Exponential stability of some linear continuous time difference systems

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Abstract

In this paper, we consider some classes of linear continuous time difference systems with discrete and distributed delays. For these infinite-dimensional systems, we derive new sufficient delay-dependent conditions for the exponential stability and exponential estimates for the solutions by using Lyapunov-Krasovskii functionals.

Key words: Continuous time difference systems, Exponential Stability, Lyapunov-Krasovskii functionals

1. Introduction

Continuous time difference systems play a fundamental role in investigating the stability properties of neutral type delay systems, whereas stability of the difference system is a necessary condition for stability of the corresponding neutral delay system [2], [9]. There are also a number of applications such as in economics, gas dynamics, lossless propagation, and models of heredity where the stability of continuous time difference systems is so important [13]. In this context, stability properties of linear continuous time difference systems have been widely studied and several stability conditions based on the spectral radius and norm of matrices have been reported [2], [9].

Lyapunov theorems for continuous time difference systems with discrete time delays have been introduced in [14], [15] and [16]. As such class of systems can be regarded like discrete time equations evolving on an appropriate infinite-dimensional space [2], the results in [14], [15], and [16] propose Lyapunov functions satisfying along solutions a first difference type condition.

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However, there are some difficulties in the application of these Lyapunov approaches to more general continuous time difference systems as, for instance, those including both discrete and distributed delay terms. The main reason of this is that the proposed functions are such that their first difference type condition, along solutions of such class of systems, include not only discrete delay terms but also distributed delay terms whose negativity cannot be directly assured.

In this paper, we propose a Lyapunov-Krasovskii approach for investigating the exponential stability of linear continuous time difference systems with discrete and distributed delay terms. We address this problem as a robust stability one. More precisely, by assuming only the stability of the discrete delay part of the system and interpreting the distributed delay term as a perturbation, we present Lyapunov-Krasovskii functionals guaranteeing the exponential stability of the whole system. Our contribution is based on the recent papers [11] and [12], where we have introduced Lyapunov-Krasovskii theorems for integral delay systems.

The paper is structured as follows: Section 2 presents the problem formulation. Some preliminary results are provided in section 3. Basic facts about solutions are given and Lyapunov-Krasovskii type stability conditions are introduced. The main results are given in section 4. First, we consider a general case for which a delay-dependent stability condition is derived. Next, a particular case of continuous time difference systems with constant but uncertain system matrices is addressed. In this case, delay-dependent conditions for the exponential stability and exponential estimates for the solutions are expressed in terms of linear matrix inequalities. Examples illustrating the results are provided in section 5. Concluding remarks end the paper.

Notation: Throughout this paper, the Euclidean norm for vectors and the induced matrix norm for matrices are used, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I stands for the identity matrix, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively. For a real symmetric matrix Q, the standard notation $Q > 0$ (respectively, $Q < 0$) is used to denote that Q is positive (respectively negative) definite.

2. Problem Formulation

Consider the following continuous time difference system:

$$x(t) = Ax(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta, \quad t \geq 0, \tag{1}$$

where $A \in \mathcal{R}^{n\times n}$ and the matrix function $G(\theta)$ has piecewise continuous bounded elements defined for $\theta \in [-h, 0]$.

Systems of the form of (1) can be found as delay approximations of the partial differential equations for describing the propagation phenomena in excitable media [1], in the stability analysis of additional dynamics introduced by some system transformations [6], in delay-dependent stability analysis of neutral type systems [4], [10], as well as in the stability analysis of some difference operators in neutral type functional differential equations [2], [9].

For the sake of simplicity of the problem formulation, let us consider that the matrix function $G(\theta)$ is a $n \times n$ constant matrix, i.e., $G(\theta) \equiv G, \forall\theta \in [-h, 0]$. In this case, it is known that (1) is asymptotically stable if the inequality

$$\|A\| + h\,\|G\| < 1$$

holds, see for instance [9]. When $G = 0$, the above inequality leads to $\|A\| < 1$ which is evidently more restrictive than Schur stability of the matrix A (all eigenvalues of the matrix lie in the open unit disc of the complex plane).

This naturally raises the following question: Is it not possible to obtain less conservative conditions by assuming that matrix A is Schur stable and considering the integral delay term as a perturbation?

Some difficulties occur if we address the problem from existing Lyapunov results for difference systems in continuous time. First we note that by the change of time $t' = t - h$, the corresponding system can be written as

$$x(t'+h) = Ax(t') + G\int_{t'}^{t'+h} x(\theta)d\theta, \quad t' \geq h. \tag{2}$$

Following [14], [15], and [16], when $G = 0$ and matrix A is Schur stable, the function $v(t') = x^T(t')Px(t')$, where P is the unique positive definite solution of the Lyapunov matrix equation: $A^TPA - P = -Q$, for a given positive definite matrix Q, is a Lyapunov function for the corresponding system. In particular, in this case, we have $\Delta v(t') \triangleq v(t'+h) - v(t') = x^T(t')\left(A^TPA - P\right)x(t') = -x^T(t')Qx(t')$.

It is clear that one cannot conclude directly the stability of (2) by using the function $v(t')$ as a Lyapunov function candidate for the system since its first difference type condition along solutions of (2) includes products of $x(t')$ and $\int_{t'}^{t'+h} x(\theta)d\theta$ which can not be compensated for negativity of $\Delta v(t')$.

We will introduce below a Lyapunov-Krasovskii approach that will give a positive answer to the above question. The method consists in a combination of the Lyapunov-Krasovskii approach that we have recently developed for integral delay systems and stability properties of linear continuous time difference systems with a discrete pure delay.

3. Preliminaries

3.1. Solutions and stability concept

In order to determine a particular solution of (1) an initial vector function $\varphi(\theta), \theta \in [-h, 0)$, should be given. We assume that φ belongs to the space of continuous vector functions $\mathcal{C}([-h,0), \mathcal{R}^n)$, equipped with the uniform convergence norm $\|\varphi\|_h = \sup_{\theta \in [-h,0)} \|\varphi(\theta)\|$.

For a given initial function $\varphi \in \mathcal{C}([-h,0), \mathcal{R}^n)$, let $x(t, \varphi), t \geq 0$, be the unique solution of (1) satisfying $x(t, \varphi) = \varphi(t), t \in [-h, 0)$. This solution has a jump discontinuity at $t = 0$ given by

$$\Delta x(0, \varphi) = x(0, \varphi) - x(-0, \varphi) = A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta - \varphi(-0).$$

As a neutral delay system, this discontinuity is propagated along the solution leading to jump discontinuities at time instants multiple of h. Except at the time instants $t = jh, j = 0, 1, 2, ...$, the solution $x(t, \varphi)$ is a continuous function of t. Clearly, if the condition

$$A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta = \varphi(-0)$$

holds, then the solution $x(t, \varphi)$ is continuous for all $t \geq -h$.

When matrix function $G(\theta)$ is continuously differentiable on the interval $[-h, 0]$, where a right-hand side continuous derivative at $-h$ and a left-hand side continuous derivative at 0 are assumed to exist, the solutions of (1) can be related with particular solutions of some neutral functional differential equations and coupled systems described by retarded functional differential equations and functional difference equations.

More precisely, consider the neutral functional differential equation

$$\frac{d}{dt}\left[z(t)-Az(t-h)\right]=G(0)z(t)-G(-h)z(t-h)-\int_{-h}^{0}\dot{G}(\theta)z(t+\theta)d\theta, \quad (3)$$

and the coupled system

$$\begin{aligned}\dot{y}_1(t) &= G(0)y_2(t)-G(-h)y_2(t-h)-\int_{-h}^{0}\dot{G}(\theta)y_2(t+\theta)d\theta, && (4)\\ y_2(t) &= Ay_2(t-h)+y_1(t). && (5)\end{aligned}$$

Denote by $z(t,\psi), t\geq 0$, the solution of (3) satisfying $z(t,\psi)=\psi(t), t\in[-h,0]$, where ψ belongs to the space of piecewise continuous vector functions $\mathcal{PC}\left(\left[-h,0\right],\mathcal{R}^n\right)$, see [2].

Similarly, denote by $y_1(t,\tilde{\varphi}_1), y_2(t,\tilde{\varphi}_2), t\geq 0$, the solutions of (4) and (5) satisfying $y_1(t,\tilde{\varphi}_1)=\tilde{\varphi}_1(t), y_2(t,\tilde{\varphi}_2)=\tilde{\varphi}_2(t), t\in[-h,0]$, where $\tilde{\varphi}_1,\tilde{\varphi}_2\in\mathcal{PC}\left(\left[-h,0\right],\mathcal{R}^n\right)$, see [5].

Lemma 1.

(i) *Assume in (1) that matrix function $G(\theta)$ is continuously differentiable on $[-h,0]$. For a given initial function $\varphi\in\mathcal{C}\left(\left[-h,0\right),\mathcal{R}^n\right)$, define the function*

$$\psi(\theta)=\begin{cases}\varphi(\theta),\theta\in\left[-h,0\right),\\ A\varphi(-h)+\int_{-h}^{0}G(\theta)\varphi(\theta)d\theta.\end{cases}$$

Then $x(t,\varphi)=z(t,\psi)$.

(ii) *For a given initial function $\psi\in\mathcal{PC}\left(\left[-h,0\right],\mathcal{R}^n\right)$, define the functions $\tilde{\varphi}_2(\theta)=\psi(\theta),\theta\in\left[-h,0\right]$, and $\tilde{\varphi}_1\in\mathcal{PC}\left(\left[-h,0\right],\mathcal{R}^n\right)$ such that*

$$\tilde{\varphi}_1(0)=\psi(0)-A\psi(-h).$$

Then $z(t,\psi)=y_2(t,\tilde{\varphi}_2)$.

Proof. See Appendix. ■

For matrix functions $G(\theta)$ being continuously differentiable on $[-h,0]$, the result of *(i)* relates the solutions of (1) with some particular solutions of (3). The result of *(ii)* relates the solutions of (3) with some particular solutions of the coupled system described by (4) and (5).

From Lemma 1 we may deduce immediately the existing relationship between special solutions of (1) and some particular solutions of the coupled system described by (4) and (5).

Lemma 2. *Assume in (1) that matrix function $G(\theta)$ is continuously differentiable on $[-h,0]$. For a given initial function $\varphi \in \mathcal{C}\left([-h,0),\mathcal{R}^n\right)$, define the functions*

$$\tilde{\varphi}_2\left(\theta\right) = \begin{cases} \varphi(\theta), \theta \in [-h,0)\,, \\ A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta, \theta = 0, \end{cases}$$

and $\tilde{\varphi}_1 \in \mathcal{PC}\left([-h,0],\mathcal{R}^n\right)$ such that $\tilde{\varphi}_1(0) = \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta$. Then $x(t,\varphi) = y_2(t,\tilde{\varphi}_2)$.

In the sequel, we shall use the following concept of stability.

Definition 1. *[2] System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that any solution of (1) satisfies the inequality*

$$\|x(t,\varphi)\| \leq \mu e^{-\alpha t}\left\|\varphi\right\|_h, \quad \forall t \geq 0. \tag{6}$$

Remark 1. *Neither the neutral functional differential equation (3) nor the coupled system described by (4) and (5) are exponentially stable. Indeed, any constant vector is a solution of (3) and therefore, by virtue of* (ii) *in Lemma 1, the coupled system described by (4) and (5) also admits constant solutions.*

The remark implies that, even in the particular case when matrix function $G(\theta)$ is continuously differentiable on $[-h,0]$, existing stability results for neutral functional differential equations [3] and coupled systems described by retarded functional differential equations and functional difference equations [5] cannot be directly applied to the stability analysis of (1).

3.2. A Lyapunov type theorem

For any $t \geq 0$ we denote the restriction of the solution $x(t,\varphi)$ on the interval $[t-h,t)$ by $x_t(\varphi) = x(t+\theta,\varphi), \theta \in [-h,0)$. When the initial function φ is irrelevant we simply write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$.

The jump discontinuities of the solutions of (1) imply that $x_t(\varphi) \in \mathcal{PC}\left([-h,0),\mathcal{R}^n\right)$ for $t \geq 0$. This means that in a Lyapunov-Krasovskii functional setting, the functionals should be defined on the infinite-dimensional space $\mathcal{PC}\left([-h,0),\mathcal{R}^n\right)$.

Theorem 3. *Let system (1) be given and assume that matrix A is Schur stable. System (1) is exponentially stable if there exists a continuous functional $v : \mathcal{PC}\left([-h,0),\mathcal{R}^n\right) \to \mathcal{R}$ such that $t \to v(x_t(\varphi))$ is differentiable for all $t \geq 0$ and satisfies the following conditions:*

1. $\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta$, *for some constants* $0 < \alpha_1 \leq \alpha_2$,
2. $\frac{d}{dt} v(x_t(\varphi)) \leq -\beta \int_{-h}^{0} \|x(t+\theta,\varphi)\|^2 d\theta$, *for some* $\beta > 0$.

Proof. Given any initial function $\varphi \in \mathcal{C}([-h,0), \mathcal{R}^n)$, it follows from the Theorem conditions that for $2\alpha = \beta\alpha_2^{-1}$ the following inequality:

$$\frac{d}{dt} v(x_t(\varphi)) + 2\alpha v(x_t(\varphi)) \leq 0, \forall t \geq 0,$$

holds. This inequality leads to

$$v(x_t(\varphi)) \leq e^{-2\alpha t} v(\varphi), \forall t \geq 0.$$

Thus it follows that for $t \geq 0$

$$\alpha_1 \int_{-h}^{0} \|x(t+\theta,\varphi)\|^2 d\theta \leq \alpha_2 e^{-2\alpha t} \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta. \tag{7}$$

From (1) one gets

$$\begin{aligned} \|x(t,\varphi) - Ax(t-h,\varphi)\|^2 &\leq \left(m_g \int_{-h}^{0} \|x(t+\theta)\| d\theta \right)^2 \\ &\leq m_g^2 h \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta, \end{aligned} \tag{8}$$

where the last inequality has been obtained by using the Cauchy-Schwarz inequality in $\mathcal{L}^2([-h,0), \mathcal{R})$ and

$$m_g = \sup_{\theta \in [-h,0]} \|G(\theta)\|.$$

Combining the inequalities (7) and (8) one obtains

$$\|x(t,\varphi) - Ax(t-h,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} m_g h \|\varphi\|_h e^{-\alpha t}, \forall t \geq 0.$$

This inequality implies

$$x(t,\varphi) - Ax(t-h,\varphi) = f(t), \tag{9}$$

where $f \in \mathcal{C}\left([0,\infty),\mathcal{R}^n\right)$ satisfies

$$\|f(t)\| \leq \mu \|\varphi\|_h e^{-\alpha t}, \quad \forall t \geq 0,$$

with

$$\mu = \sqrt{\frac{\alpha_2}{\alpha_1}} m_g h.$$

Since A is Schur stable, then there exist $\gamma > 0$ and $\sigma > 0$ such that

$$\left\|A^k\right\| \leq \gamma e^{-\sigma(kh)}, k = 0, 1, 2, \ldots.$$

From the Lemma 6 in [7] it follows that the inequality

$$\|x(t,\varphi)\| \leq \eta \|\varphi\|_h e^{-\nu t}, \quad \forall t \geq 0,$$

holds for the solutions $x(t,\varphi)$ of (9) with

$$\eta = \gamma\left(1 + \mu + \frac{\mu}{h e \varepsilon}\right) \text{ and } \nu = \min\{\sigma,\alpha\} - \varepsilon,$$

where $\varepsilon \in (0, \min\{\sigma,\alpha\})$. This implies the exponential stability of (1). ■

Remark 2. *In spite of the fact that the state $x_t(\varphi) \in \mathcal{PC}\left([-h,0),\mathcal{R}^n\right)$, the Theorem conditions guarantee the exponential stability of (1) by means of continuous and differentiable functionals.*

4. Main Results

4.1. A general case

We begin with a general case for which a simple-to-check delay-dependent stability condition is derived in the following:

Proposition 4. *Let system (1) be given and assume that matrix A is Schur stable. System (1) is exponentially stable if there exist positive definite matrices W_0 and W_1 such that*

$$h\left(\sup_{\theta\in[-h,0]} \|G(\theta)\|\right)^2 < \frac{\lambda_{\min}(W_1)}{\lambda_{\max}\left(P + PAW_0^{-1}A^T P\right)}, \tag{10}$$

with P the positive definite solution of the matrix Lyapunov equation

$$A^T PA - P = -\left(W_0 + hW_1\right). \tag{11}$$

Proof. Consider the following functional:

$$v(\varphi) = \int_{-h}^{0} \varphi^T(\theta) \left[A^T P A + W_0 + (\theta + h) W_1 \right] \varphi(\theta) d\theta, \tag{12}$$

where W_0, W_1 are positive definite matrices and P is the positive definite solution of (11).

The functional (12) satisfies the following inequalities:

$$\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta, \tag{13}$$

with $0 < \alpha_1 \leq \alpha_2$ given by

$$\alpha_1 = \lambda_{\min}(W_0) \text{ and } \alpha_2 = \lambda_{\max}(P).$$

The time derivative of the functional (12) along solutions of (1) is

$$\begin{aligned}
\frac{dv(x_t)}{dt} &= -x^T(t-h) W_0 x(t-h) + 2x^T(t-h) A^T P \int_{-h}^{0} G(\theta) x(t+\theta) d\theta \\
&+ \left(\int_{-h}^{0} G(\theta) x(t+\theta) d\theta \right)^T P \left(\int_{-h}^{0} G(\theta) x(t+\theta) d\theta \right) \\
&- \int_{-h}^{0} x^T(t+\theta) W_1 x(t+\theta) d\theta.
\end{aligned}$$

Using the Jensen integral inequality the following inequality:

$$\begin{aligned}
&\left(\int_{-h}^{0} G(\theta) x(t+\theta) d\theta \right)^T P \left(\int_{-h}^{0} G(\theta) x(t+\theta) d\theta \right) \\
&\leq h \int_{-h}^{0} x^T(t+\theta) G^T(\theta) P G(\theta) x(t+\theta) d\theta
\end{aligned} \tag{14}$$

holds. As a consequence we obtain the following upper bound for the derivative:

$$\frac{dv(x_t)}{dt} \leq -\int_{-h}^{0} \left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta) \end{array} \right] \mathcal{N}(\theta) \left[\begin{array}{c} x(t-h) \\ x(t+\theta) \end{array} \right] d\theta,$$

where

$$\mathcal{N}(\theta) = \left[\begin{array}{cc} \frac{1}{h} W_0 & -A^T P G(\theta) \\ -G^T(\theta) P A & W_1 - h G^T(\theta) P G(\theta) \end{array} \right].$$

If the inequality (10) holds then

$$\beta = \lambda_{\min}(W_1) - h\lambda_{\max}\left(P + PAW_0^{-1}A^T P\right)\left(\sup_{\theta\in[-h,0]}\|G(\theta)\|\right)^2 > 0,$$

which in turn implies

$$W_1 - hG^T(\theta)\left[P + PAW_0^{-1}A^T P\right]G(\theta) > 0, \forall\theta \in [-h, 0].$$

The above inequality is equivalent to $\mathcal{N}(\theta) > 0, \forall\theta \in [-h, 0]$, by Schur complement. Thus, if (10) holds then

$$\frac{dv(x_t)}{dt} \leq -\beta\int_{-h}^{0}\|x(t+\theta)\|^2 d\theta,$$

and the exponential stability of (1) follows. ■

Remark 3. *W_0 and W_1 are free positive definite matrices which can be used to improve the right-hand side of inequality (10).*

4.2. A particular case

Now let us consider the following continuous time difference system:

$$x(t) = (A + \Delta A)\,x(t-h) + (G + \Delta G)\int_{-h}^{0} x(t+\theta)d\theta, \tag{15}$$

where $A, G \in \mathcal{R}^{n\times n}$ are known and $\Delta A, \Delta G$ are unknown constant matrices satisfying

$$\|\Delta A\| \leq \delta \text{ and } \|\Delta G\| \leq \rho. \tag{16}$$

We here assume that the matrix $A + \Delta A$ remains Schur stable for all ΔA satisfying (16).

Proposition 5. *The uncertain system (15) is exponentially stable for all unknown matrices ΔA and ΔG satisfying (16) if there exist positive definite matrices P, W_0, W_1 and scalar $\lambda > 0$ such that the following inequalities hold:*

$$\begin{aligned} \mathcal{M}_n(P, W_0, W_1) - \lambda\mathcal{M}_{p1}(\delta,\rho) - \lambda\mathcal{M}_{p2}(\rho) &> 0, & (17)\\ A^T PA + \lambda\delta\,(2\,\|A\| + \delta)\,I - P + (W_0 + hW_1) &< 0, & (18)\\ \lambda I - P &> 0, & (19) \end{aligned}$$

where

$$\mathcal{M}_n(P, W_0, W_1) = \begin{bmatrix} \frac{1}{h}W_0 & -A^T PG \\ -G^T PA & W_1 - hG^T PG \end{bmatrix}, \tag{20}$$

$$\mathcal{M}_{p1}(\delta, \rho) = (\rho \|A\| + \delta \|G\| + \delta\rho) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{21}$$

$$\mathcal{M}_{p2}(\rho) = h\rho \left(2 \left\|G^T\right\| + \rho\right) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{22}$$

Proof. Consider the following functional:

$$v(\varphi) = \int_{-h}^{0} \varphi^T(\theta) \left[(A + \Delta A)^T P (A + \Delta A) + W_0 + (\theta + h) W_1\right] \varphi(\theta) d\theta, \tag{23}$$

where W_0, W_1 are positive definite matrices and P is the positive definite solution of the Lyapunov inequality

$$(A + \Delta A)^T P (A + \Delta A) + (W_0 + hW_1) < P. \tag{24}$$

From (23) we get the following inequalities for the functional:

$$\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta,$$

where $0 < \alpha_1 \leq \alpha_2$ are determined by

$$\alpha_1 \leq \lambda_{\min}(W_0), \tag{25}$$

$$\alpha_2 \geq \lambda_{\max}(P) (\|A\| + \|\Delta A\|)^2 + \lambda_{\max} (W_0 + hW_1). \tag{26}$$

The time derivative of the functional (23) along solutions of (15) is

$$\begin{aligned} \frac{dv(x_t)}{dt} &= -\int_{-h}^{0} x^T(t + \theta) W_1 x(t + \theta) d\theta \\ &+ x^T(t) \left[(A + \Delta A)^T P (A + \Delta A) + W_0 + hW_1\right] x(t) \\ &- x^T(t - h) \left[(A + \Delta A)^T P (A + \Delta A) + W_0\right] x(t - h). \end{aligned}$$

Taking into account the inequality (24), substituting the right-hand side of (15), and then applying the inequality (14) with $G(\theta) = G + \Delta G$, one can easily arrive at the following upper bound for the derivative:

$$\frac{dv(x_t)}{dt} \leq -\int_{-h}^{0} \left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta) \end{array}\right] \times$$
$$\times \left(\mathcal{M}_n(P, W_0, W_1) - \mathcal{M}_p(\Delta A, \Delta G)\right) \left[\begin{array}{c} x(t-h) \\ x(t+\theta) \end{array}\right],$$

where $\mathcal{M}_n(P, W_0, W_1)$ is defined by (20) and

$$\mathcal{M}_p(\Delta A, \Delta G) = \left[\begin{array}{cc} 0 & A^T P(\Delta G) + (\Delta A)^T PG + (\Delta A)^T P(\Delta G) \\ * & 2hG^T P(\Delta G) + h(\Delta G)^T P(\Delta G) \end{array}\right].$$

Here $*$ denotes the symmetric entry of the symmetric matrix.

Using Lemma 7 in the Appendix for bounding the terms involving perturbation in the matrix $\mathcal{M}_p(\Delta A, \Delta G)$ we obtain

$$\frac{dv(x_t)}{dt} \leq -\int_{-h}^{0} \left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta) \end{array}\right] \{\mathcal{M}_n(P, W_0, W_1)$$
$$-\lambda \mathcal{M}_{p1}(\delta, \rho) - \lambda \mathcal{M}_{p2}(\rho)\} \left[\begin{array}{c} x(t-h) \\ x(t+\theta) \end{array}\right] d\theta,$$

where $\mathcal{M}_{p1}(\delta, \rho)$ and $\mathcal{M}_{p2}(\rho)$ are respectively defined by (21) and (22), and $\lambda > 0$ is such that (19) holds. Clearly, if (17) holds then

$$\frac{dv(x_t)}{dt} \leq -\beta \int_{-h}^{0} \|x(t+\theta, \varphi)\|^2 d\theta,$$

where

$$\beta = \lambda_{\min}\left(\mathcal{M}_n(P, W_0, W_1) - \lambda \mathcal{M}_{p1}(\delta, \rho) - \lambda \mathcal{M}_{p2}(\rho)\right), \qquad (27)$$

and the exponential stability of the perturbed system (15).

Noting that the inequality (18) subject to the restriction (19) implies (24) the proof ends. ■

Corollary 6. *If there exist positive definite matrices P, W_0, W_1 and scalar $\lambda > 0$ such that the matrix inequalities (17)-(19) hold, then an exponential estimate for the solutions of the uncertain system (15) is given by*

$$\|x(t, \varphi)\| \leq \eta \|\varphi\|_h e^{-\nu t}, \forall t \geq 0,$$

with
$$\eta = \gamma \left(1 + \mu + \frac{\mu}{he\varepsilon}\right) \text{ and } \nu = \min\{\sigma, \alpha\} - \varepsilon,$$
where $\varepsilon \in (0, \min\{\sigma, \alpha\})$. *Here* $\gamma > 0$ *and* $\sigma > 0$ *are such that*
$$\left\| (A + \Delta A)^k \right\| \leq \gamma e^{-\sigma(kh)}, k = 0, 1, 2, ..., \tag{28}$$
and
$$\mu = h\sqrt{\frac{\alpha_2}{\alpha_1}} (\|G\| + \|\Delta G\|) \text{ and } 2\alpha = \beta \alpha_2^{-1},$$
with α_1, α_2 *and* β *given respectively by (25), (26) and (27).*

Proof. Given positive definite matrices P, W_0, W_1 and scalar $\lambda > 0$ satisfying the inequalities (17)-(19), we calculate the positive constants α_1, α_2 and β determined by (25), (26) and (27).

Noting that
$$m_g = \sup_{\theta \in [-h,0]} \|G(\theta)\| = \|G + \Delta G\| \leq \|G\| + \|\Delta G\|,$$
and that Schur stability of $A + \Delta A$ implies the existence of $\gamma > 0$ and $\sigma > 0$ such that (28) holds, the result follows directly from the proof of the Theorem 3. ■

Remark 4. *Note that the integral form of the proposed functionals (12) and (23) guarantees that along the solutions of (1) and (15), the corresponding functions* $t \to v(x_t(\varphi))$ *are continuous and differentiable for all* $t \geq 0$.

Remark 5. *It is important to point out the dependence of the functional (23) on both the nominal and uncertain matrices* A *and* ΔA *of the perturbed system. This special feature of the functional, which allows it to adapt the perturbation by exploiting the Schur stability of* $A + \Delta A$*, is in the same spirit of that introduced in [7] and [8] for investigating the robust stability of neutral delay systems having uncertainty in the corresponding difference term.*

Remark 6. *In the nominal case, when (15) does not have uncertainty*
$$x(t) = Ax(t - h) + G \int_{-h}^{0} x(t + \theta) d\theta, \tag{29}$$

delay-dependent sufficient conditions for the exponential stability can be directly derived from Proposition 5. Thus, it follows that (29) is exponentially stable if there exist positive definite matrices P, W_0 and W_1 such that

$$\begin{aligned} \begin{bmatrix} \frac{1}{h}W_0 & -A^T PG \\ -G^T PA & W_1 - hG^T PG \end{bmatrix} &> 0, && (30) \\ A^T PA - P + (W_0 + hW_1) &< 0. && (31) \end{aligned}$$

Remark 7. *When $A = 0$ in (29), delay-dependent conditions for the exponential stability of the integral delay system*

$$x(t) = G\int_{-h}^{0} x(t+\theta)d\theta, \tag{32}$$

are directly obtained from the matrix inequalities (30) and (31). Indeed, from (30) and (31) it follows that (32) is exponentially stable if there exist positive definite matrices P, W_0 and W_1 such that

$$W_1 - hG^T PG > 0 \text{ and } P > W_0 + hW_1.$$

By combining these two inequalities we arrive at the result that (32) is exponentially stable if there exist positive definite matrices W_0 and W_1 such that

$$W_1 - hG^T (W_0 + hW_1) G > 0.$$

This stability condition coincides with that obtained in [12].

5. Illustrative Examples

Example 1. *Consider the system (29) where*

$$A = \begin{pmatrix} 0.2 & 1 \\ -0.1 & -0.2 \end{pmatrix}.$$

Matrix A is Schur stable and $\|A\| = 1.0424$. Then, in principle, the known results cannot be applied to investigate the stability of (29) since the inequality $\|A\| + h\|G\| < 1$ does not hold for any delay value $h > 0$ and matrix $G \in \mathcal{R}^{2\times 2}$.

However, as suggested by a reviewer, by using similarity transformations it is still possible to use the known results for obtaining stability conditions.

Let S be a similarity transformation matrix for A and apply the state transformation $x = Sy$. Then, in the new variable, (29) is of the form

$$y(t) = Dy(t-h) + S^{-1}GS \int_{-h}^{0} y(t+\theta)d\theta,$$

where $D = S^{-1}AS$ is similar to A. By using standard calculations, one can find that

$$S = \begin{pmatrix} 0.9535 & 0 \\ -0.1907 & 0.2335 \end{pmatrix}$$

leads to

$$D = \begin{pmatrix} 0 & 0.2449 \\ -0.2449 & 0 \end{pmatrix}.$$

Since $\|D\| = 0.2449$ then the known results can now be applied. The transformed system, and hence (29), is exponentially stable if

$$\|D\| + h \left\|S^{-1}GS\right\| \leq \|D\| + h \left\|S^{-1}\right\| \|G\| \|S\| < 1$$

holds. For $h = 1$, the above condition holds if $G \in \mathcal{R}^{2\times 2}$ is such hat $\|G\| <$ 0.1774. For this delay value, by selecting matrices $W_0 = I$ and $W_1 = 1.2I$, we get from the inequality (10) of Proposition 4 that (29) is exponentially stable if $G \in \mathcal{R}^{2\times 2}$ is such that $\|G\| < 0.2082$ which significantly improves the obtained by using the known stability condition combined with similarity transformations.

Example 2. *Consider now the uncertain system (15) with nominal system matrices*

$$A = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix} \text{ and } G = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}, \tag{33}$$

and unknown constant matrices $\Delta A, \Delta G$ satisfying

$$\|\Delta A\| \leq 0.2 \text{ and } \|\Delta G\| \leq 0.3. \tag{34}$$

This nominal system is found in Example 2 of [4] and [10] as a difference operator of a neutral type delay system. Such neutral system is obtained by a model transformation technique which transform the original neutral system with discrete delay to a neutral system with distributed delay for delay-dependent stability conditions, see [10] for details.

From Remark 6 we found that the nominal system is exponentially stable for all delay values $0 < h \leq 0.7435$ *that is significantly better that the result obtained from the inequality* $\|A\| + h\|G\| < 1$ *which leads to* $0 < h < 0.5658$.

Using Proposition 5 we found that the uncertain system (15) remains exponentially stable for all matrices $\Delta A, \Delta G$ *satisfying (34) if the delay value* $0 < h \leq 0.382$.

Now we illustrate how exponential estimates for the solutions of the perturbed system (15) can be computed by using Corollary 6. For $h = 0.3$ *and* $\lambda = 0.1$, *we obtain the following solutions of the matrix inequalities (17), (18) and (19):*

$$\begin{aligned} W_0 &= \begin{pmatrix} 0.0245 & -0.0001 \\ -0.0001 & 0.0244 \end{pmatrix}, W_1 = \begin{pmatrix} 0.1177 & 0.0066 \\ 0.0066 & 0.1208 \end{pmatrix}, \\ P &= \begin{pmatrix} 0.0866 & 0.0006 \\ 0.0006 & 0.0839 \end{pmatrix}. \end{aligned}$$

Direct calculations derived from (25), (26) and (27) yields $\alpha_1 = 0.0244$, $\alpha_2 = 0.0832$ *and* $\beta = 0.0083$. *Then, the corresponding constants* μ *and* α *in Corollary 6 take the values:* $\mu = 0.2728$ *and* $\alpha = 0.0496$.

For the matrix $A + \Delta A$ *the inequality (28) holds for* $\gamma = 1$ *and* $\sigma = 1.86$. *Thus, by assuming that* $\varepsilon \in (0, \min\{\sigma, \alpha\})$ *is equal to* 0.0248 *we arrive at the following exponential upper bound for the solutions of (15):*

$$\|x(t, \varphi)\| \leq 14.7538 \|\varphi\|_{0.3} e^{-0.0248t}, \quad \forall t \geq 0.$$

6. Conclusions

In this paper, Lyapunov-Krasovskii functionals for the exponential stability of some linear continuous time difference systems with discrete and distributed delay are introduced. Delay-dependent conditions for the exponential stability and exponential estimates for the solutions are derived. The new obtained stability conditions are less conservative than existing ones. As a consequence, the results reported in this contribution can help to improve existing stability conditions of neutral type delay systems having this class of continuous time difference systems as difference operators.

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8. Appendix

Proof. [Lemma 1]

(i) Function $z(t,\psi)$ satisfies for $t \geq 0$

$$\begin{aligned}\frac{d}{dt}\left[z(t,\psi) - Az(t-h,\psi))\right] &= G(0)z(t,\psi) - G(-h)z(t-h,\psi) \\ &\quad - \int_{-h}^{0} \dot{G}(\theta)z(t+\theta,\psi)d\theta.\end{aligned}$$

Integrating the equality from 0 to t and using the fact that

$$\int_{-h}^{0} \dot{G}(\theta)\left(\int_{\theta}^{t+\theta} z(\xi,\psi)d\xi\right)d\theta = G(0)\int_{0}^{t} z(\xi,\psi)d\xi \\ -G(-h)\int_{-h}^{t-h} z(\xi,\psi)d\xi - \int_{-h}^{0} G(\theta)\left[z(t+\theta,\psi) - z(\theta,\psi)\right]d\theta \tag{35}$$

we obtain

$$\begin{aligned}z(t,\psi) &= \left[\psi(0) - A\psi(-h) - \int_{-h}^{0} G(\xi)\psi(\xi)d\xi\right] + Az(t-h,\psi) \\ &\quad + \int_{-h}^{0} G(\theta)z(t+\theta,\psi)d\theta. \qquad (36)\end{aligned}$$

By definition

$$\psi(0) = A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta = A\psi(-h) + \int_{-h}^{0} G(\theta)\psi(\theta)d\theta.$$

It then follows that $z(t,\psi)$ satisfies (1).
Assume now that $x(t,\varphi), t \geq 0$, satisfies (1). Then, we have

$$x(t,\varphi) - Ax(t-h,\varphi) = \int_{-h}^{0} G(\theta)x(t+\theta,\varphi)d\theta,$$

which implies that the difference function $x(t,\varphi) - Ax(t-h,\varphi)$ is continuously differentiable and

$$\begin{aligned}\frac{d}{dt}\left[x(t,\varphi) - Ax(t-h,\varphi)\right] &= G(0)x(t,\varphi) - G(-h)x(t-h,\varphi) \\ &\quad - \int_{-h}^{0} \dot{G}(\theta)x(t+\theta,\varphi)d\theta.\end{aligned}$$

This means that $x(t,\varphi)$ satisfies (3).
By definition function $\varphi(\theta)$ coincides with $\psi(\theta)$ for $\theta \in [-h,0)$, and

$$x(0,\varphi) = z(0,\psi) = \psi(0) = A\varphi(-h) + \int_{-h}^{0} G(\theta)\varphi(\theta)d\theta.$$

(ii) For $\psi \in \mathcal{PC}([-h,0],\mathcal{R}^n)$, the solution $z(t,\psi)$ of (3) satisfies (36). Now, function $y_1(t,\tilde{\varphi}_1)$ satisfies

$$\dot{y}_1(t,\tilde{\varphi}_1) = G(0)y_2(t,\tilde{\varphi}_2) - G(-h)y_2(t-h,\tilde{\varphi}_2) - \int_{-h}^{0} \dot{G}(\theta)y_2(t+\theta)d\theta.$$

Integrating the equality from 0 to t and taking into account the equality (35) we get

$$y_1(t,\tilde{\varphi}_1) = \left[\tilde{\varphi}_1(0) - \int_{-h}^{0} G(\theta)\tilde{\varphi}_2(\xi)d\xi\right] + \int_{-h}^{0} G(\theta)y_2(t+\theta,\tilde{\varphi}_2)d\theta,$$

hence, from (5), function $y_2(t,\tilde{\varphi}_2)$ satisfies

$$\begin{aligned}y_2(t,\tilde{\varphi}_2) &= \left[\tilde{\varphi}_1(0) - \int_{-h}^{0} G(\theta)\tilde{\varphi}_2(\xi)d\xi\right] + Ay_2(t,\tilde{\varphi}_2) \\ &\quad + \int_{-h}^{0} G(\theta)y_2(t+\theta,\tilde{\varphi}_2)d\theta. \qquad (37)\end{aligned}$$

By definition $\tilde{\varphi}_2(\theta) = \psi(\theta), \theta \in [-h,0]$ and $\tilde{\varphi}_1 \in \mathcal{PC}([-h,0],\mathcal{R}^n)$ is such $\tilde{\varphi}_1(0) = \psi(0) - A\psi(-h)$. It then follows from (36) and (37) that $y_2(t,\tilde{\varphi}_2) = z(t,\psi)$.

■

Lemma 7. *For matrices A, B, C and D the following inequality holds:*

$$\begin{bmatrix} u^T & v^T \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} P \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq \lambda \begin{bmatrix} u^T & v^T \end{bmatrix} \times$$

$$\times \left((\left\|A^T\right\| \|C\| + \kappa) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + (\left\|B^T\right\| \|D\| + \kappa) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix},$$

where $\kappa = \frac{1}{2}\left\|A^T\right\| \|D\| + \frac{1}{2}\left\|B^T\right\| \|C\|$ and $\lambda > 0$ satisfying $P < \lambda I$.

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