follows. The vector field $\sum_{i=1}^{m} \nu^i \cdot (X_i^{\text{lift}} \circ Tf)$ is added to the original spray Z so that its restriction to $Tf(T\mathbb{T}^{n-m})$ equals the image, by TTf, of the drift Δ after the addition of $\sum_{j=1}^{n-m} \nu^{j+m}\Omega_j$. In that case, Theorem 2 asserts that the nature of the uniquely defined functions ν is quite specific, in the sense that the zero-output *controlled* and *auxiliary* vector fields *are Tf-related sprays*. Since the immersed manifold $N = Tf(T\mathbb{T}^{n-m}) \subset TG$ is invariant under the zero-dynamics $Z \circ Tf + \sum_{i=1}^{m} \nu^i \cdot (X_i^{\text{lift}} \circ Tf)$, one may think of $\sum_{i=1}^{m} \nu^i \cdot (X_i^{\text{lift}} \circ Tf)$ as lifts of "force" terms that enforce *holonomic* constraints, much in the spirit of [1, Cor. 3.7.9]. In general, however, those forces are not *orthogonal* to TN; they contain components in the direction of TN, so they do not satisfy d'Alembert's principle and therefore affect the "energy content" of the zero-dynamics system. In the terminology and notation of [1, §3.7.8], $TTf \circ (\Delta + \sum_{j=1}^{n-m} \nu^{j+m}\Omega_j)$ *need not be equal to* $TP \circ S$, where P denotes the orthogonal projection of the pullback bundle $Tf^*(TTG)$ onto TN.

At this point, it is interesting to ask whether the zero-dynamics is positive-complete and, if so, whether its solutions remain in a relatively compact neighborhood of the zero section of $T\mathbb{T}^{n-m}$. Suppose, for instance, that $Z = S$ satisfies the assumptions of Theorem 2 so that the zero-dynamics writes as $\dot{\omega} = \Sigma_\omega$, with Σ a spray. The latter determines a unique *torsion-free* affine connection ∇ on $T\mathbb{T}^{n-m}$. If ∇ were the (Levi-Cività) connection of a Riemannian metric $\mathcal{G}$ on $\mathbb{T}^{n-m}$, one would deduce at once the positive-completeness of the zero-dynamics, for every compact Riemannian manifold is geodesically complete [6, Cor. 4.4]. Also, the "kinetic energy" would be constant along the solutions, so these would evolve in a relatively compact neighborhood of the zero section of $T\mathbb{T}^{n-m}$, i.e., the velocity coordinates would remain bounded. The question whether a torsion-free connection is the Levi-Cività connection of a metric is addressed, among other references, in [18], where conditions are stated in terms of the corresponding holonomy groups $\Phi(\theta)$. Recall that, for given $\theta \in T\mathbb{T}^{n-m}$, $\Phi(\theta)$ consists of all endomorphisms of $T_\theta\mathbb{T}^{n-m}$ defined via parallel transport along loops on the base which are piecewise-smooth and have θ as endpoints (see e.g. [6] for more details). Schmidt proves a general version of the following result, adapted to suit our present needs.

Proposition 3 [18] *An affine connection ∇ on $\mathbb{T}^{n-m}$ is the Levi-Cività connection of a metric on $\mathbb{T}^{n-m}$ with signature (p, q) if, and only if, there exists a non-degenerate quadratic form $\mathcal{G}$ on $T_\theta\mathbb{T}^{n-m}$, with signature (p, q), which is invariant under $\Phi(\theta)$.*

In general, determining whether a given connection ∇ satisfies the assumptions of Proposition 3 is a difficult task since ∇ need not be flat and $\mathbb{T}^{n-m}$ is not simply connected. Nevertheless, for systems underactuated by one control, for which $n - m = 1$, a simple condition can be stated.

Proposition 4 *An affine, torsion-free connection $\nabla : \Gamma(T\mathbb{T}) \longrightarrow \Gamma(T^*\mathbb{T} \otimes T\mathbb{T})$ on $T\mathbb{T}$ is the Levi-Cività connection of a pseudo-Riemannian metric if, and only if, there is a global frame $s \in \Gamma(T\mathbb{T})$ such that the one-form $A \in \Gamma(T^*\mathbb{T})$, uniquely determined by $A \otimes s = \nabla s$, is exact.*

Proof. It is easily checked that the de Rham cohomology class of A is independent of s; thus, if A is exact, then so is the form A' defined analogously by another frame $s' \in \Gamma(T\mathbb{T})$. Assume that ∇ is torsion-free and orient $\mathbb{T}$ so that $I_A := \int_{\mathbb{T}} A$ is defined. For a loop $\gamma : [a, b] \longrightarrow \mathbb{T}$ such that $\gamma(a) = \gamma(b) =: \theta$, the parallel transport of $\omega = ks(\theta) \in T_\theta\mathbb{T}$ by γ is given by $L_\gamma(\omega) = \exp(I_A)\, ks(\theta)$. Since $\dim(\mathbb{T}) = 1$, the curvature tensor R is zero and ∇ is flat. Hence, the holonomy around γ depends only on its homotopy class $[\gamma]$ in $\pi(\mathbb{T}, \theta)$, the fundamental group of $\mathbb{T}$ based at θ (cf. [6, Chap. II-9]). Since $\pi(\mathbb{T}, \theta) \simeq \mathbb{Z}$, for every such γ there exists $n \in \mathbb{Z}$ such that $L_\gamma(\omega) = \exp(nI_A)\, ks(\theta)$. Given a quadratic non-degenerate form $\mathcal{G}$ on $T_\theta\mathbb{T}$, there exists $G \neq 0$ such that $\mathcal{G}(s(\theta), s(\theta)) = G$. Thus $\mathcal{G}$ is preserved by $\Phi(\theta)$ if, and only if, for every $k, k' \in \mathbb{R}$ and every $n \in \mathbb{Z}$, $Gkk' = \mathcal{G}(ks(\theta), k's(\theta)) = G\,(\exp(nI_A))^2\, kk'$, i.e., if, and only if, $|\exp(nI_A)| = 1$ for all $n \in \mathbb{Z}$. But this is equivalent to $I_A = \int_{\mathbb{T}} A = 0$ and, in turn, since $dA = 0$, to the existence of a function $f \in C^\infty(\mathbb{T})$ such that $A = df$. Proposition 3 then implies the conclusion. ∎

5.3 Long-term behavior of the compound system

Here we approach the analysis of the closed-loop system under the assumption that the zero-dynamics is given by a spray and admits a Riemannian metric. We show that if the error dynamics has 0_e as a locally exponentially stable point (in a sense defined below), and the zero-dynamics admits a kinetic energy function, then the compact set $\{0_e\} \times Z(T\mathbb{T}^{n-m})$ is uniformly stable for the compound system, where $Z(T\mathbb{T}^{n-m})$ denotes the *zero-section of* $T\mathbb{T}^{n-m}$. Before proceeding, let us precisely define the stability notions that we shall use in the sequel. Let X be a vector field on a manifold M of dimension d. For $t_0 \in \mathbb{R}$ and $x_0 \in M$, denote by $t \mapsto \phi(t, t_0, x_0)$ the maximal solution of the initial value problem $\dot{x}(t) = X_{x(t)}$, $x(t_0) = x_0$. A point $x_0 \in M$ is said to be ***locally exponentially stable for*** X if there exists a chart (U, ψ) on M such that $\psi(x_0)$ is locally exponentially stable (in the usual sense) for the push-forward vector field $\psi_* X \in \Gamma(T\mathbb{R}^d)$.[3] Given a subset $U \subset M$, X is said to be U***-positive-complete*** if, for every $t_0 \in \mathbb{R}$ and every $x_0 \in U$, the domain of $t \mapsto \phi(t, t_0, x_0)$ contains $[t_0, +\infty)$. A subset $S \subset M$ is said to be ***positively invariant*** under X if X is S-positive-complete and, for every $t_0 \in \mathbb{R}$ and every $x_0 \in S$, $\phi(t, t_0, x_0) \in S$ for all $t \in [t_0, +\infty)$. A positively invariant, compact subset $S \subset M$ is said to be ***uniformly stable under*** X if for every neighborhood V of S there exists a neighborhood U of S such that X is U-positive-complete and, for every $t_0 \in \mathbb{R}$ and every $x_0 \in U$, one has $\phi(t, t_0, x_0) \in V$ for all $t \in [t_0, +\infty)$.

Theorem 3 *Let $Y \in \Gamma(TTG)$ be a vector field that admits $0_e \in TG$ as a locally exponentially stable equilibrium and assume that a feedback law is applied to the controlled system (4) so that the combined error and auxiliary dynamics writes as*

$$(\dot{z}, \dot{\omega}) = \left(Y_z,\ \Delta_\omega + \textstyle\sum_{i=1}^{n-m} \alpha^{i+m}(z, \omega)\Omega_{i,\omega}\right). \tag{14}$$

Assume, furthermore, that the auxiliary zero-dynamics is given by $\dot{\omega} = \Sigma_\omega = \Delta_\omega + \sum_{i=1}^{n-m} \alpha^{i+m}(0_e, \omega)\Omega_{i,\omega}$, with $\Sigma \in \Gamma(TT\mathbb{T}^{n-m})$ a spray, and that there exists a positive-definite metric tensor $\mathcal{G}$ on $\mathbb{T}^{n-m}$ such that the function $K : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}$ defined by $K(z, \omega) = \frac{1}{2}\mathcal{G}(\omega, \omega)$ is constant along the trajectories of $\dot{\omega} = \Sigma_\omega$. Then the set $\{0_e\} \times Z(T\mathbb{T}^{n-m})$, where $Z(T\mathbb{T}^{n-m})$ denotes the zero-section of $T\mathbb{T}^{n-m}$, is uniformly stable under the dynamics defined by (14).

Proof in the Appendix.

Remark 3 Under the assumptions of Theorem 3, the error decreases "exponentially fast" and the zero-dynamics is conservative, thus the conclusion appears to be intuitively clear. However, the proof of the result is slightly involved due to two facts. First, the stability notion concerns a set which cannot be covered by a single coordinate chart, so the analysis is carried out, instead, by "lifting" the system to an appropriate covering manifold. Second, the system that determines the asymptotic properties of the trajectories—the zero-dynamics—does not admit an exponentially stable equilibrium, thus ruling out the application of many of the well-known theorems regarding stability in the presence of disturbances.

Remark 4 It is interesting to observe that Theorem 3 pertains to the stability of $\{0_e\} \times Z(T\mathbb{T}^{n-m})$ under the combined target- and *auxiliary*- (as opposed to *zero*-) dynamics. In view of the marginal stability (or conservativeness) of the zero-dynamics, one may legitimately wonder whether the target system velocities may grow indefinitely whenever the error $z(t)$ does not vanish asymptotically but remains bounded, as in the case when measurement noise is present. Naturally, a thorough robustness analysis should be carried out to settle this still unanswered question; however, simulations seem to suggest that in such case the target (and auxiliary) system velocities remain bounded, with a bound that seems to depend on the error bound.

[3] We use this—voluntarily naive—definition for simplicity; however, since there is no canonical way to define exponential stability of a point for systems evolving on general manifolds, alternative definitions may be envisaged.

6 Example

Consider the so-called extended chained form (ECF) system

$$\ddot{q}^1 = u^1, \quad \ddot{q}^2 = u^2, \quad \ddot{q}^3 = u^1 q^2. \tag{15}$$

This is a second-order, two-input system on $\mathbb{R}^3$ which locally represents, modulo a static-feedback transformation, the dynamics of a class of underactuated mechanical systems. An example is the idealized, three degree-of-freedom, prismatic-prismatic-rotational (PPR) manipulator with *passive* rotational joint (cf. [4]). The ECF fails to satisfy Brockett's condition, hence the asymptotic stabilization of any equilibrium point by continuous state-feedback is ruled out. Thus the point-stabilization problem is more difficult to solve for the ECF than for other underactuated mechanical systems; e.g. for systems akin to the *inverted pendulum*—where the passive joint is subject to gravity—local asymptotic stabilization of the upright equilibrium position can be easily carried out via a linear approximation at that point. Consider the Lie group G with underlying manifold structure $\mathbb{R}^3$ and group multiplication $\widehat{\mu}(x,y) = (\ x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1\ )$. We take $q = \mathrm{id}_G$ and consider natural (global) coordinates $(TG, (q, \dot{q}))$ on the tangent group TG. Using these coordinates, the group operation on TG is

$$\mu(x,y) = (x^1 + y^1,\, x^2 + y^2,\, x^3 + y^3 + x^2 y^1,\, x^4 + y^4,\, x^5 + y^5,\, x^6 + y^6 + x^2 y^4 + x^5 y^1)$$

Now, defining vector fields $X_1, X_2 \in \Gamma(TG)$ by $X_{1,q} = \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^3}$ and $X_{2,q} = \frac{\partial}{\partial q^2}$, we see that (15) defines the target system as a second-order system on TG of the form (4) provided we consider, for $v = (q, \dot{q})$,

$$Z_v = \sum_{i=1}^{3} \dot{q}^i \frac{\partial}{\partial q^i}, \quad X_{1,v}^{\mathrm{lift}} = \frac{\partial}{\partial \dot{q}^1} + q^2 \frac{\partial}{\partial \dot{q}^3}, \quad X_{2,v}^{\mathrm{lift}} = \frac{\partial}{\partial \dot{q}^2}.$$

Clearly, the vector field Z is a spray—the geodesic spray of a Euclidean metric on $\mathbb{R}^3$, indeed. It is also easy to check that both X_i and X_i^{lift} $(i = 1,2)$ are left-invariant and satisfy $[X_1, X_2] = -\frac{\partial}{\partial q^3}$, so that $\mathrm{Lie}(\{X_1, X_2\})(q) = T_q G$ for every $q \in G$. Therefore one may apply the methodology of [14], recalled in Section 3, to construct a transverse function $f : \mathbb{T} \longrightarrow G$ for the system $\dot{q} = u^1 X_{1,q} + u^2 X_{2,q}$. Using angular coordinates (U, θ) on $\mathbb{T}$, the method in [14] yields $f(\theta) = \big(\ \varepsilon \sin(\theta),\ \varepsilon \cos(\theta),\ \frac{1}{4}\varepsilon^2 \sin(2\theta)\ \big)$, with $\varepsilon > 0$ arbitrary. (For brevity, in the sequel we write $\mathrm{s} = \sin$ and $\mathrm{c} = \cos$.) The transversality condition (3) amounts to the determinant of the matrix with columns $X_{1,f(\theta)}$, $X_{2,f(\theta)}$ and $f'(\theta)$ being constant equal to $-\frac{1}{2}\varepsilon^2$. Using natural coordinates $(\theta, \dot{\theta})$ for $T\mathbb{T}$, the value of the associated tangent mapping Tf at $\omega = (\theta, \dot{\theta}) \in T_\theta \mathbb{T}$ is

$$Tf(\omega) \quad = \quad \Big(\ \varepsilon \mathrm{s}(\theta),\ \varepsilon \mathrm{c}(\theta),\ \tfrac{1}{4}\varepsilon^2 \mathrm{s}(2\theta),\ \varepsilon \mathrm{c}(\theta)\dot{\theta},\ -\varepsilon \mathrm{s}(\theta)\dot{\theta},\ \tfrac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\dot{\theta} \Big).$$

Let us now verify that Tf satisfies the vertical transversality condition (6). Considering natural coordinates $(\theta, \dot{\theta}, \alpha_L, \alpha_H)$ for $TT\mathbb{T}$, one first evaluates *the tangent* of Tf at a vertical vector $\alpha \in \ker(T_\omega \pi_{\mathbb{T}}) \subset T_\omega T\mathbb{T}$. Since $T_\omega \pi_{\mathbb{T}}$ maps $(\theta, \dot{\theta}, \alpha_L, \alpha_H)$ to (θ, α_L), α is in the kernel of $T_\omega \pi_{\mathbb{T}}$ if, and only if, it has the form $\alpha = (\theta, \dot{\theta}, 0, \alpha_H)$, so for simplicity we take $\tilde{\alpha} = (\theta, \dot{\theta}, 0, 1)$. Carrying out the operations one obtains

$$TTf(\tilde{\alpha}) \quad = \quad \big(\ 0,\ 0,\ 0,\ \varepsilon \mathrm{c}(\theta), -\varepsilon \mathrm{s}(\theta),\ \tfrac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\ \big).$$

Now, X_1^{lift}, X_2^{lift} and $T_\omega Tf(\tilde{\alpha})$ span $(T_{Tf(\omega)} TG)^{\mathrm{vert}}$. To see this, observe that any vector in the latter is of the form $\sum_{i=1}^{3} \alpha^i \frac{\partial}{\partial \dot{q}^i}$, that is, its first three components are zero. Hence the claim holds if the determinant of the submatrix consisting of the lower three rows of the matrix with columns $X_{1,Tf(\omega)}^{\mathrm{lift}}$, $X_{2,Tf(\omega)}^{\mathrm{lift}}$ and $TTf(\tilde{\alpha})$ does not vanish. But this is exactly the matrix $(X_{1,f(\theta)}, X_{2,f(\theta)}, f'(\theta))$ considered above, with determinant equal to $-\frac{1}{2}\varepsilon^2$, so Tf indeed satisfies (6).

We define the auxiliary system (7) on $T\mathbb{T}$ by

$$\ddot{\theta} = w, \tag{16}$$

and the corresponding error $z = \mu(v, Tf(\omega)^{-1})$

$$
\begin{aligned}
z \;=\; \Big(\; & q^1 - \varepsilon \mathrm{s}(\theta),\ q^2 - \varepsilon \mathrm{c}(\theta),\ q^3 + \tfrac{1}{4}\varepsilon^2 \mathrm{s}(2\theta) - q^2 \varepsilon \mathrm{s}(\theta),\ \dot{q}^1 - \varepsilon \mathrm{c}(\theta)\dot{\theta}, \\
& \dot{q}^2 + \varepsilon \mathrm{s}(\theta)\dot{\theta},\ \dot{q}^3 - \dot{q}^2 \varepsilon \mathrm{s}(\theta) - q^2 \varepsilon \mathrm{c}(\theta)\dot{\theta} + \tfrac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\dot{\theta} \quad \Big).
\end{aligned}
$$

Differentiating this expression we get the error dynamics

$$
\dot{z} = F(z, \omega) + \textstyle\sum_{i=1}^{3} u^i G_i(z, \omega), \tag{17}
$$

with $u^3 = w$ and the components of F and the G_is given by

$$
\begin{aligned}
F(z, \omega) &= \Big(z^4,\ z^5,\ z^6,\ \varepsilon \mathrm{s}(\theta)\dot{\theta}^2,\ \varepsilon \mathrm{c}(\theta)\dot{\theta}^2,\ \tfrac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot{\theta}^2 + \varepsilon \mathrm{s}(\theta) z^2 \dot{\theta}^2 - 2\varepsilon \mathrm{c}(\theta) z^5 \dot{\theta} \Big) \\
G_1(z, \omega) &= (\ 0,\ 0,\ 0,\ 1,\ 0,\ z^2 + \varepsilon \mathrm{c}(\theta)\ ) \\
G_2(z, \omega) &= (\ 0,\ 0,\ 0,\ 0,\ 1,\ -\varepsilon \mathrm{s}(\theta)\ ) \\
G_3(z, \omega) &= \big(\ 0,\ 0,\ 0,\ -\varepsilon \mathrm{c}(\theta),\ \varepsilon \mathrm{s}(\theta),\ -\tfrac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z^2\ \big).
\end{aligned}
$$

Each of the G_is, as well as F, is a family of vector fields on TG indexed by $\omega = (\theta, \dot{\theta}) \in T_\theta \mathbb{T}$. Clearly, $F(\cdot, \omega)$ is second order whereas $G_i(\cdot, \omega)$ is vertical ($i = 1, 2, 3$), thus the error dynamics (17) is second order for all $\omega \in T\mathbb{T}$, as anticipated by Proposition 2.

To construct a control law as outlined in Section 5, and Theorem 1 in particular, we select for the desired dynamics a second-order vector field $S \in \Gamma(TTG)$ having 0 as an exponentially stable equilibrium, for instance

$$
S_z = (\ z^4,\ z^5,\ z^6,\ -k_1 z^1 - k_2 z^4,\ -k_1 z^2 - k_2 z^5,\ -k_1 z^3 - k_2 z^6\ ),
$$

where the control gains k_1, k_2 are strictly positive. The control design now reduces to searching for a function $u : TG \times T\mathbb{T} \longrightarrow \mathbb{R}^3$ such that $F(z, \omega) + \sum_{i=1}^{3} u^i(z, \omega) G_i(z, \omega) = S_z$ for all $(z, \omega) \in TG \times T\mathbb{T}$. Inspecting the structure of the error dynamics (17), one concludes that this problem is equivalent to solving (11), which in this case boils down to solving for u in the following matrix equation

$$
\begin{pmatrix}
1 & 0 & -\varepsilon \mathrm{c}(\theta) \\
0 & 1 & \varepsilon \mathrm{s}(\theta) \\
z^2 + \varepsilon \mathrm{c}(\theta) & -\varepsilon \mathrm{s}(\theta) & -\tfrac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z^2
\end{pmatrix} u =
\begin{pmatrix}
-\dot{\theta}^2 \varepsilon \mathrm{s}(\theta) - k_1 z^1 - k_2 z^4 \\
-\dot{\theta}^2 \varepsilon \mathrm{c}(\theta) - k_1 z^2 - k_2 z^5 \\
-\tfrac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot{\theta}^2 - \varepsilon \mathrm{s}(\theta) z^2 \dot{\theta}^2 + 2\varepsilon \mathrm{c}(\theta) z^5 \dot{\theta} - k_1 z^3 - k_2 z^6
\end{pmatrix}.
$$

This equation is smoothly solvable since invertibility of the coefficient of u is equivalent to invertibility of the matrix ensuring vertical transversality of Tf; its determinant, in particular, is equal to $\frac{1}{2}\varepsilon^2$.

In accordance with Theorem 2, the zero-dynamics of the compound system is an affine connection control system with no potential term, defined by a spray Σ:

$$
\dot{\omega} = \Sigma_\omega = (\ \dot{\theta},\ -\sin(2\theta)\dot{\theta}^2\ ). \tag{18}
$$

Σ determines a torsion-free affine connection on $T\mathbb{T}$ given by $\nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} = \sin(2\theta) \frac{\partial}{\partial \theta}$. If $s = \frac{\partial}{\partial \theta}$ is a local section on the domain of (U, θ), then $\kappa_s(\nabla) = \sin(2\theta) d\theta$, which is exact, so its cohomology class is zero. By Proposition 4, ∇ is the Levi-Civitá connection of a family of metrics on $T\mathbb{T}$, namely $g_\theta = A e^{-\cos(2\theta)} d\theta \otimes d\theta$, $A > 0$. Indeed, the (geodesic) Euler-Lagrange equation $\nabla_{\dot{\theta}} \dot{\theta} = 0$ associated with the Lagrangian $L(\omega) = \frac{1}{2} g_\theta(\dot{\theta}, \dot{\theta}) = \frac{1}{2} A e^{-\cos(2\theta)} \dot{\theta}^2$ precisely coincides with the zero-dynamics (18) for any A. Since $(L \circ \omega)' = 0$, the (kinetic) energy is a conserved quantity and, given that it is bounded with respect to θ and depends quadratically on $\dot{\theta}$, it follows that $\dot{\theta}(t)$ remains bounded for all $t \in [t_0, \infty)$. Therefore, both $Tf(\omega(t))$ and $v(t)$ converge to a bounded neighborhood of the zero section in TG, the extent of which, as shown in Theorem 3, depends on the initial conditions.

7 Concluding remarks

In this paper we study *vertical transversality*, a property exhibited by tangent mappings of the transverse functions on tori introduced in [13]. A framework is then outlined which allows one to cast practical point-stabilization problems for underactuated (left-invariant) mechanical systems on Lie groups. Whereas the framework does not provide solutions to those problems "automatically," under appropriate assumptions it leads to practical stabilizers in the configuration (or base) trajectories. By comparison with [11], besides giving technical details, here we prove that if the target system is a simple mechanical system on a Lie group, then the zero-dynamics is an affine connection control system. For systems underactuated by one control, we give conditions for the zero-dynamics to admit a kinetic energy function which remains invariant along its trajectories. We also show that in such case, if the initial error and velocities are sufficiently small, then the solutions exist for all positive times and ultimately exhibit velocities that may be made arbitrarily small by taking sufficiently "small" initial conditions for the error. Part of our current research aims at introducing dissipation into the compound dynamics—via the use of so-called generalized transverse functions, introduced in [15]—in order to ensure that the velocities of the controlled and auxiliary systems converge to zero.

8 Appendix: Technical lemmas and proofs

Lemma 1 *Let $f : M \longrightarrow N$ be a mapping of class C^2. Then:*

(i) *TTf maps vertical (tangent) vectors to vertical vectors.*

(ii) *If $v, w \in TM$ satisfy $\pi_M(v) = \pi_M(w)$, then $TTf(\text{lift}(v,w)) = \text{lift}(Tf(v), Tf(w))$.*

(iii) *For $v \in TM$, $\text{lift}(v, \cdot) : T_{\pi_M(v)}M \longrightarrow T_vTM^{\text{vert}}$ is a vector space isomorphism.*

(iv) *If f is an immersion, then so is Tf.*

(v) *If X is a second-order vector field defined along a curve $\omega : (t_0, t_1) \longrightarrow TM$ by $X_{\omega(t)} = \dot{\omega}(t)$, then $Tf \circ \omega$ satisfies a second-order differential equation.*

Proof. **(i)** Let $v \in TM$ and $\xi \in T_vTM^{\text{vert}}$. From $T\pi_N \circ TTf = Tf \circ T\pi_M$ one gets $T\pi_N \circ TTf(\xi) = Tf \circ T\pi_M(\xi) = 0$, hence $TTf(\xi) \in T_{Tf(v)}TN^{\text{vert}}$. **(ii)** Let $v, w \in TM$ satisfy $\pi_M(v) = \pi_M(w)$ and define $\gamma_{v,w}(t) = v + tw$ so that $\text{lift}(v,w) = T_0\gamma_w \left(\frac{\partial}{\partial r}\big|_0\right)$. By linearity of Tf on fibers, $(Tf \circ \gamma_{v,w})(t) = Tf(v + wt) = Tf(v) + tTf(w) = \gamma_{Tf(v),Tf(w)}$. Therefore $TTf(\text{lift}(v,w)) = T(Tf \circ \gamma_{v,w}) \left(\frac{\partial}{\partial r}\big|_0\right) = \text{lift}(Tf(v), Tf(w))$. **(iii)** This is well known (cf. e.g. [2]). **(iv)** Let $v \in TM$ and $\alpha \in \ker(T_vTf) \subset T_vTM$. We have $Tf \circ T\pi_M = T\pi_N \circ TTf$, hence $Tf(T\pi_M(\alpha)) = T\pi_N(TTf(\alpha)) = 0$. But f is an immersion, so Tf is injective. Thus $T\pi_M(\alpha) = 0$, so α is vertical and, by (i), there exists $w \in T_{\pi_M(v)}M$ such that $\alpha = \text{lift}(v,w)$. Using (ii), $0 = TTf(\alpha) = TTf(\text{lift}(v,w)) = \text{lift}(Tf(v), Tf(w))$, so $Tf(w) = 0$ by the injectivity of $\text{lift}(Tf(v), \cdot\,)$. Since Tf is injective, $w = 0$, so $\alpha = \text{lift}(v, 0) = 0 \in T_vTM$. Hence T_vTf is injective for every $v \in TM$ and Tf is an immersion. **(v)** Let Y be defined along $Tf \circ \omega$ by setting, for $t \in (t_0, t_1)$, $Y_{Tf\circ\omega(t)} = T_t(Tf \circ \omega) \left(\frac{\partial}{\partial r}\big|_t\right)$. Let us prove that Y is a second order vector field along $Tf \circ \omega$. First, the mapping $\theta = \pi_M \circ \omega : (t_0, t_1) \longrightarrow M$ defines a curve on M whose time-derivative is precisely ω, for if $t \in (t_0, t_1)$, then $\dot{\theta}(t) = T\pi_M \circ T\omega \left(\frac{\partial}{\partial r}\big|_t\right) = T\pi_M(X_{\omega(t)}) = \omega(t)$, since X is second order. Then, using $\pi_N \circ Tf = f \circ \pi_M$ one gets, for every $t \in (t_0, t_1)$, $T\pi_N \circ Y(Tf \circ \omega(t)) = T\pi_N((Tf \circ \omega)'(t)) = T\pi_N \circ T(Tf \circ \omega)\left(\partial/\partial r|_t\right) = T(\pi_N \circ Tf \circ \omega)\left(\partial/\partial r|_t\right) = T(f \circ \pi_M \circ \omega)\left(\partial/\partial r|_t\right) = Tf \circ T\theta\left(\partial/\partial r|_t\right) = Tf \circ \omega(t)$, so Y is second order along $Tf \circ \omega$, as claimed. ∎

Lemma 2 *Let G be a Lie group and TG its tangent group. If $X \in \Gamma(TG)$ is left-invariant then so is $X^{\text{lift}} \in \Gamma(TTG)$.*

Proof. We denote translations and multiplication on G by $\widehat{L}, \widehat{R}, \widehat{\mu}$ and, on TG, by L, R, μ. The canonical projection $TG \longrightarrow G$ is denoted by π. Recall that $X \in \Gamma(TG)$ is left-invariant if, and only if, $T\widehat{L}_g(X_h) = X_{gh}$ for every $g, h \in G$. Under the assumption that $u, v \in TG$, we shall show that $TL_u(X_v^{\text{lift}}) = X_{uv}^{\text{lift}}$. First note that $L_u \circ \gamma_{v,X_{\pi(v)}} = \mu\left(u, v + tX_{\pi(v)}\right) = T\widehat{L}_{\pi(u)}\left(v + tX_{\pi(v)}\right) + T\widehat{R}_{\pi(v+tX_{\pi(v)})}(u) =$

$T\widehat{L}_{\pi(u)}(v)+tT\widehat{L}_{\pi(u)}(X_{\pi(v)})+T\widehat{R}_{\pi(v)}(u)=uv+tT\widehat{L}_{\pi(u)}(X_{\pi(v)})=uv+tX_{\pi(u)\pi(v)}=\gamma_{uv,X_{\pi(uv)}}(t)$, where we used $\pi(v+tX_{\pi(v)})=\pi(v)$ and $\pi(u)\pi(v)=\pi(uv)$, as well as the left invariance of X. Therefore, $TL_u(X_v^{\mathrm{lift}})=TL_u(T\gamma_{v,X_{\pi(v)}}\left(\frac{\partial}{\partial r}\big|_0\right))=T(L_u\circ\gamma_{v,X_{\pi(v)}})\left(\frac{\partial}{\partial r}\big|_0\right)=X_{uv}^{\mathrm{lift}}$, as was to be shown. ∎

8.1 Proof of Theorem 2.

We assume that $f:\mathbb{T}^{n-m}\longrightarrow G$ is transverse to $\{X_1,\dots,X_m\}$ near $e\in G$ and the sum in (3) is direct. Thus $Tf:T\mathbb{T}^{n-m}\longrightarrow TG$ is vertically transverse to $\{X_1^{\mathrm{lift}},\dots,X_m^{\mathrm{lift}}\}$ and satisfies (6) for every $\omega\in T\mathbb{T}^{n-m}$. Let $\{\Lambda_1,\dots,\Lambda_{n-m}\}\subset\Gamma(T\mathbb{T}^{n-m})$ represent a global frame for the trivial bundle $T\mathbb{T}^{n-m}$. From Lemma 1-(iii), $\{\Lambda_1^{\mathrm{lift}},\dots,\Lambda_{n-m}^{\mathrm{lift}}\}$ is a global frame for the vertical subbundle $(TT\mathbb{T}^{n-m})^{\mathrm{vert}}$; hence, since Ω_j is vertical and smooth, there exist smooth functions $\lambda_j^i:T\mathbb{T}^{n-m}\longrightarrow\mathbb{R}$, $i,j=1,\dots,n-m$, such that $\Omega_j=\sum_{i=1}^{n-m}\lambda_j^i\Lambda_i^{\mathrm{lift}}$, $j=1,\dots,n-m$. In preparation for the sequel of the proof, we recall a standard procedure (see e.g. [19, Prop. 1.35]) which locally extends mappings defined along immersed manifolds. Consider manifolds L,M,N and mappings $F:L\longrightarrow M$ and $h:L\longrightarrow N$, and assume that T_pF is injective for some $p\in L$. Let d_L and d_M denote the dimensions of L and M respectively. Then there exists an open neighborhood $U\subset L$ of p such that $F|_U$ is injective, and there exists a cubic-centered coordinate system (V,φ) for M about $F(p)$ for which $F(U)$ is a slice, that is, (V,φ) satisfies $F(p)\in V$, $\varphi(F(p))=0\in\mathbb{R}^{d_M}$ and $\varphi(F(U))=(-\varepsilon,\varepsilon)^{d_L}\times\{0\}\subset\mathbb{R}^{d_L}\times\mathbb{R}^{d_M-d_L}$ for some $\varepsilon>0$. Now, if $\pi:\mathbb{R}^{d_M}\longrightarrow\mathbb{R}^{d_M}$ denotes the projector that maps $(x^1,\dots,x^{d_M})$ to $(x^1,\dots,x^{d_L},0,\dots,0)$, then the mapping $\widetilde{h}=h\circ(F|_U)^{-1}\circ\varphi^{-1}\circ\pi\circ\varphi$ is smooth, is defined on V, and satisfies $\widetilde{h}(F(U))=h\circ(F|_U)^{-1}\circ\varphi^{-1}\circ\pi\circ\varphi(F(U))=h\circ(F|_U)^{-1}\circ F(U)=h(U)$, since $\varphi^{-1}\circ\pi\circ\varphi(F(U))=F(U)$. In other words, $\widetilde{h}$ is explicitly constructed to "extend" h so that the following diagram commutes

$$\begin{tikzcd} U\subset L \arrow[rr, "F|_U"] \arrow[dr, "h|_U"'] & & V\subset M \arrow[dl, "\widetilde{h}"] \\ & N & \end{tikzcd}$$

Let $\omega\in T\mathbb{T}^{n-m}$ and set $\theta=\pi_{\mathbb{T}^{n-m}}(\omega)$. Applying the extension procedure, with $F=f$, $h=Tf\circ\Lambda_j$, $L=\mathbb{T}^{n-m}$, $M=G$ and $N=TG$, and using the assumption that $T_\theta f$ is injective, one deduces the existence of open sets $U\subset\mathbb{T}^{n-m}$ and $V\subset G$, as well as vector fields $\widetilde{\Lambda}_j$ defined on V, such that $\theta\in U$ and

$$\widetilde{\Lambda}_j\circ f|_U=Tf\circ\Lambda_j|_U,\quad j=1,\dots,n-m. \tag{19}$$

(In the terminology of [19, Def. 1.51], $\widetilde{\Lambda}_j$ is a local C^∞ extension of Λ_j). Moreover, by continuity of f, U can be taken so small that, by virtue of the transversality property (3), $T_qG=\mathrm{span}\{X_{1,q},\dots,X_{m,q},\widetilde{\Lambda}_{1,q},\dots,\widetilde{\Lambda}_{n-m,q}\}$ for every $q\in V$. It follows from Lemma 1-(iii) that, together with $X_1^{\mathrm{lift}},\dots,X_m^{\mathrm{lift}}$, the lifted vector fields $\widetilde{\Lambda}_1^{\mathrm{lift}},\dots,\widetilde{\Lambda}_{n-m}^{\mathrm{lift}}$, defined on $\widetilde{W}=\pi_G^{-1}(V)\subset TG$, constitute a frame for the vertical bundle over $\widetilde{W}$:

$$T_vTG^{\mathrm{vert}}=\mathrm{span}\{X_{1,v}^{\mathrm{lift}},\dots,X_{m,v}^{\mathrm{lift}},\widetilde{\Lambda}_{1,v}^{\mathrm{lift}},\dots,\widetilde{\Lambda}_{n-m,v}^{\mathrm{lift}}\},\quad\forall v\in\widetilde{W}. \tag{20}$$

Let $\sigma:T\mathbb{T}^{n-m}\longrightarrow\mathbb{R}^n$ be the mapping with components given by $\sigma^i=\nu^i$ for $i=1,\dots,m$ and $\sigma^{j+m}=\sum_{k=1}^{n-m}\nu^{k+m}\lambda_k^j$, $j=1,\dots,n-m$. In the extension procedure described above we take $h=\sigma$, $L=T\mathbb{T}^{n-m}$, $M=TG$ and $N=\mathbb{R}^n$, and replace F by Tf, the tangent mapping of which is injective by virtue of Lemma 1-(iv), to deduce the existence of open neighborhoods $\widetilde{U}$ of ω and $\widetilde{V}$ of $Tf(\omega)$, as well as a mapping $\widetilde{\sigma}:\widetilde{V}\longrightarrow\mathbb{R}^n$ such that $\widetilde{\sigma}^i\circ Tf|_{\widetilde{U}}=\sigma^i|_{\widetilde{U}}$, $\quad i=1,\dots,n$. Again, by continuity of Tf, $\widetilde{U}$ can be taken so small that $\widetilde{V}\subset\widetilde{W}$, so that $\widetilde{\Lambda}_1^{\mathrm{lift}},\dots,\widetilde{\Lambda}_{n-m}^{\mathrm{lift}}$ and the functions $\widetilde{\sigma}^i$ are defined on $\widetilde{V}\subset TG$.

Using the ingredients above, in particular the verticality and smoothness of P, along with (20), we ascertain the existence of a smooth mapping $\widetilde{a}=(\widetilde{a}^1,\dots,\widetilde{a}^n):\widetilde{V}\longrightarrow\mathbb{R}^n$ such that

$$P=\textstyle\sum_{i=1}^m\widetilde{a}^iX_i^{\mathrm{lift}}+\sum_{j=1}^{n-m}\widetilde{a}^{j+m}\widetilde{\Lambda}_j^{\mathrm{lift}}. \tag{21}$$

From this definition and from $[C, X_i^{\rm lift}] = -X_i^{\rm lift}$ for $i = 1, \dots, m$, it follows that

$$\begin{array}{rcl} [C,P] & = & \sum_{i=1}^m(-\widetilde{a}^i + C(\widetilde{a}^i))X_i^{\rm lift} + \sum_{j=1}^{n-m}(-\widetilde{a}^{j+m} + C(\widetilde{a}^{j+m}))\widetilde{\Lambda}_j^{\rm lift} \\ & = & -P + \sum_{i=1}^m C(\widetilde{a}^i)X_i^{\rm lift} + \sum_{j=1}^{n-m} C(\widetilde{a}^{j+m})\widetilde{\Lambda}_j^{\rm lift}. \end{array} \tag{22}$$

Now, we claim that the vector fields

$$\Sigma = \Delta + \sum_{j=1}^{n-m}(\sigma^{j+m} - a^{j+m})\Lambda_j^{\rm lift} \quad \text{and} \quad \Pi = \sum_{j=1}^{n-m} a^{j+m}\Lambda_j^{\rm lift}, \tag{23}$$

with $a = \widetilde{a} \circ Tf|_{\widetilde{U}}$, satisfy the properties in the statement. By definition of $\mathcal{P}$, we see that $\mathcal{P} \circ P \circ Tf = TTf \circ \Pi$, hence the proof reduces to showing that Σ is a spray. Since Σ is a second-order vector field, as follows immediately from its definition, it suffices to prove that $[\widehat{C}, \Sigma] = \Sigma$, where $\widehat{C} = C^{\mathbb{T}^{n-m}}$ denotes the Liouville vector field associated with $\mathbb{T}^{n-m}$. Using the definition of Σ and the fact that Δ is a spray we obtain

$$\begin{array}{rcl} [\widehat{C}, \Sigma] & = & [\widehat{C}, \Delta] + \sum_{j=1}^{n-m}[\widehat{C}, (\sigma^{j+m} - a^{j+m})\Lambda_j^{\rm lift}] \\ & = & \Delta + \sum_{j=1}^{n-m}(-(\sigma^{j+m} - a^{j+m})\Lambda_j^{\rm lift} + \widehat{C}(\sigma^{j+m} - a^{j+m})\Lambda_j^{\rm lift}). \end{array} \tag{24}$$

As suggested by this equation, in order to prove the claim we shall find an expression for $\widehat{C}(\sigma^{j+m} - a^{j+m})$, the Lie derivative of $\sigma^{j+m} - a^{j+m}$ in the direction of $\widehat{C}$, $j = 1, \dots, n-m$. We have

$$\begin{array}{rcl} \left[C, Z + \sum_{i=1}^m \widetilde{\sigma}^i X_i^{\rm lift}\right] \circ Tf|_{\widetilde{U}} & = & [C, S+P] \circ Tf|_{\widetilde{U}} + \sum_{i=1}^m [C, \widetilde{\sigma}^i X_i^{\rm lift}] \circ Tf|_{\widetilde{U}} \\ & = & S \circ Tf|_{\widetilde{U}} + [C,P] \circ Tf|_{\widetilde{U}} + \\ & & \sum_{i=1}^m((-\widetilde{\sigma}^i + C(\widetilde{\sigma}^i))X_i^{\rm lift}) \circ Tf|_{\widetilde{U}}, \end{array} \tag{25}$$

and

$$\begin{array}{rcl} TTf \circ \left[\widehat{C}, \Delta + \sum_{j=1}^{n-m} \sigma^{j+m}\Lambda_j^{\rm lift}\right] & = & TTf \circ \Big([\widehat{C}, \Delta] + \\ & & \sum_{j=1}^{n-m}(\sigma^{j+m}[\widehat{C}, \Lambda_j^{\rm lift}] + \widehat{C}(\sigma^{j+m})\Lambda_j^{\rm lift})\Big) \\ & = & TTf \circ \Delta + \\ & & \sum_{j=1}^{n-m} TTf \circ ((-\sigma^{j+m} + \widehat{C}(\sigma^{j+m}))\Lambda_j^{\rm lift}). \end{array} \tag{26}$$

Now, from (13) and the definition of $\widetilde{\sigma}^i$ it follows that $[\widehat{C}, \Delta + \sum_{j=1}^{n-m} \sigma^{j+m}\Lambda_j^{\rm lift}]$ and $[C, Z + \sum_{i=1}^m \widetilde{\sigma}^i X_i^{\rm lift}]$ are Tf-related, hence the respective members of (25) and (26) are equal. Equating the right-hand-sides of (25) and (26), and then replacing P and $[C,P]$ by their equivalent expressions as given by (21) and (22), respectively, we obtain

$$\begin{array}{c} \Big(S + \sum_{i=1}^m(C(\widetilde{a}_i + \widetilde{\sigma}_i) - (\widetilde{a}_i + \widetilde{\sigma}_i))X_i^{\rm lift} + \sum_{j=1}^{n-m}(C(\widetilde{a}^{j+m}) - \widetilde{a}^{j+m})\widetilde{\Lambda}_j^{\rm lift}\Big) \circ Tf|_{\widetilde{U}} = \\ TTf \circ \Delta + \sum_{j=1}^{n-m} TTf \circ ((-\sigma^{j+m} + \widehat{C}(\sigma^{j+m}))\Lambda_j^{\rm lift}). \end{array} \tag{27}$$

Now, the addition and then subtraction of $(P + \sum_{i=1}^m \widetilde{\sigma}^i X_i^{\rm lift}) \circ Tf|_{\widetilde{U}}$ to the left-hand-side of (27), and the addition and then subtraction of $TTf \circ \left(\sum_{j=1}^{n-m} \sigma^{j+m}\Lambda_j^{\rm lift}\right)$ to its right-hand-side, yields an equation where like terms can be cancelled using again the equality of (25) and (26). After simplifying it we get

$$\begin{array}{c} \sum_{i=1}^m((C(\widetilde{a}^i + \widetilde{\sigma}^i) - 2(\widetilde{a}^i + \widetilde{\sigma}^i))X_i^{\rm lift}) \circ Tf|_{\widetilde{U}} + \sum_{j=1}^{n-m}((C(\widetilde{a}^{j+m}) - 2\widetilde{a}^{j+m})\widetilde{\Lambda}_j^{\rm lift}) \circ Tf|_{\widetilde{U}} = \\ TTf \circ \left(\sum_{j=1}^{n-m}(\widehat{C}(\sigma^{j+m}) - 2\sigma^{j+m})\Lambda_j^{\rm lift}\right). \end{array} \tag{28}$$

Using the fact that $\widehat{C}$ and C are Tf-related, along with the definitions of tangent mapping and of $a = \widetilde{a} \circ Tf|_{\widetilde{U}}$, one obtains

$$\begin{array}{rcl} C(\widetilde{a}^{j+m}) \circ Tf|_{\widetilde{U}} & = & (TTf \circ \widehat{C})(\widetilde{a}^{j+m}) \\ & = & \widehat{C}(\widetilde{a}^{j+m} \circ Tf|_{\widetilde{U}}) \\ & = & \widehat{C}(a^{j+m}), \qquad j = 1, \dots, n-m. \end{array}$$

Moreover, using the definition of lift of a vector field

$$\begin{aligned}\widetilde{\Lambda}_j^{\text{lift}} \circ Tf|_{\widetilde{U}}(\omega) &= \text{lift}(Tf(\omega), \widetilde{\Lambda}_{j,f(\theta)}) \qquad (\text{since } \pi_G(Tf(\omega)) = f(\theta))\\ &= \text{lift}(Tf(\omega), Tf(\Lambda_{j,\theta})) \qquad (\text{by (19)})\\ &= TTf(\text{lift}(\omega, \Lambda_{j,\theta})) \qquad (\text{by Lemma 1})\\ &= TTf(\Lambda_{j,\omega}^{\text{lift}}),\end{aligned}$$

thus $\widetilde{\Lambda}_j^{\text{lift}} \circ Tf|_{\widetilde{U}} = TTf \circ \Lambda_j^{\text{lift}}$, $j = 1, \ldots, n-m$. These expressions, along with the fiberwise linearity of TTf, enable us to write (28) as

$$\sum_{i=1}^{m}((C(\widetilde{a}^i + \widetilde{\sigma}^i) - 2(\widetilde{a}^i + \widetilde{\sigma}^i))X_i^{\text{lift}}) \circ Tf|_{\widetilde{U}} \quad + \\ \sum_{j=1}^{n-m}(\widehat{C}(a^{j+m} - \sigma^{j+m}) - 2(a^{j+m} - \sigma^{j+m}))(TTf \circ \Lambda_j^{\text{lift}}) = 0.$$

In view of the vertical transversality condition (6), the coefficient of $TTf \circ \Lambda_j^{\text{lift}}$ must be zero, which implies that $\widehat{C}(a^{j+m} - \sigma^{j+m}) = 2(a^{j+m} - \sigma^{j+m})$, $j = 1, \ldots, n-m$. Plugging these equations into (24) we conclude that $[\widehat{C}, \Sigma] = \Sigma$, as required. ∎

8.2 Proof of Theorem 3.

For the sake of brevity, we omit a number of details which are straightforward to fill in. Let $\kappa = n - m$ and $M = TG \times T\mathbb{T}^\kappa$, so that $\dim(M) = 2n + 2\kappa$. Let $p : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{T}^\kappa$ be the universal covering of $\mathbb{T}^\kappa$ (here we consider only *smooth* coverings and identify coverings *up to isomorphism*, cf. e.g. [3, Chap. 1] for more details). It is easily seen that $Tp : T\widetilde{\mathbb{T}}^\kappa \longrightarrow T\mathbb{T}^\kappa$ is the smooth universal covering of $T\mathbb{T}^\kappa$. The canonical projection $\pi : \mathbb{R}^\kappa \longrightarrow \mathbb{R}^\kappa/\mathbb{Z}^\kappa$ defines a smooth covering isomorphic to $p : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{T}^\kappa$, i.e., there exist diffeomorphisms $\alpha : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{R}^\kappa$ and $\alpha' : \mathbb{T}^\kappa \longrightarrow \mathbb{R}^\kappa/\mathbb{Z}^\kappa$ such that $\pi \circ \alpha = \alpha' \circ p$. Clearly, $(\widetilde{\mathbb{T}}^\kappa, \alpha)$ is a global coordinate chart which naturally induces a global chart $(T\widetilde{\mathbb{T}}^\kappa, \widetilde{\alpha})$ on $T\widetilde{\mathbb{T}}^\kappa$. Let (O, ϕ) be a chart on TG such that $0_e \in O$ and $\phi(0_e) = 0$. Finally, let $\widetilde{M} := TG \times T\widetilde{\mathbb{T}}^\kappa$, $P(z, \vartheta) = (z, Tp(\vartheta))$ and $\psi(z, \vartheta) = (\phi(z), \widetilde{\alpha})$. Thus defined, $P : \widetilde{M} \longrightarrow M$ is a smooth covering whereas $\psi = (\psi^1, \ldots, \psi^{2n+2\kappa}) : O \times T\widetilde{\mathbb{T}}^\kappa \subset \widetilde{M} \longrightarrow \mathbb{R}^{2n+2\kappa}$ is a chart on $\widetilde{M}$. These coordinates will be labeled as $x = (x^1, \ldots, x^{2n}) = (\psi^1, \ldots, \psi^{2n})$, $\theta = (\theta^1, \ldots, \theta^\kappa) = (\psi^{2n+1}, \ldots, \psi^{2n+\kappa})$ and $\dot{\theta} = (\dot{\theta}^1, \ldots, \dot{\theta}^\kappa) = (\psi^{2n+\kappa+1}, \ldots, \psi^{2n+2\kappa})$. (This choice corresponds to the θ^is representing angle-like functions and the $\dot{\theta}^i$s "angular velocities.") It is straightforward, albeit somewhat laborious, to show that our particular choice of P and ψ makes then following claims hold: **(C1)** $\psi(P^{-1}(\{0_e\} \times Z(T\mathbb{T}^\kappa))) = \{0\} \times \mathbb{R}^\kappa \times \{0\}$, i.e., the preimage of $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ by P is a subset of $\widetilde{M}$ whose coordinates are of the form $(0, \theta, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^\kappa \times \mathbb{R}^\kappa$; **(C2)** if V is a neighborhood of $\{0_e\} \times Z(T\mathbb{T}^\kappa)$, then there exist compact, convex neighborhoods $V_1 \subset \mathbb{R}^{2n}$ and $V_2 \subset \mathbb{R}^\kappa$ of the respective origins such that $V_1 \times \mathbb{R}^\kappa \times V_2 \subset \psi(P^{-1}(V))$; **(C3)** For $i = 1, \ldots, 2n+2\kappa$, there is a well-defined vector field $P_* \frac{\partial}{\partial \psi^i}$, on an open subset of M, which is P-related to the ith coordinate vector field $\frac{\partial}{\partial \psi^i}$, that is, $(P_* \frac{\partial}{\partial \psi^i}) \circ P = TP \circ \frac{\partial}{\partial \psi^i}$; **(C4)** if $V_1 \subset \mathbb{R}^{2n}$ and $V_2 \subset \mathbb{R}^\kappa$ are compact, then so is $P(\psi^{-1}(V_1 \times \mathbb{R}^\kappa \times V_2)) \subset M$; and **(C5)** Let $\widehat{K}$ be the representative of P^*K (the pullback of $\widehat{K}$ by P) in the ψ coordinates, then there exist smooth functions $g_{i,j} : \mathbb{R}^\kappa \longrightarrow \mathbb{R}$ such that $\widehat{K}(x, \theta, \dot{\theta}) = \frac{1}{2} g_{i,j}(\theta)\dot{\theta}^i\dot{\theta}^j$ (using Einstein's summation convention, which remains in effect in the sequel of the proof).

Let $A \in \Gamma(TM)$ the vector field whose value at $(z, \omega) \in M$ is given by the right-hand side of (14)—with the usual identification $T(TG \times T\mathbb{T}^\kappa) \simeq TTG \times TT\mathbb{T}^\kappa$. The set $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ is invariant under A, which entails that $\mathcal{I} := \{0\} \times \mathbb{R}^\kappa \times \{0\}$ is invariant under $\widehat{A} = \psi_*(P^*A)$, the representative of the pullback P^*A in the ψ coordinates. In view of the previous assumptions, at this point the proof reduces to establishing the following claim: For any open neighborhood $\mathcal{V} = V_1 \times \mathbb{R}^\kappa \times V_2$ of $\mathcal{I}$, there is a neighborhood $\mathcal{U} = U_1 \times \mathbb{R}^\kappa \times U_2$ of $\mathcal{I}$ such that $\widehat{A}$ is $\mathcal{U}$-positive complete and, for any initial condition $(x_0, \theta_0, \dot{\theta}_0) \in \mathcal{U}$, the integral curve $(x(t), \theta(t), \dot{\theta}(t))$ of $\widehat{A}$, initialized at $(x_0, \theta_0, \dot{\theta}_0)$, satisfies $(x(t), \theta(t), \dot{\theta}(t)) \in \mathcal{V}$ for all $t \geq 0$. In order to prove the claim, assume, without loss of generality, that $V_1 \subset \mathbb{R}^{2n}$ and $V_2 \subset \mathbb{R}^\kappa$ are compact neighborhoods of the respective origins and let $\mathcal{V} = V_1 \times \mathbb{R}^\kappa \times V_2$.

Define $\xi \in \Gamma(TM)$ by $\xi_{(z,\omega)} = \left(0_z, \sum_{i=1}^{\kappa}(\alpha^{i+m}(z,\omega) - \alpha^{i+m}(0_e,\omega))\Omega_{i,\omega}\right)$, so that $A = (S_z, \Sigma_\omega) + \xi_{(z,\omega)}$ and let $\widehat{\xi}$ be the representative of $P^*\xi$ in the coordinates ψ. Since K does not depend explicitly on z, and $\mathcal{G}$ is invariant under Σ, one has $P^*(A(K)) = P^*(\xi(K)) = P^*\xi(P^*K)$. By smoothness of the compound system, for any initial condition in its domain, the integral curve $(x(t), \theta(t), \dot\theta(t))$ of $\widehat{A}$ is defined for t in some neighborhood of 0. For any such t, the derivative of $W : t \mapsto P^*K \circ \psi^{-1}(x(t), \theta(t), \dot\theta(t))$ can then be computed as follows

$$\begin{aligned} W'(t) &= P^*\xi|_{\psi^{-1}(x(t),\theta(t),\dot\theta(t))}(P^*K) \\ &= \textstyle\sum_{i=1}^{\kappa} \left(\frac{\partial(P^*K)}{\partial\theta^i} \cdot 0 + \frac{\partial(P^*K)}{\partial\dot\theta^i} (P^*\xi)^{2n+\kappa+i} \right)\Big|_{\psi^{-1}(x(t),\theta(t),\dot\theta(t))} \\ &=: \mu \circ \psi^{-1}(x(t), \theta(t), \dot\theta(t)), \end{aligned}$$

with $\mu \in C^\infty(\widetilde{M})$ given by $\mu = \sum_{i=1}^{\kappa} \partial(P^*K)/\partial\dot\theta^i \cdot (P^*\xi)^{2n+\kappa+i}$. Since $\xi_{(0,\omega)} = 0$ for all $\omega \in T\mathbb{T}^\kappa$, then $\widehat{\xi}_{(0,\theta,\dot\theta)} = 0$ for all $(\theta, \dot\theta) \in \mathbb{R}^\kappa \times \mathbb{R}^\kappa$. Therefore, using a Taylor expansion with remainder, along with the fact that $\widehat{K}$ is quadratic in the $\dot\theta^i$s, one obtains:

$$\begin{aligned} W'(t) = \Bigg(\Bigg(&\frac{\partial^2\mu}{\partial\dot\theta^k\partial x^i} \circ \psi^{-1}(0, \theta(t), 0) + \frac{1}{2} \frac{\partial^3\mu}{\partial\dot\theta^k\partial x^j\partial x^i} \circ \psi^{-1}(c_1 x(t), \theta(t), 0) \cdot x^j(t) \; + \\ &\frac{1}{2} \frac{\partial^3\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial x^i} \circ \psi^{-1}(0, \theta(t), c_2\dot\theta(t)) \cdot \dot\theta^\ell(t) \\ &+ \frac{1}{4} \frac{\partial^4\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial x^j\partial x^i} \circ \psi^{-1}(c_1 x(t), \theta(t), c_2\dot\theta(t)) \cdot \dot\theta^\ell(t) \cdot x^j(t) \Bigg) \dot\theta^k(t) \Bigg) x^i(t), \end{aligned}$$

for some reals $c_1, c_2 \in (0, 1)$. Without loss of generality, we assume that ϕ was chosen so that, in view of the assumption of local exponential stability of 0_e for S_z, there exist reals $C_1, C_2 > 0$ such that $|x^i(t)| \le \|x(t)\| \le C_1\|x_0\|e^{-C_2 t}$ for $i = 1, \dots, 2n$. Moreover, since $\widehat{K}$ is positive-definite, quadratic in the $\dot\theta^i$s, and independent of x, there exist constants $C_3, C_4 > 0$ such that $C_3 W(t)^{\frac{1}{2}} \le |\dot\theta^k(t)| \le C_4 W(t)^{\frac{1}{2}}$ for $k = 1, \dots, \kappa$ and $t \ge 0$. For $i = 1, \dots, 2n$ and $k = 1, \dots, \kappa$ let

$$\begin{aligned} N_{i,k} = \max \Bigg\{ \Bigg| &\frac{\partial^2\mu}{\partial\dot\theta^k\partial x^i} \circ \psi^{-1}(0, \theta, 0) + \frac{1}{2} \frac{\partial^3\mu}{\partial\dot\theta^k\partial x^j\partial x^i} \circ \psi^{-1}(x, \theta, 0) \cdot x^j \\ &+ \frac{1}{2} \frac{\partial^3\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial x^i} \circ \psi^{-1}(0, \theta, \dot\theta) \cdot \dot\theta^\ell \\ &+ \frac{1}{4} \frac{\partial^4\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial x^j\partial x^i} \circ \psi^{-1}(x, \theta, \dot\theta) \cdot \dot\theta^\ell \cdot x^j \Bigg| : (x, \theta, \dot\theta) \in V_1 \times \mathbb{R}^\kappa \times V_2 \Bigg\}. \end{aligned}$$

That these maxima exist is an easily deduced consequence of (C3) and (C4), along with the compactness of V_1 and V_2. Let $C_5 = \max\{N_{i,k} : i = 1, \dots, 2n,\ k = 1, \dots, \kappa\}$. Note that if $a \in \mathbb{R}$, then $a \le a^2 + 1$, hence $W(t)^{1/2} \le W(t) + 1$ for every t for which $W(t)$ is defined. Therefore

$$\begin{aligned} W'(t) &\le C_5 |\dot\theta^k(t)||x^i(t)| \\ &\le C_4 C_5 (W(t))^{\frac{1}{2}} |x^i(t)| \\ &\le C_1 C_4 C_5 (W(t) + 1)\|x_0\|e^{-C_2 t}. \end{aligned}$$

Let $\varepsilon > 0$ be such that $\{x \in \mathbb{R}^{2n} : \|x\| < \varepsilon\} \subset V_1$ and $\{\dot\theta \in \mathbb{R}^\kappa : \|\dot\theta\| < \varepsilon\} \subset V_2$. If $\|x_0\| < \varepsilon/C_1$ then $\|x(t)\| < \varepsilon$ for $t \ge 0$. Now, the initial value problem $\dot y(t) = C_1C_4C_5(y(t) + 1)\|x_0\|e^{-C_2 t}$, $y(0) = W(0) = W_0$, admits the unique solution $y(t) = -1 + (1 + W_0)\exp\left(-\frac{C_1C_4C_5}{C_2}\|x_0\|(e^{-C_2 t} - 1)\right)$. From the "Comparison Lemma" (cf. e.g. [5, Lemma 2.5]), $W(t) \le y(t) \le L(x_0, W_0) := -1 + (1 + W_0)\exp\left(\frac{C_1C_4C_5}{C_2}\|x_0\|\right)$. Since L is continuous and $\lim_{(x_0,W_0)\to(0,0)} = 0$, there exists $\delta > 0$ such that if $0 < \|x_0\|, W_0 < \delta$, then $W(t) \le L(x_0, W_0) < \frac{\varepsilon^2}{\kappa C_4^2}$ for $t \ge 0$. Using $\kappa C_3^2 W(t) \le \|\dot\theta(t)\|^2 \le \kappa C_4^2 W(t)$, $t \ge 0$, we conclude that if $\|\dot\theta_0\| < \delta_2 := \sqrt{\kappa\delta}C_3$, then $W_0 < \delta$ and, in turn, $\|\dot\theta(t)\|^2 < \varepsilon^2$. Thus, taking x_0 and $\dot\theta_0$ so that $\|x_0\| < \delta_1 := \min\{\varepsilon/C_1, \delta\}$ and $\|\dot\theta_0\| < \delta_2$, one has $(x(t), \theta(t), \dot\theta(t)) \in V_1 \times \mathbb{R}^\kappa \times V_2$ for all $t \ge 0$, which completes the proof. ■

References

[1] R. Abraham and J.E. Marsden. *Foundations of Mechanics.* Addison-Wesley Publ. Co., Inc., 2nd. edition, 1985.

[2] J. Grifone. Structure presque tangente et connexions I. *Annales de l'Institut Fourier*, 22(1):287–334, 1972.

[3] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[4] J. Imura, K. Kobayashi, and T. Yoshikawa. Nonholonomic control of 3 link planar manipulator with a free joint. In *IEEE Conf. on Decision and Control (CDC)*, pages 1435–1436, Kobe, Japan, December 1996.

[5] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, Inc., Upper Saddle River, NJ, 2nd. edition, 1996.

[6] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry (Volume I)*. Wiley Classics Library. John Wiley and Sons, Inc., 1996 (reprint of the 1963 edition).

[7] A.D. Lewis. When is a mechanical control system kinematic? In *IEEE Conf. on Decision and Control (CDC)*, pages 1162–1167, Phoenix, USA, December 1999.

[8] A.D. Lewis. Affine connection control systems. In *Proc. of the IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 128–133, Princeton, NJ, March 2000.

[9] D.A. Lizárraga. Obstructions to the existence of universal stabilizers for smooth control systems. *Mathematics of Control, Signals, and Systems*, 16:255–277, 2003.

[10] D.A. Lizárraga, P. Morin, and C. Samson. Non-Robustness of Continuous Homogeneous Stabilizers for Affine Control Systems. In *IEEE Conf. on Decision and Control (CDC)*, volume 1, pages 855–860, Phoenix, AZ, December 1999.

[11] D.A. Lizárraga and J.M. Sosa. Vertically transverse functions as an extension of the transverse function control approach for second-order systems. In *IEEE Conference on Decision and Control and European Control Conference ECC*, pages 7290–7295, Sevilla, Spain, December 2005.

[12] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, Second edition, 1999.

[13] P. Morin and C. Samson. A characterization of the Lie Algebra Rank Condition by Transverse Periodic Functions. *SIAM Journal on Control and Optimization*, 40(4):1227–1249, 2001.

[14] P. Morin and C. Samson. Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Transactions on Automatic Control*, 48(9):1496–1508, September 2003.

[15] P. Morin and C. Samson. Practical and asymptotic stabilization of chained systems by the transverse function control approach. *SIAM Journal on Control and Optimization*, 43(1):32–57, 2004.

[16] P. Morin and C. Samson. Control of underactuated mechanical systems by the transverse function approach. In *IEEE Conference on Decision and Control and European Control Conference ECC*, pages 7208–7213, Sevilla, Spain, December 2005.

[17] P. Morin and C. Samson. Control with transverse functions and a single generator of underactuated mechanical systems. In *IEEE Conf. on Decision and Control (CDC)*, pages 6110–6115, San Diego, CA, December 2006.

[18] B.G. Schmidt. Conditions on a Connection to be a Metric Connection. *Commun. math Phys*, 29:55–59, 1973.

[19] F.W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*, volume 94 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Inc., 1983.