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On the Transversality of Functions at the core of the Transverse Function Approach to Control

David A. Lizárraga

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Abstract A precise relationship is established between *transversality*, as understood in differential topology, and the functions lying at the core of the transverse function approach to control. The latter, introduced in the context of nonholonomic systems by Morin and Samson in the early 2000s, is based on the construction and properties of functions that are "transverse" to a set of vector fields in a sense *formally* similar to—although, strictly speaking, different from—the classical notion of transversality. In this paper, the domain of definition of transverse functions is first extended from multidimensional torii to more general manifolds. Then it is shown that a function $f : M \longrightarrow Q$ is "transverse" to a set of vector fields which (locally) span a distribution D on Q if, and only if, its tangent mapping Tf is transverse to D, the latter regarded as a *submanifold* of the tangent bundle TQ. It is shown, furthermore, that these two equivalent conditions are in turn equivalent to transversality of Tf to D *along the zero section* of TM. These results are then used to rigorously state and prove the intuitively clear fact that when M is compact and D is a distribution on Q, the set of mappings of M into Q that are transverse to D is open in the strong (or "Whitney C^∞") topology on the space $C^\infty(M,Q)$.

Keywords Transversality · Transverse Function Approach · Weak / Strong Topology

1 Introduction

In recent years, the *transverse function* approach to control has been developed, as an alternative to time-varying and other feedback techniques, for the control of nonholonomic and underactuated mechanical systems. Many of these systems share the property of being *critical* in the sense that, as a consequence of so-called "Brockett's necessary condition" (or

David Antonio Lizárraga Navarro
Division of Applied Mathematics
Instituto Potosino de Investigación Científica y Tecnológica
Camino a la Presa San José 2055
Col. Lomas 4a sección CP 78216
San Luis Potosí, SLP, México
Tel.: (52-444) 834-2000
Fax: (52-444) 834-2010
E-mail: D.Lizarraga@ipicyt.edu.mx

generalizations thereof, cf. [1,3], [31, Chap. 5]), they admit equilibria that cannot be asymptotically stabilized by means of continuous state feedback. Moreover, the corresponding trajectory tracking problems, which arise in a number of control applications, turn out to be difficult to solve as well. During the decade of the 1990s, considerable research was devoted to the development of feedback strategies to deal with both point stabilization and trajectory tracking for critical systems. As far as point stabilization is concerned, the critical nature of the system calls for techniques more elaborated than continuous pure state feedback, among which one finds *time-varying* and *discontinuous* state feedback. The development of time-varying feedback for critical systems has evolved in a number of directions, some of them spawned by the need to overcome limitations of previously proposed strategies. For instance, the control laws presented in [28], and extended in [4] to cover a larger class of systems, were *differentiable* functions of both the state and time, which entailed *slow convergence* rates for the closed-loop trajectories [25]. In order to improve convergence rates, the control laws were required to be at most Hölder-continuous—hence typically nonsmooth—at the stabilized point [7], and elegant approaches were developed to use *homogeneity* tools to systematically obtain *continuous* (time-varying) stabilizers from differentiable ones [17,27, 18]. Nonetheless, the homogeneous, continuous control laws thus obtained carried along another drawback that affected their robustness with respect to unmodeled dynamics, namely, arbitrarily "small" perturbations in the system vector fields could render the stabilized point unstable [14]. The trajectory tracking problem, on the other hand, may be addressed by means of standard state or output feedback techniques (cf. e.g. [29,2,26,32,11,12]), but the solutions available typically require that the tracked trajectories exhibit a form of "persistency of excitation." Although the latter condition occurs in several guises, it usually rules out trajectories that converge to a point, so the tracking of constant trajectories (i.e., "point stabilization") is excluded from the outset. Conceptually, thus, point stabilization and trajectory tracking were frequently addressed as two different problems in the literature. What is more, a result from [13] shows that for some critical systems, the construction of "universal" stabilizers capable of stabilizing *every* system trajectory—including equilibria—is a hopeless goal when one does not have *a priori* information on the nature of the trajectory being stabilized. In this context, the transverse function approach emerged [19,20] as an attempt to tackle point stabilization and trajectory tracking problems in a unified setting and, at the same time, to circumvent the limitations associated with the existing solutions. Essentially, the approach is based on two premises, the first of which is the relaxation of the control objective from convergence to the desired point to merely convergence to a given *neighborhood* of that point, while the second consists in the adjunction of an auxiliary system whose state is compared with that of the target system via the use of a "transverse function." For detailed discussions on the transverse function approach to control, its underlying principles and its implementation, the reader may wish to consult e.g. [19–21]. In this paper, however, we shall focus on a slightly more conceptual aspect of so-called "transverse functions," one that has to do with their *transversality* in the differential-topological sense of the word. To be more precise, let us recall the context in which Morin and Samson [21] define the notion of function transverse to a *distribution* (or to a *set of vector fields*). Let Q be a (real, smooth) n-dimensional manifold and consider a set $\boldsymbol{X} = \{X_1, \dots, X_m\} \subset \Gamma(TQ)$, $m \leq n$, of smooth vector fields on Q. The elements of $\boldsymbol{X}$ locally span a distribution $D_x = \operatorname{span}\{X_1(x), \dots, X_m(x)\}$ and determine a driftless control system (sometimes called a "nonholonomic system") for which a point-stabilization or trajectory tracking problem is to be solved, namely,

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x). \tag{1}$$

The theory underlying the transverse function approach relies on functions transverse to the distribution D, and on a result characterizing the existence of such functions. Specifically, given a smooth manifold M, a smooth mapping $f : M \longrightarrow Q$ is said to be ***transverse to*** D (or ***transverse to*** X) if, for every $\theta \in M$,

$$T_{f(\theta)}Q = Tf(T_\theta M) + D_{f(\theta)}. \tag{2}$$

Morin and Samson originally introduced transverse functions defined on κ-dimensional torii ($\kappa \geq n-m$) and taking values in (arbitrarily small subsets of) $\mathbb{R}^n$, which corresponds to setting $M = \mathbb{T}^\kappa$ and $Q = \mathbb{R}^n$ in the above definition. Then they characterized, in a constructive way via homogeneous approximations, the existence of those functions in terms of the non-integrability of D:

Theorem 1 [19] *Let D be a constant-rank distributions on a neighborhood of a point $x \in Q$. Then $\mathrm{Lie}(D)(x) = T_xQ$ if, and only if, there exists $\kappa \geq n-m$ such that, for every neighborhood U of x, there exists a function $f : \mathbb{T}^\kappa \longrightarrow U$ transverse to D.*

In the above statement, $\mathrm{Lie}(D)(x)$ is the *involutive closure* of D at x, that is, the subspace of T_xQ spanned by all D-valued vector fields and their iterated Lie brackets evaluated at x. Paraphrased in the language of control theory, this theorem states that the *local accessibility* of (1) at x (as evidenced by the *complete nonintegrability* of D near x), is equivalent to the existence of a function $f : \mathbb{T}^\kappa \longrightarrow Q$, whose image is contained in an arbitrarily small neighborhood U of x, such that the image of $T_\theta\mathbb{T}^\kappa$ by the tangent mapping Tf is supplementary to D at $f(\theta)$. Under additional assumptions, $Tf(T_\theta\mathbb{T}^\kappa)$ may actually be *complementary* to $D_{f(\theta)}$ in the sense that the sum in the right-hand-side of (2) is direct.

For every $\theta \in \mathbb{T}^\kappa$, $Tf(T_\theta\mathbb{T}^\kappa)$ adds "extra" dimensions to the control distribution $D_{f(\theta)}$ so as to span the whole tangent space $T_{f(\theta)}Q$. The latter is a particularly useful property, exploited in [20–22,16] and other references for the construction of feedback laws. Interestingly, the control solutions developed on the basis of transverse functions imply the use of *variable-frequency oscillators*, hence the suitability of multidimensional torii as state spaces for those oscillators. More recently, however, it has been observed that depending on the configuration manifold and the dynamic structure of the system under control, alternative domains of definition for the transverse function may lead to simpler computations or to farther-reaching conclusions; e.g. [10,23], where transverse functions are defined on special orthogonal groups. This naturally raises the question as to how general these domains should be. Clearly, if one wishes to preserve salient features of the transverse function approach to control, including its ability to ensure practical stability of trajectories, one may typically be led to considering compact manifolds (without boundary). Perhaps another feature one may wish to preserve are the nice algebraic properties of torii derived from their Lie group structure, in which case one might opt for compact Lie groups. By contrast, as it shall become apparent below, from a mathematical standpoint nothing prevents one from defining transversality of mappings on general manifolds. Hence, a definition is adopted in this paper for which the domain of a transverse function is a general manifold and, only when it becomes relevant to obtain more particular results, attention shall be focused on *compact* manifolds.

Now, as observed in [19, Rmk. 1], the condition in (2) is *reminiscent of* a condition occurring in the classical definition of *transversality* (see e.g. [8, § 3.2]). Indeed, given smooth manifolds M and N, a subset $K \subset M$, and a submanifold $S \subset N$, a class C^1 mapping $f : M \longrightarrow N$ is said to be ***transverse to*** S ***along*** K (denoted $f \pitchfork_K S$) if, for every $\theta \in K$,

$$f(\theta) \in S \quad \Longrightarrow \quad T_{f(\theta)}N = Tf(T_\theta M) + T_{f(\theta)}S. \tag{3}$$

When $K = M$, one simply says that f ***is transverse to*** S and usually writes $f \pitchfork S$. It is worth pointing out, in the above definitions, the different nature of the objects to which the function f is said to be transverse, namely a *distribution* on Q as opposed to a *submanifold* of N. When the need arises to emphasize this distinction, the term ***Morin–Samson function for*** D (or ***for*** X) shall be employed to refer to a function f that satisfies the first "transversality" definition, namely that (2) holds for every $\theta \in M$.

In spite of their distinct nature, condition (2) is clearly reminiscent of (3) and, in fact, their similarity motivated the nomenclature from [19], whereby f is qualified of being "transverse to $\boldsymbol{X}$." One naturally wonders, however, about the essence of that similarity and whether transversality of f to $\boldsymbol{X}$ may be defined equivalently in terms of transversality of f to some submanifold $S \subset N$. For example, given M, N, D and $f : M \longrightarrow N$ as above, one might be tempted to say that f is a Morin–Samson function for $\boldsymbol{X}$ if and only if there exists a submanifold $S \subset N$, with $f(M) \subset S$ and $T_{f(\theta)}S = D_{f(\theta)}$ for every $\theta \in M$[1], such that $f \pitchfork S$. To see that the latter formulation could not hold in general, consider a nonintegrable distribution $D \subset T\mathbb{R}^3$ of constant rank 2, and a function $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ transverse to D, the existence of which is guaranteed by Theorem 1. If $S \subset \mathbb{R}^3$ were a submanifold such that $f(\mathbb{T}) \subset S$ and $T_{f(\theta)}S \subset D_{f(\theta)}$ for every $\theta \in M$, then $Tf(T_\theta\mathbb{T}) \subset T_{f(\theta)}S \subset D_{f(\theta)}$. One would then have $T_{f(\theta)}\mathbb{R}^3 = Tf(T_\theta\mathbb{T}) + D_{f(\theta)} \subset D_{f(\theta)}$ for $\theta \in M$, a contradiction. This highlights the difficulties that occur when one tries to characterize Morin–Samson functions for D in terms of transversality of f to a submanifold of Q, and *vice versa*. In fact, to the extent of our knowledge it was not clear, thus far, how the two transversality notions could be rigorously related. In this paper we address this issue and show that, rather than focusing on f and its transversality (both in the sense of being transverse "to $\boldsymbol{X}$" and "to a submanifold whose tangent space is contained in D along the image of f"), one should simultaneously consider **(i)** transversality of f to $\boldsymbol{X}$ and **(ii)** transversality of its associated *tangent* mapping Tf to D—the latter regarded as a *submanifold of* TQ. That conditions (i) and (ii) are indeed equivalent is, roughly stated, the content of one of the main results below. This result is then used to obtain, as a corollary of the transversality of Tf to D when M is compact, a proof of the openness, in the strong (or Whitney C^∞-) topology on $C^\infty(M,Q)$, of the set of functions transverse to X in the sense of (2).

The paper is organized as follows. A number of technical notions necessary for the ensuing developments are recalled in Section 2. In Section3, a link between transversality and transverse functions is established and the first main result of the paper is stated and proved. In Section 4, the study of the genericness of Morin–Samson functions is studied and its openness within a space of smooth mappings endowed with a particular topology is established along with some auxiliary results. Concluding remarks and lines of possible future research are presented in Section 5. Finally, elementary notions about vector bundles and the weak and strong topologies are briefly recalled in the Appendix, which also contains some technical lemmas.

2 Preliminary recalls

Throughout the paper, unless otherwise specified, *manifold* refers to a real, paracompact, boundaryless, finite-dimensional, connected manifold of class C^∞. The term *submanifold* refers to an *embedded submanifold*. Mappings and distributions are assumed to be *smooth*

[1] Observe that the condition "$T_{f(\theta)}S = D_{f(\theta)}$ for every $\theta \in M$" is weaker than (local) integrability of D; indeed, the tangent space to S at points away from the image of f is not required to coincide with D.

(i.e., of class C^∞) unless explicitly indicated. Throughout, the ground field is $\mathbb{R}$, so the adjective "real" shall be omitted systematically. If Q is a manifold, $\pi_Q : TQ \longrightarrow Q$ denotes its tangent bundle and $\Gamma(TQ)$ the set of smooth vector fields on Q. If U is a submanifold of Q, TU denotes the subbundle of TQ with underlying set $\pi_Q^{-1}(U)$, and $\Gamma(TU)$ the set of vector fields defined on U. On $\mathbb{R}^n$ one has ***canonical coordinates*** $r = (r_1, \ldots, r_n)$, also denoted (r_i), defined by $r_i(x) = x_i$ $(i = 1, \ldots, n)$. The remaining sections are largely based on notions from vector bundle theory. The reader who wishes to recall basic definitions of vector bundles may consult Appendix 6.1 or, for more detailed discussions, standard references such as [30, 8, 33].

2.1 Distributions

Let Q be a manifold. A ***distribution D on Q*** is a disjoint union of the form $\bigsqcup_{q\in Q} D_q$, where D_q is a vector subspace of T_qQ for every $q \in Q$. The distribution is said to be *smooth* if, for every $p \in Q$, there is a neighborhood U of p and a set of vector fields $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TU)$ such that $D_q = \operatorname{span}\{X_1(q), \ldots, X_m(q)\}$ for every $q \in U$. In such case, $\boldsymbol{X}$ is called a (local) ***frame for D on U***, and one says that ***D is of rank m on U***. If $\dim(D_q) = m$ for every $q \in Q$, D is said to have constant rank equal to m, and in that case one writes $\operatorname{rank}(D) = m$. As shown in the following lemma, a constant-rank distribution $D \subset TQ$ admits a structure of a closed submanifold of TQ and a vector subbundle of $\pi_Q : TQ \longrightarrow Q$.

Lemma 1 *A constant-rank distribution D on Q is a vector subbundle of (TQ, π_q, Q) with base space Q, closed as a submanifold of TQ*

Proof That D is a submanifold of TQ and a vector subbundle of $\pi_Q : TQ \longrightarrow Q$ with base space Q follows from [30, Prop. 2.1.18]. Let $m = \operatorname{rank}(D)$. If $m = n$, then $D = TQ$, and hence D is closed. Now assume that $m < n$, let $v \in TQ \setminus D$, and set $p = \pi_Q(v)$. Since D is smooth, there exists an open neighborhood $U \subset Q$ of p and a local frame $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TU)$ for D on U. By virtue of Prop. 2.1.17 in [30], there exists a neighborhood W of p and vector fields $X_{m+1}, \ldots, X_n$ in $\Gamma(TW)$ such that $\{X_1(q), \ldots, X_n(q)\}$ is a basis of T_qQ for every $q \in W$. As a result, there exist vector bundle coordinates $(x_i, u_i) : \pi_Q^{-1}(W) \longrightarrow W \times \mathbb{R}^n$, where the u_i are dual to the $X_j|_{\pi_Q^{-1}(W)}$ in the sense that $u_i(X_j(q)) = \delta_{ij}$ for every $q \in W$ and all $i, j \in \{1, \ldots, n\}$. Then, $w \in \pi_Q^{-1}(W) \cap D$ if, and only if, $(x_i(w), u_i(w)) \in x(W) \times \mathbb{R}^n$ and $u_{m+1}(w) = \cdots = u_n(w) = 0$. In particular, $u_j(v) \neq 0$ for some $j \in \{m+1, \ldots, n\}$ since $v \in \pi_Q^{-1}(W) \setminus D$. Thus, if $\varepsilon \in (0, u_j(v))$ then every point in B, the open ball of radius ε centered on $(x_i(w), u_i(w))$, has nonzero jth component. Since (x_i, u_i) is a homeomorphism, the image of B by $(x_i, u_i)^{-1}$ is a neighborhood of v contained in $TQ \setminus D$, proving that the complement of D is open. Therefore, D is closed. $\square$

2.2 Vertical and transverse subbundles, vertical lift

Let $\pi : E \longrightarrow Q$ be a vector bundle. At every point $u \in E$, the kernel of $T_u\pi$ is a subspace of T_uE, called the ***vertical space over*** u, and denoted by $V_u(TE)$. The union of the $V_u(TE)$, as u ranges over E, admits a structure that turns it into a vector subbundle $V(TE)$, called the ***vertical subbundle*** of $\pi_E : TE \longrightarrow E$. A section of $V(TE)$ is called a ***vertical vector field*** on E. Given a mapping $f : M \longrightarrow Q$, the ***pullback of E by f***, denoted $f^*(E)$, is a vector bundle over M with total space given by the fibered product $E \times_Q M := \{(u, m) \in E \times M : \pi(u) =$

$f(m)\}$, projection $(u,m) \mapsto u$, and vector bundle structure induced naturally by the given data (cf. [30, Ch. 1]). In particular, one may consider $\pi^*(TQ)$, the pullback of $\pi_Q : TQ \longrightarrow Q$ by $\pi : E \longrightarrow Q$, and define a mapping $\tau : TE \longrightarrow \pi^*(TQ)$ by $\tau(a) = (T\pi(a), \pi_E(a))$. One readily checks that τ is surjective, hence one has a short exact sequence of morphisms over the identity id_E:

$$0 \longrightarrow V(TE) \xrightarrow{i} TE \xrightarrow{\tau} \pi^*(TQ) \longrightarrow 0 \, , \tag{4}$$

where $i : V(TE) \longrightarrow TE$ denotes the inclusion and "0" the zero vector bundle over E. The paracompactness of Q and [8, Thm. 2.2] guarantee that the sequence splits, so there is a section $j : \pi^*(TQ) \longrightarrow TE$, called an ***Ehresmann connection on*** E, which allows one to define a ***horizontal subbundle*** $H(TE) := j(\pi^*(TQ))$ such that $TE = V(TE) \oplus H(TE)$. Since, in that case, $H(TE)$ is diffeomorphic with $\pi^*(TQ)$, the latter may be thought of as an abstract complement to $V(TE)$. For this reason, $\pi^*(TQ)$ is referred to in this context as the ***transverse bundle***. It is important to stress, however, that in general *the sequence in (4) does not split canonically*, so the specification of an Ehresmann connection, or equivalently, the identification of $\pi^*(TQ)$ with a particular horizontal subbundle $H(TE)$, defines additional structure on $\pi : E \longrightarrow Q$.

Given $q \in Q$ and $u \in E_q$, E_q is canonically isomorphic with $V_u(TE)$ via the "vertical lift" map $\Lambda^E_u : E_q \longrightarrow V_u(TE)$. Recall that if $\{e_i\}$ is a basis of E_q with dual basis $\{x_i\}$, then, for every $u \in E_q$, there is a canonical isomorphism $E_q \longrightarrow T_u(E_q)$ given by $\phi : \sum_i v_i e_i \mapsto \sum_i v_i \partial/\partial x_i|_u$. For $u, v \in E_q$, define the curve $\gamma_{u,v} : \mathbb{R} \longrightarrow E_q : t \mapsto u + tv$ and let $\lambda(u,v)$ be its derivative at $t = 0$. Under these conditions one has

$$\lambda(u,v) = \left.\frac{d}{dt}\right|_0 (\gamma_{u,v}(t)) = \phi\left(\lim_{t\to 0} \frac{\gamma_{u,v}(t) - \gamma_{u,v}(0)}{t}\right) = \sum_i v_i \left.\frac{\partial}{\partial x_i}\right|_u ,$$

so $\lambda(u,\cdot) : E_q \longrightarrow T_u(E_q)$ obviously is an isomorphism. Denoting by $\iota_q : E_q \hookrightarrow E$ the inclusion, its tangent mapping at u is, as stated in [30, Lem. 3.1.2], an isomorphism $T_u\iota_q : T_u(E_q) \longrightarrow V_u(TE)$. Thus the composition $\Lambda^E_u = T_u\iota_q \circ \lambda(u,\cdot)$ is an isomorphism of E_q onto $V_u(TE)$. $\Lambda^E_u(v)$ is called the ***vertical lift of*** v ***by*** u.

The following lemma states well known facts, namely, that TE admits a canonical splitting into vertical and horizontal subbundles *along its zero section*, that tangent maps of morphisms respect that splitting, and that vertical lift maps commute with morphisms. The proof of the lemma is brief and is included for completeness.

Lemma 2 *Let $\pi : E \longrightarrow Q$ be a vector bundle and let $z \in \Gamma(E)$ be its zero section.* ***(i)*** *The space T_0E, tangent to E at any point in $Z(E) := z(Q)$, admits a canonical splitting $T_0E = V_0(TE) \oplus H_0(TE)$, with $H_0(TE) = Tz(T_{\pi(0)}Q)$.* ***(ii)*** *If, in addition, $\rho : F \longrightarrow R$ is a vector bundle and $f : E \longrightarrow F$ is a morphism, then*

(a) $Tf|_{Z(E)}$ preserves the splitting of (i), i.e., $T_0f(V_0(TE)) \subset V_0(TF)$ and $T_0f(H_0(TE)) \subset H_0(TF)$.
(b) For every $q \in Q$ and all $v, w \in E_q$, $Tf \circ \Lambda^E_v(w) = \Lambda^F_{f(v)} \circ f(w)$, i.e., the following diagram (with the obvious restrictions for f and Tf) commutes:

$$\begin{tikzcd}
V_v(TE) \arrow[r, "Tf"] & V_{f(v)}(TF) \\
E_q \arrow[u, "\Lambda^E_v"] \arrow[r, "f"] & F_{\rho\circ f(v)} \arrow[u, "\Lambda^F_{f(v)}"']
\end{tikzcd}$$

Proof **(i)** One has $\pi \circ z = \mathrm{id}_Q$, so $T\pi \circ Tz$ is a monomorphism and the dimensions of $V_0(TE)$ and $H_0(TE)$ are both equal to $\dim(Q)$. Now, if $v \in V_0(TE) \cap H_0(TE)$ then, by definition of $V_0(TE)$, $T\pi(v) = 0$ whereas $v = Tz(w)$ for some $w \in T_{\pi(0)}Q$. It follows that $0 = T\pi(v) = T\pi \circ Tz(w)$, and hence, $w = 0$ and $v = 0$. Therefore $\dim(V_0(TE) + H_0(TE)) = 2\dim(N) = \dim(T_0E)$, as required. **(ii)(a)** f is a morphism over some base mapping, say $\hat{f} : Q \longrightarrow R$, and satisfies $\rho \circ f = \hat{f} \circ \pi$, so $T\rho(Tf(V(TE))) = T\hat{f} \circ T\pi(V(TE)) = \{0\}$. Hence, $Tf(V(TE)) \subset V(TF)$ and, in particular, $T_0f(V_0(TE)) \subset V_0(TF)$. Now, since f is linear on every fiber, $f \circ z = 0$, so $f \circ z(Q) \subset \hat{z}(R)$, with $\hat{z} \in \Gamma(F)$ the zero section of F. Therefore, $T_0f(Tz(T_{\pi(0)}Q)) = T_0(f \circ z)(T_{\pi(0)}Q) \subset T_0(\hat{z}(R)) = T\hat{z}(T_{\rho(0)}R)$ since $\hat{z}$ is an embedding. Thus, $T_0f(H_0(TE)) \subset H_0(TF)$. **(ii)(b)** Let $q \in Q$, let $v, w \in E_q$ and define $\gamma_{v,w} : \mathbb{R} \longrightarrow E : t \mapsto v + tw$ so that $\Lambda_v^E(w) = T_0\gamma_{v,w}(\partial/\partial r|_0)$. Since f is linear on every fiber, $f \circ \gamma_{v,w}(t) = f(v) + tf(w) = \gamma_{f(v),f(w)}(t)$. Therefore, $Tf \circ \Lambda_v^E(w) = T_0(f \circ \gamma_{v,w})(\partial/\partial r|_0) = T_0\gamma_{f(v),f(w)}(\partial/\partial r|_0) = \Lambda_{f(v)}^F(f(w))$. $\square$

A consequence of the above lemma is that if $(D, \pi|_D, \pi(D))$ is a vector subbundle of $\pi : E \longrightarrow Q$, then for every $q \in Q$ and every $v \in D_q$ one has $\Lambda_v^D = \Lambda_v^E|_{D_q}$. Indeed, if $i : D \hookrightarrow E$ denotes the inclusion, so that $\pi|_D = \pi \circ i$, then $T(\pi|_D) = T\pi \circ Ti$, so $\ker T(\pi|_D) \subset \ker T\pi$ and hence $V(TD) \subset V(TE)$. Now, setting $E = D$, $F = E$, $f = i$ and $\rho = \pi$ in Lemma 2**(ii)(b)**, one gets $Ti \circ \Lambda_v^D = \Lambda_v^E \circ i|_{D_q} = \Lambda_v^E|_{D_q}$. But $\Lambda_v^D(D_q) \subset V_v(TD)$ and $Ti|_{TD} = \mathrm{id}_{TD}$, hence $\Lambda_v^D = \Lambda_v^E|_{D_q}$. Therefore, the *vertical subbundle of TD at v exactly equals the image of D_q by the vertical map on E*:

$$V_v(TD) = \Lambda_v^E(D_q). \tag{5}$$

For notational simplicity, the super-index (e.g. "E" in Λ^E) shall sometimes be omitted when the domain of the vertical lift map is clear from the context.

3 Transversality and transverse functions

As anticipated above, the condition that $f : M \longrightarrow Q$ be a Morin–Samson function for D may be expressed in terms of transversality of its tangent mapping Tf to $D \subset TQ$. This means that one should focus on a function defined "one tangent level higher" and check whether it is transverse to the *same distribution D* but regarded *as a submanifold of TQ*. In an attempt to address the solution of control problems for systems that are not kinematically reducible, or more generally, for "second-order systems," a particular avenue in the study of *tangent mappings* of Morin–Samson functions was initiated in [15] and further pursued in [16]. Indeed, it was shown in [16, Prop. 1] that the tangent mapping of a function f transverse to a distribution D exhibits a property referred to in that reference as *vertical transversality* with respect to the *vertical lift* of D. As it will become apparent in the proof of Theorem 2, vertical transversality may be thought of as guaranteeing "one half" of the transversality of Tf to D.

The following proposition extends [16, Prop. 1] by proving that its converse also holds: if a bundle mapping $F : TM \longrightarrow TQ$ is vertically transverse to the vertical lift of D, and equals the tangent of a function $f : M \longrightarrow Q$, then f is transverse to D. It also states that it suffices to consider vertical transversality of Tf *along the zero section* $Z(TM)$. The bridge between the notations in this paper and in reference [16] is as follows. Suppose that $\{X_1, \ldots, X_m\} \subset \Gamma(TU)$ is a local frame for D on an open set U. Given $v \in \pi_Q^{-1}(U) \subset TQ$, let $q = \pi_Q(v) \in U$ and define *vertical vector fields* in $\Gamma(TTQ)$ whose values at v are given by

$X^{\text{lift}}_{i,v} := \Lambda^{TQ}_v(X_i(q))$ $(i = 1,\dots,m)$. By linearity of Λ^{TQ}_v, $\operatorname{span}\{X^{\text{lift}}_{1,v},\dots,X^{\text{lift}}_{m,v}\} = \Lambda^{TQ}_v(D_q)$. From (5) it follows that

$$\operatorname{span}\{X^{\text{lift}}_{1,v},\dots,X^{\text{lift}}_{m,v}\} = V_v(TD).$$

The expression on the left-hand member reflects the notation in [16, Prop. 1]; the one on the opposite member corresponds to the notation in the following proposition.

Proposition 1 *Let M and Q be manifolds, let $f : M \longrightarrow Q$ be a mapping and let D be a distribution on Q. The following conditions are equivalent*

(i) For every $\theta \in M$, (2) holds true.
(ii) For every $\omega \in TM$,

$$V_{Tf(\omega)}(TTQ) = TTf(V_\omega(TTM)) + V_{Tf(\omega)}(TD). \tag{6}$$

(iii) For every $\omega \in Z(TM)$, (6) holds true.

Proof **((i) $\Rightarrow$ (ii))**: Assume that (i) holds. Let $\omega \in TM$, $\theta = \pi_M(\omega)$ and $a \in V_{Tf(\omega)}(TTQ)$. Set $\hat{a} = \Lambda^{-1}_{Tf(\omega)}(a) \in T_{f(\theta)}Q$ so that, by (2), $\hat{a} = Tf(\hat{\omega}) + v$, with $\hat{\omega} \in T_\theta M$ and $v \in D_{f(\theta)}$. Thus $a = \Lambda_{Tf(\omega)}(Tf(\hat{\omega})+v) = TTf(\Lambda_\omega(\hat{\omega})) + \Lambda_{Tf(\omega)}(v)$, where we have used the linearity of $\Lambda_{Tf(\omega)}$ and Lemma 2**(ii)(b)**. Since $\Lambda_\omega(\hat{\omega}) \in V_\omega(TTM)$ and $\Lambda_{Tf(\omega)}(v) = \Lambda^D_{Tf(\omega)}(v) \in V_{Tf(\omega)}(TD)$, then (ii) holds.
((ii) $\Rightarrow$ (iii)): This implication is trivial since $Z(TM) \subset TM$.
((iii) $\Rightarrow$ (i)): Assume that (iii) holds and let $\theta \in M$, $\omega = z(\theta) \in T_\theta M$ and $v \in T_{f(\theta)}Q$. One has $\Lambda_{Tf(\omega)}(v) \in V_{Tf(\omega)}(TTQ)$, so there exist $\alpha \in V_\omega(TTM)$ and $a \in V_{Tf(\omega)}(TD)$ such that $\Lambda_{Tf(\omega)}(v) = TTf(\alpha) + a$. Since Λ^E_v is an isomorphism for every vector bundle E and every $v \in E$, there exist elements $\hat{\omega} \in T_\theta M$ and $\hat{v} \in D_{f(\theta)}$ such that $\alpha = \Lambda_\omega(\hat{\omega})$ and $a = \Lambda_{Tf(\omega)}(\hat{v})$. One has

$$\begin{aligned}
\Lambda_{Tf(\omega)}(v) &= TTf(\alpha) + a \\
&= TTf(\Lambda_\omega(\hat{\omega})) + \Lambda_{Tf(\omega)}(\hat{v}) \\
&= \Lambda_{Tf(\omega)}(Tf(\hat{\omega})) + \Lambda_{Tf(\omega)}(\hat{v}), \quad \text{(by Lemma 2}\textbf{(ii)(b)}\text{)},
\end{aligned}$$

hence, applying $\Lambda^{-1}_{Tf(\omega)}$ to both members, one gets $v = Tf(\hat{\omega}) + \hat{v}$, with $\hat{\omega} \in T_\theta M$ and $\hat{v} \in D_{f(\theta)}$. Therefore (2) holds for every $\theta \in M$, as was to be shown. □

The following, the main result in this section, establishes that f is a Morin–Samson function for D if, and only if, $Tf \pitchfork D$.

Theorem 2 *Let Q be a manifold, let D be a distribution on Q and let $f : M \longrightarrow Q$ be a mapping. Then (2) holds for every $\theta \in M$ if, and only if, $Tf \pitchfork D$.*

Proof By definition, $Tf : TM \longrightarrow TQ$ is transverse to $D \subset TQ$ if, for every $\omega \in TM$, one has

$$Tf(\omega) \in D \quad \Longrightarrow \quad T_{Tf(\omega)}TQ = T_\omega Tf(T_\omega TM) + T_{Tf(\omega)}D. \tag{7}$$

Let us first prove that if (2) holds for every $\theta \in M$, then Tf is transverse to D. Let $\omega \in TM$ and assume that $Tf(\omega) \in D$. Obviously, if suffices to prove that

$$T_{Tf(\omega)}TQ \subset T_\omega Tf(T_\omega TM) + T_{Tf(\omega)}D. \tag{8}$$

To that effect, consider the short exact sequence

$$0 \longrightarrow V(TTQ) \xrightarrow{i} TTQ \xrightarrow{\tau} \pi_Q^*(TQ) \longrightarrow 0\ , \tag{9}$$

where $i : V(TTQ) \hookrightarrow TTQ$ is the canonical inclusion of the vertical subbundle, $\pi_Q^*(TQ)$ is the transverse bundle, and $\tau : a \mapsto (T\pi_Q(a), \pi_{TQ}(a))$. By [8, Thm. 2.2] and the assumption that Q is paracompact, the sequence splits, so there exist a section $j : \pi_Q^*(TQ) \longrightarrow TTQ$ of τ (an Ehresmann connection on TQ) and a retraction $\sigma : TTQ \longrightarrow V(TTQ)$ of i, such that $V(TTQ) \oplus \pi_Q^*(TQ)$ is diffeomorphic with TTQ via $(\alpha, \beta) \mapsto (i(\alpha), j(\beta))$. Thus, one has projection morphisms $\mathscr{V} = i \circ \sigma$ and $\mathscr{H} = j \circ \tau$, and every element $a \in TTQ$ writes, in a unique way, as $a = \mathscr{V}(a) + \mathscr{H}(a)$. One immediately checks that $V(TTQ) = \mathscr{V}(TTQ)$ and $V(TD) = \mathscr{V}(TD)$.

Let $a \in T_{Tf(\omega)}TQ$. Denoting by $\iota : D \hookrightarrow TQ$ the inclusion, the projection $\pi_Q|_D = \pi_Q \circ \iota$ is obviously surjective and so is $T(\pi_Q|_D)$, thus the mapping $\tau|_{TD} : \xi \mapsto (T(\pi_Q|_D)(\xi), \pi_{TQ}(\xi))$ is an epimorphism of TD onto $\pi_Q^*(TQ)$. Therefore, since $\tau(a) \in \pi_Q^*(TQ)$, there exists $\xi \in T_{Tf(\omega)}D$ such that $\tau(\xi) = \tau(a)$. Consequently, $\mathscr{H}(\xi) = j \circ \tau(\xi) = j \circ \tau(a) = \mathscr{H}(a)$.

Note that $\mathscr{V}(a) - \mathscr{V}(\xi) = \mathscr{V}(a - \xi) \in V_{Tf(\omega)}(TTQ)$. However, according to Proposition 1 one has

$$V_{Tf(\omega)}(TTQ) = TTf(V_\omega(TTM)) + V_{Tf(\omega)}(TD),$$

hence there exist $\alpha \in V_\omega(TTM)$ and $\zeta \in V_{Tf(\omega)}(TD)$ such that $TTf(\alpha) + \zeta = \mathscr{V}(a) - \mathscr{V}(\xi)$. Thus,

$$a = \mathscr{V}(a) + \mathscr{H}(\xi) = \mathscr{V}(a) + \mathscr{H}(\xi) + \mathscr{V}(\xi) - \mathscr{V}(\xi) = TTf(\alpha) + \zeta + \xi.$$

Since $\alpha \in T_\omega TM$ and $\zeta + \xi \in T_{Tf(\omega)}D$, it follows that (8) holds, so Tf is transverse to D.

Now assume that Tf is transverse to D, so that (7) holds for every $\omega \in TM$, and let $\theta \in M$. Taking, in particular, $\omega = 0 \in T_\theta M$ and recalling that $D_{f(\theta)}$ is a vector subspace of $T_{f(\theta)}Q$, we see that $Tf(\omega) = 0 \in Tf(T_\theta M) \cap D_{f(\theta)}$, hence $Tf(\omega) \in D$. Consider the splitting $T_0TQ = V_0(TTQ) \oplus H_0(TTQ)$ described in Lemma 2-**(i)**, and define the corresponding projections $\mathscr{V} : T_0TQ \longrightarrow V_0(TTQ)$ and $\mathscr{H} : T_0TQ \longrightarrow H_0(TTQ)$. We claim that

$$TTf(V_0(TTM)) = \mathscr{V}(TTf(T_0TM)). \tag{10}$$

Indeed, if $a \in TTf(V_0(TTM))$, then there is $\alpha \in V_0(TTM) \subset T_0TM$ such that $a = TTf(\alpha)$. Now, in view of Lemma 2**(ii)(b)**, TTf maps vertical vectors to vertical vectors, so $a \in V_0(TTQ)$ and $a = \mathscr{V}(a) = \mathscr{V}(TTf(\alpha))$. Conversely, if $a \in \mathscr{V}(TTf(T_0TM))$, then there is $\alpha \in T_0TM$ such that $a = \mathscr{V}(TTf(\alpha))$. But $TTf(\mathscr{H}(\alpha)) \subset H_0(TTQ)$ by Lemma 2**(ii)(a)**, so $\mathscr{V}(TTf(\mathscr{H}(\alpha))) = 0$ and $a = \mathscr{V}(TTf(\mathscr{V}(\alpha))) = TTf(\mathscr{V}(\alpha))$, with $\mathscr{V}(\alpha) \in V_0(TTM)$. In view of (7) and (10) one has

$$\begin{aligned} V_0(TTQ) &= \mathscr{V}(TTf(T_0TM)) + \mathscr{V}(T_0D) \\ &= TTf(V_0(TTM)) + V_0(TD). \end{aligned}$$

Together, the latter equation and Proposition 1 imply that $Tf(T_\theta M) + D_{f(\theta)} = T_{f(\theta)}Q$. Therefore, (2) holds for every $\theta \in M$. □

Remark 1 If $f : M \longrightarrow Q$ were *any* function such that $Tf(TM) \cap D = \emptyset$, one would have $Tf \pitchfork D$, so Theorem 2 would vacuously imply that f is a Morin–Samson function for D. Of course, this cannot occur since each fiber D_q, as a vector subspace of T_qQ, contains the zero vector and Tf is a morphism mapping $Z(TM)$ into $Z(TQ)$. Thus the preimage of D by Tf necessarily contains the zero section, $Z(TM) \subset Tf^{-1}(D)$. More can be said about this preimage, for a fundamental result based on transversality states that if $f : M \longrightarrow N$ is transverse to a submanifold $S \subset N$, then $f^{-1}(S) \subset M$ is a submanifold whose codimension equals that of S (cf. e.g. [5, Thm. 2.4.4]). Hence, if f is a Morin–Samson function for D then $Tf \pitchfork D$, so $Tf^{-1}(D) \subset TM$ is a submanifold with $\mathrm{codim}(Tf^{-1}(D)) = \mathrm{codim}(D)$. By construction $\dim(Tf^{-1}(D)) = 2\dim(M) - \dim(Q) + \mathrm{rank}(D)$. In particular, if $\dim(M) = \mathrm{corank}(D)$, then $\dim(Tf^{-1}(D)) = \dim(M)$. As a matter of fact, in this case the preimage $Tf^{-1}(D)$ *equals the zero section* $Z(TM)$ since $\dim(Z(TM)) = \dim(M)$ and, by virtue of (2), for every $\omega \in TM$, $Tf(\omega) \in D$ implies $\omega = 0$, i.e., $\omega \in Z(TM)$. In more general cases, however, the latter implication need not hold and the zero section $Z(TM)$ may be *properly* contained in the submanifold $Tf^{-1}(D)$.

Remark 2 Inspection of the second part of the proof of Theorem 2 reveals that, via the assumption $\omega = 0$, only elements $\omega \in Z(TM)$ were actually considered. Therefore, that argument shows that if $Tf \pitchfork_{Z(TM)} D$, i.e., if (7) holds for every $\omega \in Z(TM)$, then (2) is true for every $\theta \in M$. The following corollary is an immediate consequence of this observation and of Theorem 2.

Corollary 1 *(2) holds for every $\theta \in M \Longleftrightarrow Tf \pitchfork D \Longleftrightarrow Tf \pitchfork_{Z(TM)} D$.*

The following example illustrates the previous two results.

Example 1 On $Q = \mathbb{R}^3$, consider canonical coordinates $q = (q_1, q_2, q_3)$ and the following distribution (where column vector notation is used in this example for conciseness):

$$D_q = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ q_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -q_1 \end{pmatrix} \right\}.$$

A parametrization $\phi : Q \times \mathbb{R}^2 \longrightarrow TQ$ of D as a submanifold of TQ is given by $(q,d) \mapsto d_1(\partial/\partial q_1 + q_2 \partial/\partial q_3)|_q + d_2(\partial/\partial q_2 - q_1 \partial/\partial q_3)|_q$. Equivalently, D is the submanifold of TQ with underlying set

$$D = \left\{ (q_1, q_2, q_3, d_1, d_2, d_1 q_2 - d_2 q_1) : d \in \mathbb{R}^2 \right\}.$$

Hence, the tangent space to D at $v \in D$ is the following distribution on TQ of rank 5:

$$T_vD = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ q_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -q_1 \end{pmatrix} \right\} \tag{11}$$

Note that $T_vD \subset T_vTQ$. Using an angular coordinate θ on $\mathbb{T}$, along with the abbreviated notation $\mathrm{s} = \sin$ and $\mathrm{c} = \cos$, a Morin–Samson function for D is described by

$$f(\theta) = \left(\varepsilon\, \mathrm{s}(\theta),\ \varepsilon\, \mathrm{c}(\theta), \frac{1}{4}\varepsilon^2\, \mathrm{s}(2\theta) \right)$$

for any $\varepsilon > 0$. In the induced vector bundle coordinates $(\theta, \dot{\theta})$ on $T\mathbb{T}$, if $\omega = (\theta, \dot{\theta}) \in T_\theta \mathbb{T}$ then:

$$Tf(T_\theta \mathbb{T}) = \operatorname{span}\left\{ \begin{pmatrix} \varepsilon\,\mathrm{c}(\theta) \\ -\varepsilon\,\mathrm{s}(\theta) \\ \frac{1}{2}\varepsilon^2\,\mathrm{c}(2\theta) \end{pmatrix} \right\}.$$

One readily verifies that

$$\begin{aligned} Tf(T_\theta \mathbb{T}) + D_{f(\theta)} &= \operatorname{span}\left\{ \begin{pmatrix} \varepsilon\,\mathrm{c}(\theta) \\ -\varepsilon\,\mathrm{s}(\theta) \\ \frac{1}{2}\varepsilon^2\,\mathrm{c}(2\theta) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \varepsilon\,\mathrm{c}(\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\varepsilon\,\mathrm{s}(\theta) \end{pmatrix} \right\} \\ &= T_{f(\theta)}\mathbb{R}^3 \end{aligned}$$

for the determinant of the matrix formed by the column-vectors enclosed in braces equals $-\frac{1}{2}\varepsilon^2 \neq 0$. Hence, for all $\theta \in \mathbb{T}$, (2) holds and f satisfies $\operatorname{rank}(T_\theta f) = \dim(Q) - \operatorname{rank}(D) = 1$. In the induced vector bundle coordinates $(\theta, \dot{\theta}, \alpha_1, \alpha_2)$ on $TT\mathbb{T}$, the value TTf at $\alpha = (\theta, \dot{\theta}, \alpha_1, \alpha_2) \in T_\omega T\mathbb{T}$ is

$$TTf(\alpha) = \begin{pmatrix} \varepsilon \cos(\theta)\alpha_1 \\ -\varepsilon \sin(\theta)\alpha_1 \\ \frac{1}{2}\varepsilon^2 \cos(2\theta)\alpha_1 \\ -\varepsilon \sin(\theta)\dot{\theta}\alpha_1 + \varepsilon \cos(\theta)\alpha_2 \\ -\varepsilon \cos(\theta)\dot{\theta}\alpha_1 - \varepsilon \sin(\theta)\alpha_2 \\ -\varepsilon^2 \sin(2\theta)\dot{\theta}\alpha_1 + \frac{1}{2}\varepsilon^2 \cos(2\theta)\alpha_2 \end{pmatrix}$$

Let $\omega = (\theta, \dot{\theta})$ and assume that $Tf(\omega) \in D$. As observed in Remark 1, this assumption and condition (2) imply that $\omega \in Z(T\mathbb{T})$, so $\dot{\theta} = 0$. Hence

$$TTf(T_\omega T\mathbb{T}) = \operatorname{span}\left\{ \begin{pmatrix} \varepsilon \cos(\theta)\alpha_1 \\ -\varepsilon \sin(\theta)\alpha_1 \\ \frac{1}{2}\varepsilon^2 \cos(2\theta)\alpha_1 \\ \varepsilon \cos(\theta)\alpha_2 \\ -\varepsilon \sin(\theta)\alpha_2 \\ \frac{1}{2}\varepsilon^2 \cos(2\theta)\alpha_2 \end{pmatrix} : \alpha \in \mathbb{R}^2 \right\}.$$

Comparing the vector obtained by setting $\alpha = (0,1) \in \mathbb{R}^2$ in the latter expression with the description of $T_{Tf(\omega)}D$ given in (11), it is clear—again by virtue of (2)—that the dimension of $TTf(T_\omega T\mathbb{T}) + T_{Tf(\omega)}D$ equals $\dim(T_{Tf(\omega)}TQ) = 6$. Hence (7) holds for every $\omega \in TM$, so $Tf \pitchfork D$. Obviously, this also shows that $Tf \pitchfork_{Z(TM)} D$.

Remark 3 Corollary 1 implies that if f fails to be a Morin–Samson function for D because (2) does not hold for some $\theta \in TM$, then there exists ω in the zero section $Z(TM)$ such that (7) does not hold. One may wonder, however, whether the same condition implies the existence of $\omega \neq 0$ such that (7) fails, and the answer is in the negative, as shown in the following example.

Example 2 For $M = \mathbb{R}$ and $Q = \mathbb{R}^2$, let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $f(\theta) = (0, \theta^2/2)$ and define a distribution $D \subset TQ$ by $D_q = \operatorname{span}\{\partial/\partial q_1|_q\}$.

Then $T_\theta f(\dot{\theta}\partial/\partial\theta|_\theta) = \theta\dot{\theta}\partial/\partial q_2|_{f(\theta)}$. If we let $\hat{\theta} = 0$ and $\hat{\omega} = \dot{\theta}\partial/\partial\theta|_{\hat{\theta}}$, with $\dot{\theta} \neq 0$, we see that $Tf(\hat{\omega}) = 0 \in D_{f(\hat{\theta})}$ whereas, obviously, $Tf(T_{\hat{\theta}}M) + D_{f(\hat{\theta})} \subsetneq T_{f(\hat{\theta})}Q$. However, if $\alpha = \alpha_1 \partial/\partial\theta|_{\hat\omega} + \alpha_2 \partial/\partial\dot\theta|_{\hat\omega} \in T_{\hat\omega}TM$, then $TTf(\alpha) = \dot\theta\alpha_1 \partial/\partial\dot q_2|_{Tf(\hat\omega)}$. Since $\dot\theta \neq 0$ and

$$T_{Tf(\hat\omega)}D = \operatorname{span}\left\{\partial/\partial q_1|_{Tf(\hat\omega)}, \partial/\partial q_2|_{Tf(\hat\omega)}, \partial/\partial \dot q_1|_{Tf(\hat\omega)}\right\},$$

one has $TTf(T_{\hat\omega}TM) + T_{Tf(\hat\theta)}D = T_{Tf(\hat\theta)}TQ$. Therefore, (2) fails to be true for $\hat\theta \in M$, but (7) holds for every $\omega \in T_{\hat\theta}M \setminus \{0\}$.

4 Openness of sets of transverse functions

Transversality provides a unified approach to make precise sense of notions such as *genericness* and *structural stability* in a variety of contexts (see e.g. [8,6,5]). In particular, its use in differential topology and analysis has led to important results on the density and openness of different classes of mappings in function spaces equipped with diverse topologies. It is thus natural to expect transversality of tangent mappings of Morin–Samson functions to produce analogous results, and some steps in that direction are taken in this section. These results might pave the way to future, possibly deeper findings on existence and genericness of Morin–Samson functions on domains that include more general manifolds than torii.

The main result in this section has to do with structural stability of Morin–Samson functions on *compact* manifolds. Stated in more topological terms, given a compact manifold M, and a distribution D on Q, the set of functions that are transverse to D is open in the strong topology on $C^\infty(M,Q)$[2]. This is but one among several possibilities to rigorously state and prove the intuitive fact that, if a function f is transverse to a distribution D, then the mapping obtained by smoothly deforming f is still transverse to D provided the "size" of the deformation is "sufficiently small."

There are two key elements to the main result of this section. The first, Theorem 3, is a version of the classical "transversality theorem" taken from [8], a statement about the density and openness of the set of C^k mappings that are transverse *along a compact* subset of their domain to a given submanifold of their codomain:

Theorem 3 ("Transversality theorem" [8, Thm. 3.2.2.1]) *Let M, N be manifolds, let $K \subset M$ be a compact set, and let $S \subset N$ be a closed submanifold. Then the set $\{g \in C^k(M,N) : g \pitchfork_K S\}$ is dense and open in $C^k_W(M,N)$ for $1 \leq k \leq \infty$.*

In the above statement and in the sequel, the index "W," as in $C^k_W(M,N)$, indicates that the given set is equipped with the weak (or *compact-open*) topology. The second element, stated and proved below as Proposition 2, establishes that when M is compact, the tangent functor defines a continuous mapping $f \mapsto Tf$ of $C^\infty_S(M,Q)$ into $C^\infty_W(TM,TQ)$, where "S" is used to indicate that the set is equipped with the *strong* (or "Whitney C^∞") topology. Of course, for compact M, the strong and weak topologies on $C^\infty(M,Q)$ are the same, but a word may be in order about the choice of the weak topology for $C^\infty(TM,TQ)$. Indeed, since the weak topology does not control the behavior "at infinity" of smooth maps on a noncompact manifold such as TM (see e.g. [8, Chap. 2] or [5, II§3]), the usually preferred choice is the strong topology, all the more so that a number of important results—including those stating that some notable subsets of mappings are open—are based on the strong topology. Nonetheless, the control "at infinity" provided by the latter topology on $C^\infty(TM,TQ)$ comes at a

[2] The construction of the weak topology on the function space $C^k(M,N)$, which coincides with the strong topology when M is compact, is recalled in Appendix 6.2.

price: the topology is too large and the notion of convergence of a sequence in that topology is overly restrictive. In particular, one easily shows that Proposition 2 *no longer holds* if both function spaces are equipped with the *strong* topology, even for compact M. Fortunately, the version of the transversality theorem cited above is strong enough to require only the weak topology on $C^\infty(TM,TQ)$, thus enabling our proof of Proposition 2 and ultimately leading to Theorem 4, the main result in this section.

Theorem 4 *Let M be compact and let $f \in C^\infty(M,Q)$ be a Morin–Samson function for a smooth, constant-rank distribution D on Q. Then there exists a neighborhood of f in $C^\infty_S(M,Q)$ all of whose elements are Morin–Samson functions for D.*

Proof Under the stated assumptions, f satisfies (2) for every $\theta \in M$; by Theorem 2, Tf is transverse to D. In particular, $Tf \pitchfork_{Z(TM)} D$, so $Tf \in \mathscr{F} = \{F \in C^\infty(TM,TQ) : F \pitchfork_{Z(TM)} D\}$. Being the image of M by the embedding $z : M \longrightarrow TM$, $Z(TM)$ is a compact submanifold of TM. Also, by virtue of Lemma 1, D is a closed submanifold of TQ, so Theorem 3 implies that $\mathscr{F}$ is dense and open in $C^\infty_W(M,N)$. Thus $\mathscr{F}$ contains an open neighborhood $\mathscr{V}$ of Tf and, since $f \mapsto Tf$ is continuous by Proposition 2, there exists an open neighborhood $\mathscr{U}$ of f in $C^\infty_S(M,Q)$ such that $T(\mathscr{U}) \subset \mathscr{V}$. Consequently, if $g \in \mathscr{U}$, then $Tg \in \mathscr{V}$, so g is a Morin–Samson function for D. $\square$

Proposition 2 *The mapping $C^\infty_S(M,Q) \longrightarrow C^\infty_W(TM,TQ)$ defined by $f \mapsto Tf$ is continuous.*

Proof Let $f \in C^\infty(M,Q)$ and let $\mathscr{V} \subset C^\infty_W(TM,TQ)$ be open and such that $Tf \in \mathscr{V}$. Since a basic element of the topology equals the finite intersection of subbasic sets, it suffices to consider only subbasic sets. Thus one may assume that $\mathscr{V} = \mathscr{N}^1(Tf,(U,\varphi),(V,\psi),K,\varepsilon)$, where (U,φ) and (V,ψ) are charts on TM and TQ, respectively, $K \subset U$ is compact, $Tf(K) \subset V$, and $\varepsilon > 0$. By Lemma 4 (cf. the Appendix), we may further assume that (U,φ) and (V,ψ) are vector bundle charts, induced by charts $(\hat{U},\hat{\varphi})$ and $(\hat{V},\hat{\psi})$ on M and Q, respectively. Let $\hat{K} = \pi_M(K) \subset \hat{U}$. Since $f \circ \pi_M = \pi_Q \circ Tf$, we have $f(\hat{K}) = \pi_Q(Tf(K)) \subset \hat{V}$. Denote by $(q_i,\dot{q}_i)$ $(i = 1,\dots,n)$ the coordinates defined by (U,φ). Given a function $h \in C^1(M,Q)$, set $\hat{h} = \hat{\psi} \circ h \circ \hat{\varphi}^{-1}$ and let $\widetilde{Th} = \psi \circ Th \circ \varphi^{-1}$ be the representative of Th in the charts (U,φ) and (V,ψ). By definition of vector bundle chart, $\widetilde{Tf}(q,\dot{q}) = (\hat{f}_j(q), \sum_{i=1}^n \frac{\partial \hat{f}_j}{\partial q_i}(q)\dot{q}_i)$ $(j = 1,\dots,m)$. Now, since $\widetilde{Tf}(\varphi(K))$ is a compact subset of the open set $\psi(V)$, there exists $\eta > 0$ such that the open η-neighborhood of $\widetilde{Tf}(\varphi(K))$ is contained in $\psi(V)$, i.e., for every $y \in \widetilde{Tf}(\varphi(K))$ and every $z \in \mathbb{R}^{2m}$, $\|y-z\|_\infty < \eta$ implies $z \in \psi(V)$, with $\|z\|_\infty = \max\{|z_i| : i = 1,\dots,2m\}$. On the other hand, let $M = \max\{|\dot{q}_i| : (q,\dot{q}) \in \varphi(K),\ i = 1,\dots,n\}$ and $\delta = \frac{1}{n}\min\{\eta,\varepsilon,\frac{\varepsilon}{M}\}$. Assume that $g \in C^2(M,Q)$ with $\|\hat{g}-\hat{f}\|^2_{\hat{K}} < \delta$ (for the definition of $\|\cdot\|^2_{\hat{K}}$ cf. Appendix 6.2). Then, in particular, $|\hat{g}_j(q) - \hat{f}_j(q)| < \delta$, $\left|\frac{\partial \hat{g}_j}{\partial q_i}(q) - \frac{\partial \hat{f}_j}{\partial q_i}(q)\right| < \delta$ and $\left|\frac{\partial^2 \hat{g}_j}{\partial q_i \partial q_k}(q) - \frac{\partial^2 \hat{f}_j}{\partial q_i \partial q_k}(q)\right| < \delta$ for all $j = 1,\dots,m$, $i,k = 1,\dots,n$ and $q \in \hat{K}$. The expressions for the first partial derivatives of the components of $\widetilde{Tg} - \widetilde{Tf}$ with respect to q_i are

$$\left(\frac{\partial \hat{g}_j}{\partial q_i}(q) - \frac{\partial \hat{f}_j}{\partial q_i}(q),\ \sum_{l=1}^n \left(\frac{\partial^2 \hat{g}_j}{\partial q_i \partial q_l}(q) - \frac{\partial^2 \hat{f}_j}{\partial q_i \partial q_l}(q)\right)\dot{q}_l\right),$$

whereas those for the derivatives with respect to $\dot{q}_i$ are

$$\left(0,\ \frac{\partial \hat{g}_j}{\partial q_i}(q) - \frac{\partial \hat{f}_j}{\partial q_i}(q)\right)$$

($j = 1,\dots,m$; $i = 1,\dots,n$). We note that, since $|\dot{q}_i| \le M$ for all $(q,\dot{q}) \in \varphi(K)$ and all $i = 1\dots,n$,

$$\begin{aligned}\left|\sum_{l=1}^{n}\left(\frac{\partial^2 \hat{g}_j}{\partial q_i \partial q_l}(q) - \frac{\partial^2 \hat{f}_j}{\partial q_i \partial q_l}(q)\right)\dot{q}_l\right| &\le \sum_{l=1}^{n}\left|\left(\frac{\partial^2 \hat{g}_j}{\partial q_i \partial q_l}(q) - \frac{\partial^2 \hat{f}_j}{\partial q_i \partial q_l}(q)\right)\right||\dot{q}_l| \\ &< n\delta M \\ &\le \varepsilon\end{aligned}$$

for $(q,\dot{q}) \in \varphi(K)$. Thus, the above assumptions on g and δ imply that

$$\|\psi \circ Tg \circ \varphi^{-1} - \psi \circ Tf \circ \varphi^{-1}\|_{\varphi(K)} = \|\widetilde{Tg} - \widetilde{Tf}\|_{\varphi(K)} < \varepsilon.$$

Similarly one deduces that $\|\widetilde{Tg}(q,\dot{q}) - \widetilde{Tf}(q,\dot{q})\|_\infty < \eta$, so $\widetilde{Tg}(q,\dot{q}) \in \psi(V)$ for $(q,\dot{q}) \in \varphi(K)$, that is, $Tg(K) \subset V$. Therefore, if $g \in \mathscr{N}^2(f,(\hat{U},\hat{\varphi}),(\hat{V},\hat{\psi}),\hat{K},\delta)$, then $Tg \in \mathscr{V}$. Let $\mathscr{U} = \mathscr{N}^2(f,(\hat{U},\hat{\varphi}),(\hat{V},\hat{\psi}),\hat{K},\delta) \cap C^\infty(M,N)$. Since M is compact, the weak and strong topologies on $C^\infty(M,Q)$ coincide, so $\mathscr{U}$ is an open neighborhood of f in $C^\infty_S(M,Q)$. Consequently, $f \mapsto Tf$ is continuous. $\square$

5 Concluding remarks

In this paper an attempt is made to clarify the precise relationship between *transversality*, as defined in differential topology, and the functions involved in the transverse function approach to control. Although the functions originally introduced in that approach were defined on torii, here we consider more general domains, and some of our results, e.g. Proposition 1, Theorem 2 and Corollary 1, even hold for noncompact domains. By contrast, *compactness* of the domain of the transverse function is critically required in the proof of Theorem 4. An interesting problem that remains open is the characterization of the existence of Morin–Samson functions: Given manifolds M and Q, with M compact, and a distribution $D \subset TQ$, characterize the existence of functions $f : M \longrightarrow Q$ such that $Tf \pitchfork D$. In its original formulation, in which the image of the function may be made "arbitrarily small," the existence problem is of a *global–local* nature. Let us illustrate this point by focusing on the special case $\dim(M) = \operatorname{corank}(D)$ which, incidentally, corresponds to a typical scenario in many practical applications. A consequence of this assumption is that f satisfies (2) for all $\theta \in M$ only if f is an immersion, so the *global* properties of M are likely to impose conditions on the existence of f. On the other hand, the image $f(M)$ may be contained in an arbitrarily small neighborhood of a given point $q \in Q$, hence only the *local* structure of the distribution D near q is determinant. In other words, the obstructions to the existence of Morin–Samson functions in a given case might be related to the *topology* of the domain M and the *geometry* of the distribution $D \subset TQ$. A concrete example of the global–local nature alluded to before is the necessary condition expounded in Theorem 1 of [19]. Roughly speaking, this condition states that the local integrability of D near a point precludes the existence of a transverse function with image arbitrarily close to that point. A simple adaptation of the proof of that result immediately yields the following corollary.

Corollary 2 *Let M, Q be manifolds, with M compact, and let D be a constant-rank distribution on a neighborhood of a point $x \in Q$. If there exists $\kappa \ge n - m$ such that for every neighborhood U of x there is a mapping $f : M \longrightarrow U$ transverse to D, then* $\mathrm{Lie}(D)(x) = T_xQ$.

For noncompact manifolds the condition is no longer necessary: if $M = \mathbb{R}$ and D is the distribution which foliates $\mathbb{R}^3$ by "horizontal planes" $D_q = \operatorname{span}\{\partial/\partial r_1|_q, \partial/\partial r_2|_q\}$, then $f(\theta) = (0,0,\varepsilon \arctan(\theta))$ defines a Morin–Samson function for D with image arbitrarily close to 0 by an appropriate choice of $\varepsilon > 0$. Moreover, replacing M by $[0,1] \subset \mathbb{R}$ exemplifies that the corollary breaks down even for *compact* manifolds *with boundary* (as defined in e.g. [8, Chap. 1], [6, Chap. 2])[3]. Hence, topological properties of the domain, such as being compact or having a boundary, and geometric properties of the distribution, such as its local integrability, influence the existence of transverse functions. One may venture to speculate that, for the more general case, the characterization of the topological and geometric properties governing the existence of Morin–Samson functions might require the use of more specialized tools, including finer topological and geometric invariants of M and D, respectively. Finally, another difficult problem is the *explicit construction* of transverse functions defined on more general manifolds, as initially explored in [23]. It is to be hoped that transversality properties, such as those explored in this paper, contribute to solving problems in these two directions.

6 Appendix

6.1 Vector bundles, vector subbundles, morphisms

A natural way to define vector bundles is via charts, mimicking the construction of manifolds. Although the definition recalled below is that of *smooth* vector bundle, this requirement may be relaxed by systematically replacing C^∞ by C^k $(0 \le k < \infty)$. Let E, Q be manifolds, $\pi : E \longrightarrow Q$ a C^∞ mapping, and F a finite-dimensional vector space. A ***vector bundle chart*** (U, φ) on E, with domain U and typical fiber F, is given by an open set $U \subset Q$ and a homeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ such that $\pi = p_1 \circ \varphi$, where $p_1 : (x,y) \mapsto x$. For each $q \in U$, one has a homeomorphism $\varphi_q : p_2 \circ \varphi|_{\pi^{-1}(\{q\})} : \pi^{-1}(\{q\}) \longrightarrow F$, with $p_2 : (x,y) \mapsto y$, which turns $\pi^{-1}(\{q\})$ into a vector space diffeomorphic with F. Two vector bundle charts (U, φ) and (V, ϕ) are ***C^∞-compatible*** if the homeomorphism $\psi_q \circ \varphi_q^{-1} : F \longrightarrow F$ is linear for every $q \in U \cap V$ (so it belongs to $\mathrm{GL}(F)$), and the assignment $q \mapsto \psi_q \circ \varphi_q^{-1} : U \cap V \longrightarrow \mathrm{GL}(F)$ is C^∞. A ***vector bundle atlas*** $\mathscr{A}$ on E is a collection of vector bundle charts $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ such that the U_λ cover Q and every pair of charts in $\mathscr{A}$ are C^∞-compatible. A ***vector bundle structure*** on (E, π, M) is given by a vector bundle atlas which is maximal with respect to C^∞-compatibility, i.e., whenever (V, ψ) is a vector bundle chart on E, C^∞-compatible with $(U, \varphi) \in \mathscr{A}$, then $(V, \psi) \in \mathscr{A}$. The vector bundle shall be referred to as (E, π, Q), as $\pi : E \longrightarrow Q$, or simply as E when the remaining data are clear from the context. Given an n-dimensional manifold Q, the vector bundle chart (U, φ) on the tangent bundle $\pi_Q : TQ \longrightarrow Q$ ***(naturally) induced*** by a chart $(\hat{U}, \hat{\varphi})$ on Q is defined by $U = \pi_Q^{-1}(\hat{U})$ and $\varphi(v) = (\hat{\varphi} \circ \pi_Q(v), d\hat{\varphi}(v))$, where $d\hat{\varphi}$ is regarded as an $\mathbb{R}^n$-valued 1-form on U with components $(d\hat{\varphi}_1, \ldots, d\hat{\varphi}_n)$. Let $\pi : E \longrightarrow Q$ be a vector bundle. A ***section*** of E is a mapping $s : Q \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_Q$. The set of smooth sections of E is denoted by $\Gamma(E)$. The ***fiber above*** $q \in Q$ is $E_q := \pi^{-1}(\{q\})$; although its vector space structure depends on the choice of a vector bundle chart, the zero vector $0 \in E_q$ is canonically defined. The term ***zero section*** of E is used to refer to the map $z \in \Gamma(E)$ given by $z : q \mapsto 0$, or to its image $Z(E) := z(Q) \subset E$. A ***vector subbundle*** D of (E, π, Q) is a submanifold $D \subset E$ such that $(D, \pi|_D, \pi(D))$, with the vector bundle atlas obtained from the atlas of (E, π, Q)

[3] Observe, however, that *manifolds with boundary* are not manifolds in the sense understood in this paper.

by restricting the domains of its charts to D, is itself a vector bundle. Given vector bundles $\pi : E \longrightarrow Q$, $\rho : F \longrightarrow R$, and a mapping $\varphi : Q \longrightarrow R$, one says that $\Phi : E \longrightarrow F$ is a ***vector bundle morphism*** (over φ), in this paper referred to simply as *morphism (over φ)*, if $\varphi \circ \pi = \rho \circ \Phi$ and $\Phi_q := \Phi|_{E_q} : E_q \longrightarrow F_{\varphi(q)}$ is linear for every $q \in Q$. A ***monomorphism*** is an injective morphism; an ***epimorphism*** is a surjective morphism. If $\operatorname{rank}(\Phi_q)$ is constant for all $q \in Q$, Φ is said to have constant rank. For a constant-rank morphism Φ, $\ker(\Phi)$ and $\operatorname{im}(\Phi)$ are subbundles of $\pi : E \longrightarrow Q$ and $\rho : F \longrightarrow R$, with fibers $\ker(\Phi_q)$ and $\operatorname{im}(\Phi_q)$, respectively. Thus, if $\Phi : E \longrightarrow F$ and $\Psi : F \longrightarrow G$ are constant-rank morphisms, one has a short sequence of morphisms

$$E \xrightarrow{\Phi} F \xrightarrow{\Psi} G \,.$$

As usual, the short sequence is said to be ***exact*** (at F) if $\operatorname{im}(\Phi) = \ker(\Psi)$.

6.2 The weak topology on function spaces

Given manifolds M and N, the set $C^k(M,N)$ of mappings of class C^k $(0 \le k \le \infty)$ on M into N is endowed, in this paper, with either the *weak* (or "compact-open") or the *strong* (or "Whitney C^k") topology. Nonetheless, solely the construction of the weak topology is recalled below since in this work *only the weak topology* is considered in cases when M is *noncompact* and, as is well known, both topologies coincide whenever M is compact.

Following [9], given an open set $U \subset \mathbb{R}^n$, a compact set $K \subset U$, and a function $f \in C^k(U,\mathbb{R}^m)$, $1 \le k < \infty$, we define

$$\|f\|_K^k = \max\left\{ \left| \frac{\partial^{|\alpha|} f_i}{\partial r_1^{\alpha_1} \cdots \partial r_n^{\alpha_n}}(x) \right| : x \in K,\ 1 \le i \le m,\ 0 \le |\alpha| \le k \right\},$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ an n-tuple of nonnegative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The following lemma is a simple observation based on the compactness of the set K in the definition of $\|\cdot\|_K^k$, so its proof is omitted.

Lemma 3 *Fix n, m and k in $\mathbb{N}$, and suppose that $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open; let $K \subset U$ be compact and let $f \in C^k(U,\mathbb{R}^m)$ be such that $f(K) \subset V$. Suppose that $\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^n$ and $\psi : V \longrightarrow \psi(V) \subset \mathbb{R}^m$ are diffeomorphisms of class C^k. Then there exist real numbers $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_K^k \le \|\psi \circ f \circ \varphi^{-1}\|_{\varphi(K)}^k \le c_2 \|f\|_K^k.$$

Let M,N be manifolds and consider: **(a)** a function $f \in C^k(M,N)$; **(b)** charts (U,φ) and (V,ψ) on M and N, respectively; **(c)** a compact set $K \subset U$ such that $f(K) \subset V$; and **(d)** a real number $\varepsilon > 0$. Let $\mathscr{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)$ denote the set

$$\left\{ g \in C^k(M,N) : g(K) \subset V,\ \|\psi \circ f \circ \varphi^{-1} - \psi \circ g \circ \varphi^{-1}\|_{\varphi(K)}^k < \varepsilon \right\}.$$

The collection $\mathscr{S}$ of all sets of the form $\mathscr{N}^k(F,(U,\varphi),(V,\psi),K,\varepsilon)$ clearly covers $C^k(M,N)$, so it is a subbasis (cf. e.g. [24]). The topology generated by the subbasis $\mathscr{S}$ is referred to as the ***weak topology on*** $C^k(M,N)$. The ***weak topology on*** $C^\infty(M,N)$ is the union, for $k \ge 0$, of the topologies induced by the inclusions $C^\infty(M,N) \longrightarrow C^k(M,N)$.

The remainder of this appendix contains proofs of auxiliary technical lemmas. Although a tangent bundle admits very general charts from its structure as a manifold, vector bundle charts induced by charts on its base manifold are especially convenient to work with. The following lemma states that the class of vector bundle charts is rich enough to define the weak topology on the set of C^k mappings $F : TM \longrightarrow TQ$.

Lemma 4 *In the definition of the weak topology on $C^k(TM,TQ)$ one may consider only vector bundle charts on TM and TQ, induced by charts on M and Q, respectively, and still obtain the same topology.*

Proof The proof appeals to an elementary fact stated below (without proof) as Lemma 7. Let $\mathscr{S}$ be the subbasis used in the standard definition of the weak topology $\mathscr{T}$, and let $\mathscr{S}'$ be the collection of sets defined analogously but considering only bundle charts on TM and TQ. Clearly, $\mathscr{S}'$ is a subbasis since its elements cover $C^k(TM,TQ)$. Denoting by $\mathscr{T}'$ the topology generated by $\mathscr{S}'$, we shall show that $\mathscr{T} = \mathscr{T}'$.
($\subset$) Let $g \in C^k(TM,TQ)$ and suppose that $g \in \mathscr{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)$, with (U,φ) and (V,ψ) arbitrary charts on TM and TQ, respectively, and $f(K) \subset V$. In particular, $g(K) \subset V$. By Lemma 5, there exist *finite* families $((U_\alpha,\varphi_\alpha))_{\alpha\in A}$, $((V_\beta,\psi_\beta))_{\beta\in B}$ of bundle charts on TM and TQ, respectively, as well as a family $(K_\alpha)_{\alpha\in A}$ such that: **(i)** $U_\alpha \cap U \neq \emptyset$ and $V_\beta \cap V \neq \emptyset$ for $\alpha \in A$, $\beta \in B$; **(ii)** $K = \bigcup_{\alpha\in A} K_\alpha$; and **(iii)** for $\alpha \in A$, $K_\alpha \subset U_\alpha$ and $g(K_\alpha) \subset V_\beta$ for some $\beta \in B$. For each $\beta \in B$, set $A_\beta = \{\alpha \in A : g(K_\alpha) \subset V_\beta\}$. Fix $\beta \in B$ and for every $\alpha \in A_\beta$ denote by $(\hat{U}_\alpha,\hat{\varphi}_\alpha)$ and $(\hat{V}_\beta,\hat{\psi}_\beta)$ the charts on M and Q that naturally induce the charts $(U_\alpha,\varphi_\alpha)$ and (V_β,ψ_β), respectively. Let $\eta_\alpha = \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U) \longrightarrow \mathbb{R}^{2n}$ and $\theta_\beta = \psi \circ \psi_\beta^{-1} : \psi_\beta(V_\beta \cap V) \longrightarrow \mathbb{R}^{2m}$ ($n = \dim(M)$, $m = \dim(Q)$) be the respective coordinate transformations. By the assumption on g, $d = \varepsilon - \|\psi\circ g\circ\varphi^{-1} - \psi\circ f\circ\varphi^{-1}\|^k_{\varphi(K)}$ is strictly positive. Let $h \in C^k(TM,TQ)$ and denote by $h_{\alpha,\beta} = \psi_\beta \circ h \circ \varphi_\alpha^{-1}$ its representative in the coordinates φ_α and ψ_β. Likewise, let $g_{\alpha,\beta} = \psi_\beta \circ g \circ \varphi_\alpha^{-1}$. One has $\psi\circ h\circ\varphi^{-1} = \theta_\beta \circ h_{\alpha,\beta} \circ \eta_\alpha^{-1}$ and $\psi\circ g\circ\varphi^{-1} = \theta_\beta \circ g_{\alpha,\beta} \circ \eta_\alpha^{-1}$. Hence,

$$\|\psi\circ h\circ\varphi^{-1} - \psi\circ g\circ\varphi^{-1}\|^k_{\varphi(K)} = \|\theta_\beta \circ (h_{\alpha,\beta} - g_{\alpha,\beta}) \circ \eta_\alpha^{-1}\|^k_{\varphi(K)}.$$

From Lemma 3, there exists $c_1 > 0$ such that

$$\|\psi\circ h\circ\varphi^{-1} - \psi\circ g\circ\varphi^{-1}\|^k_{\varphi(K)} \leq c_1 \|h_{\alpha,\beta} - g_{\alpha,\beta}\|^k_{\varphi_\alpha(K_\alpha)}.$$

Let $\delta_1 = d/c_1$ and assume that $\|h_{\alpha,\beta} - g_{\alpha,\beta}\|^k_{\varphi_\alpha(K_\alpha)} < \delta_1$. Then, using the triangle inequality:

$$\begin{aligned}\|\psi\circ h\circ\varphi^{-1} - \psi\circ f\circ\varphi^{-1}\|^k_{\varphi(K)} &\leq \|\psi\circ h\circ\varphi^{-1} - \psi\circ g\circ\varphi^{-1}\|^k_{\varphi(K)} \\ &\quad + \|\psi\circ g\circ\varphi^{-1} - \psi\circ f\circ\varphi^{-1}\|^k_{\varphi(K)} \\ &< \varepsilon.\end{aligned}$$

On the other hand, given that $K_\alpha \subset K$, $g(K) \subset V$ and $g(K_\alpha) \subset V_\beta$, one has $g(K_\alpha) \subset V \cap V_\beta$. Therefore the set $g_{\alpha,\beta}(\varphi_\alpha(K_\alpha))$ is compact and contained in $\psi_\beta(V \cap V_\beta)$. Thus there exists $\delta_2 > 0$ such that, for every $y \in g_{\alpha,\beta}(\varphi_\alpha(K_\alpha))$ and every $z \in \mathbb{R}^{2m}$, $\|y - z\|_\infty < \delta_2$ implies $z \in \psi_\beta(V \cap V_\beta)$, with $\|z\|_\infty = \max\{|z_i| : i = 1,\ldots,2m\}$. Now suppose that $\|h_{\alpha,\beta} - g_{\alpha,\beta}\|^k_{\varphi_\alpha(K_\alpha)} < \delta_2$. This implies, in particular, that for every $x \in \varphi_\alpha(K_\alpha)$, $\|h_{\alpha,\beta}(x) - g_{\alpha,\beta}(x)\|_\infty < \delta_2$, so $h_{\alpha,\beta}(x) \in \psi_\beta(V \cap V_\beta)$; hence $h(K_\alpha) \subset V \cap V_\beta \subset V$. Set $\delta_{\alpha,\beta} = \min\{\delta_1,\delta_2\}$ and define $\delta = \min\{\delta_{\alpha,\beta} : \beta \in B,\, \alpha \in A_\beta\}$. By construction of δ, for every $\beta \in B$ and $\alpha \in A_\beta$, if $h \in$

$\mathcal{N}^k(g,(U_\alpha,\varphi_\alpha),(V_\beta,\psi_\beta),K_\alpha,\delta)$, then $\|\psi\circ h\circ\varphi^{-1}-\psi\circ f\circ\varphi^{-1}\|^k_{\varphi(K)}<\varepsilon$ and $h(K)=h(\bigcup_{\alpha\in A}K_\alpha)\subset V$, so $h\in\mathcal{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)$. It follows that

$$S'=\bigcap_{\beta\in B}\bigcap_{\alpha\in A_\beta}\mathcal{N}^k(g,(U_\alpha,\varphi_\alpha),(V_\beta,\psi_\beta),K_\alpha,\delta)$$

satisfies $g\in S'$, $S'\in\mathcal{S}'$ and $S'\subset\mathcal{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)$. Thus $\mathcal{T}\subset\mathcal{T}'$.
($\supset$) Let $g\in C^k(TM,TQ)$ and suppose that $g\in\mathcal{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)$, with (U,φ) and (V,ψ) bundle charts on TM and TQ, respectively. Since any bundle chart is a chart, $\mathcal{N}^k(f,(U,\varphi),(V,\psi),K,\varepsilon)\in\mathcal{S}$, and by Lemma 7, $\mathcal{T}\supset\mathcal{T}'$. Therefore $\mathcal{T}=\mathcal{T}'$. □

Lemma 5 *Let M and N be manifolds, let $U\subset M$ and $V\subset N$ be open, let $K\subset U$ be compact and let $f\in C^0(TM,TN)$ satisfy $f(K)\subset V$. There exist finite families $((U_\alpha,\varphi_\alpha))_{\alpha\in A}$ and $((V_\beta,\psi_\beta))_{\beta\in B}$ of vector bundle charts on TM and TN, respectively, and a family $(K_\alpha)_{\alpha\in A}$ of compact sets such that: **(i)** $U_\alpha\cap U\neq\emptyset$ and $V_\beta\cap V\neq\emptyset$ for all $\alpha\in A$ and all $\beta\in B$; **(ii)** $K=\bigcup_{\alpha\in A}K_\alpha$; **(iii)** For every $\alpha\in A$, $K_\alpha\subset U_\alpha$ and there exists $\beta\in B$ such that $f(K_\alpha)\subset V_\beta$.*

Proof Clearly, $\pi_N(f(K))$ is compact, so there exists a finite family $((\hat V_\beta,\hat\psi_\beta))_{\beta\in B}$ of charts on N such that $\pi_N(f(K))\subset\bigcup_{\beta\in B}\hat V_\beta$ and $\pi_N(f(K))\cap\hat V_\beta\neq\emptyset$ for every $\beta\in B$. For $\beta\in B$, let (V_β,ψ_β) be the vector bundle chart on TN induced by $(\hat V_\beta,\hat\psi_\beta)$. In particular, $V_\beta=\pi_N^{-1}(\hat V_\beta)$ and $V\cap V_\beta\neq\emptyset$. By construction, $(V_\beta)_{\beta\in B}$ is a cover of $f(K)$, hence Lemma 6, stated below, implies the existence of a family of compact sets $(D_\beta)_{\beta\in B}$ such that $f(K)=\bigcup_{\beta\in B}D_\beta$ and $D_\beta\subset V_\beta$ for every $\beta\in B$. Now let $\beta\in B$. The set $f^{-1}(D_\beta)\cap K$ is closed and contained in K, so it is compact. Hence $\pi_M(f^{-1}(D_\beta)\cap K)$ is compact as well, so there exists a finite set Γ_β and a family $((\hat U^\beta_\gamma,\hat\varphi^\beta_\gamma))_{\gamma\in\Gamma_\beta}$ of charts on M whose domains cover that set and satisfy $\pi_M(f^{-1}(D_\beta)\cap K)\cap\hat U^\beta_\gamma\neq\emptyset$ for all $\gamma\in\Gamma_\beta$. For each $\gamma\in\Gamma_\beta$, let $(U^\beta_\gamma,\varphi^\beta_\gamma)$ be the vector bundle chart on TM induced by $(\hat U^\beta_\gamma,\hat\varphi^\beta_\gamma)$. In particular, $U^\beta_\gamma=\pi_M^{-1}(\hat U^\beta_\gamma)$ and $U\cap U^\beta_\gamma\neq\emptyset$. Let $\beta\in B$. By construction, $(U^\beta_\gamma)_{\gamma\in\Gamma_\beta}$ is a cover of $f^{-1}(D_\beta)\cap K$ so, by Lemma 6 again, there exists a family $(K^\beta_\gamma)_{\gamma\in\Gamma}$ of compact sets such that $f^{-1}(D_\beta)\cap K=\bigcup_{\gamma\in\Gamma_\beta}K^\beta_\gamma$ and $K^\beta_\gamma\subset U^\beta_\gamma$ for every $\gamma\in\Gamma_\beta$. One has $K=\bigcup_{\beta\in B}\bigcup_{\alpha\in\Gamma_\beta}K^\beta_\gamma$. Indeed, let $v\in K$. Then $f(v)\in f(K)$, so there exists $\beta\in B$ such that $f(v)\in D_\beta$. Therefore $v\in f^{-1}(D_\beta)\cap K$, so there exists $\gamma\in\Gamma_\beta$ such that $v\in K^\beta_\gamma$. Moreover, for every $\beta\in B$ and every $\gamma\in\Gamma_\beta$, $f(K^\beta_\gamma)\subset V_\beta$. Indeed, let $w\in f(K^\beta_\gamma)$, so that $w=f(v)$ for some $v\in K^\beta_\gamma$. Hence $v\in f^{-1}(D_\beta)\cap K$, so $f(v)\in D_\beta\subset V_\beta$. Let $A=\bigsqcup_{\beta\in B}\Gamma_\beta$. Clearly, A is finite and for every $\alpha\in A$ there exist $\beta\in B$ and $\gamma\in\Gamma_\beta$ such that $\alpha=(\beta,\gamma)$; setting $K_\alpha=K^\beta_\gamma$, $U_\alpha=U^\beta_\gamma$ and $\varphi_\alpha=\varphi^\beta_\gamma$, we see that $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$, $(V_\beta,\psi_\beta)_{\beta\in B}$ and $(K_\alpha)_{\alpha\in A}$ satisfy the stated requirements. □

Lemma 6 ("Compact Shrinking Lemma.") *Let X be a locally compact, Hausdorff topological space, let $K\subset X$ be compact and let $(U_\alpha)_{\alpha\in A}$ be a finite open cover of K. Then there exists a family $(K_\alpha)_{\alpha\in A}$ of compact sets such that $K=\bigcup_{\alpha\in A}K_\alpha$ and $K_\alpha\subset U_\alpha$ for every $\alpha\in A$.*

Proof Given a set $S\subset X$, let S' denote the set of limit points of S. Since A is finite, we may assume, without loss of generality, that $A=\{1,\ldots,n\}$. The family $(K_\alpha)_{\alpha\in A}$ shall be built inductively. Set $K_1=\mathrm{cl}(K\cap U_1)\setminus\bigcup_{i=2}^n U_i$. If $K_1=\emptyset$, then K_1 is compact; otherwise, let $x\in K_1$, so $x\in\mathrm{cl}(K\cap U_1)$ but $x\notin\bigcup_{i=2}^n U_i$. Suppose that $x\notin K\cap U_1$; then $x\in(K\cap U_1)'$, so x is a limit point of K. Since K is closed, $x \in K$ and thus $x \notin U_1$. But then $x \notin \bigcup_{i=1}^{n} U_i$; a contradiction. Thus $x \in K \cap U_1$ and, since K_1 is the complement of an open set in a closed one, K_1 is itself closed, hence compact. Now let $x \in K$ and assume that $x \notin \bigcup_{i=2}^{n} U_i$. Then $x \in U_1$, so $x \in K_1$. Thus $K \subset K_1 \cup \bigcup_{i=2}^{n} U_i$. Now suppose that, for some $\alpha \in \{1,\ldots,n\}$ and every $i \in \{1,\ldots,\alpha\}$, compact sets $K_i \subset U_i$ have been defined such that $K \subset \bigcup_{i=1}^{\alpha} K_i \cup \bigcup_{i=\alpha+1}^{n} U_i$. Let $K_{\alpha+1} = \mathrm{cl}(K \cap U_{\alpha+1}) \setminus \bigcup_{i=\alpha+2}^{n} U_i$. By an argument analogous to that used in the construction of K_1, $K_{\alpha+1}$ is compact and $K_{\alpha+1} \subset K \cap U_{\alpha+1}$. Now let $x \in K$ and suppose that $x \notin \bigcup_{i=1}^{\alpha} K_i$ and $x \notin \bigcup_{i=\alpha+2}^{n} U_i$. Then, by the inductive assumption, $x \in U_{\alpha+1}$, so $x \in K_{\alpha+1}$. Hence $K \subset \bigcup_{i=1}^{\alpha+1} K_i \cup \bigcup_{i=\alpha+2}^{n} U_i$. By this process, one ends up with a family $(K_\alpha)_{\alpha \in A}$ of compact sets such that $K = \bigcup_{i=1}^{n} K_i$ and $K_\alpha \subset U_\alpha$ for every $\alpha \in A$. □

Lemma 7 *(Comparison of topologies using subbases). Let X be a set and let $\mathscr{S}$ and $\mathscr{S}'$ be subbases which generate topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively. Then $\mathscr{T} \subset \mathscr{T}'$ if, and only if, for every $x \in X$ and every $S \in \mathscr{S}$, $x \in S$ implies the existence of $S' \in \mathscr{S}'$ such that $x \in S'$ and $S' \subset S$.*

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