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Generation of multi-scroll attractors without equilibria via piecewise linear systems

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In this paper, we present a new class of dynamical system without equilibria which possesses a multiscroll attractor. It is a piecewise-linear system which is simple, stable, displays chaotic behavior and serves as a model for analogous non-linear systems. We test for chaos using the 0-1 Test for Chaos from Gottwald and Melbourne [SIAM J. Appl. Dyn. Syst. **8**(1), 129–145 (2009)].
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Piecewise-linear (PWL) systems are switching systems composed of linear affine subsystems along with a rule which determines the acting subsystem. These systems are known to be capable of producing chaotic attractors, as in the well studied Chua's circuit. This paper discusses the generation of multiscroll attractors without equilibria based on PWL systems.

I. INTRODUCTION

In recent years, the study of dynamical systems with complicated dynamics but without equilibria has attracted attention. Since the first dynamical system of this kind with a chaotic attractor was introduced in Ref. 1 (Sprott case A), several works have investigated this topic. Several three-dimensional (3D) autonomous dynamical systems which exhibit chaotic attractors and whose associated vector fields present quadratic nonlinearities have been reported, for example, Ref. 2 and the 17 NE systems given in Ref. 3. Also, four-dimensional (4D) autonomous systems have been exhibited, such as the four-wing non-equilibrium chaotic system Ref. 7 and the systems with hyperchaotic attractors (equivalently two positive Lyapunov exponents) in Refs. 5 and 6, whose vector fields have quadratic and cubic nonlinearities.

These studies have not been restricted to integer orders; in Ref. 4, a new fractional-order chaotic system without equilibrium points was presented; this system represents the fractional-order counterpart of the integer-order system NE₆ studied in Ref. 3.

The first piecewise linear system without equilibrium points with an hyperchaotic attractor was reported in Ref. 8. This system was based on the diffusion-less Lorenz system by approximating the quadratic nonlinearities with the sign and absolute value functions. Recently, in Ref. 9, a class of PWL dynamical systems without equilibria was reported. The attractors exhibited by these systems can be easily shifted along the x axis by displacing the switching plane.

In fact, the attractors generated by these systems without equilibria fulfill the definition of hidden attractor given in Ref. 10.

In this paper, we present a new class of PWL dynamical system without equilibria that generates a multiscroll attractor. It is based on linear affine transformations of the form $Ax + B$, where A is a 3×3 matrix with two complex conjugate eigenvalues with positive real part and the third eigenvalue either zero, slightly negative or positive. The construction for A having a zero eigenvalue is presented in Section II, and the case when no eigenvalue of A is zero due to a perturbation is presented in Section III; numerical simulations are provided for both cases.

II. PWL DYNAMICAL SYSTEM WITH MULTISCROLL ATTRACTOR: SINGULAR MATRIX CASE

We will first present the construction where A has two complex conjugate eigenvalues with positive real part and the third eigenvalue is zero. In section III, we will study the effect of allowing the zero eigenvalue to become slightly negative $-\epsilon$ or slightly positive ϵ for a parameter $\epsilon > 0$. In order to introduce the new class of PWL dynamical system, we first review some useful theorems used in the construction. The next theorem gives necessary and sufficient conditions for the absence of equilibria in a system based on a linear affine transformation. These systems will be used as sub-systems for the PWL construction.

Theorem 1.⁹ *Given a dynamical system based on an affine transformation of the form $\dot{x} = Ax + B$, where $x \in \mathbb{R}^n$ is the state vector, $B \in \mathbb{R}^n$ is a nonzero constant vector, and $A \in \mathbb{R}^{n \times n}$ is a linear operator, the system possesses no equilibrium point if and only if*

- A is not invertible and
- B is linearly independent of the set of non-zero vectors comprised by columns of the operator A .

The next theorem tells us, for a specific type of linear operator, which vectors are linearly independent of the non-zero column vectors of the associated matrix of the operator which will help us to fulfill the conditions in Theorem 1.

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Theorem 2. Suppose $A \in \mathbb{R}^{3 \times 3}$ is a linear operator whose characteristic polynomial has zero as a single root (so that zero has algebraic multiplicity one). Then, any eigenvector associated with zero is linearly independent of the column vectors of the matrix A .

Proof. By considering the Jordan Canonical Form, there is an invariant subspace U of two dimensions corresponding to the other non-zero eigenvalues (either two real eigenvalues or a complex conjugate pair of eigenvalues $a + ib, a - ib$ with $b \neq 0$) and a one dimensional invariant subspace V corresponding to the eigenvalue $\lambda = 0$. Thus, $U \oplus V = \mathbb{R}^3$ and A restricted to U is invertible; hence, $AU \subset U$. If $x \in \mathbb{R}^3$, then we may write (uniquely) $x = x_u \oplus x_v$, where $x_u \in U$ and $x_v \in V$. Then, $Ax = Ax_u + Ax_v = Ax_u$ as $Ax_v = 0$. Since $Ax_u \subset U$, we cannot solve $Ax = w$, where $w \in V$ is a nonzero vector. \square

We will illustrate our mechanism for producing multi-scroll chaotic attractors via a detailed example.

A. Example

Consider a dynamical system whose associated vector field is the linear system

$$\dot{x} = Ax, \tag{1}$$

where $x \in \mathbb{R}^3$ is the state vector, and $A \in \mathbb{R}^{3 \times 3}$ is a linear operator. The matrix A has eigenvalues $\lambda_i, i = 1, 2, 3$, where λ_1, λ_2 are complex conjugate eigenvalues with positive real part while $\lambda_3 = 0$.

To illustrate this class of dynamical systems with multi-scroll attractors, consider the model system with a vector field of the form (1) in \mathbb{R}^3 whose matrix A is given as follows:

$$A = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = [a_1, a_2, a_3], \tag{2}$$

where a_1, a_2 , and a_3 are the column vectors of the matrix A and we suppose $m > 0$ and $n \neq 0$. The eigenvectors V associated with the eigenvalue $\lambda = 0$ are given as follows:

$$V = (0, 0, v)^T, \tag{3}$$

with $v \neq 0$.

Note that A is of rank two and its column space equals the two-dimensional unstable subspace $\langle a_1, a_2 \rangle$. We will now consider the vector field formed by adding a vector $k_1 a_1 + k_2 a_2$ in the span of the column vectors of the linear operator A

$$\dot{x} = Ax + k_1 a_1 + k_2 a_2. \tag{4}$$

Using the matrix A given by (2), we have the following linear system:

$$\dot{x} = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 + k_1 \\ x_2 + k_2 \\ x_3 \end{bmatrix}. \tag{5}$$

Considering the following change of variables $y_1 = x_1 + k_1, y_2 = x_2 + k_2$, and $y_3 = x_3$, the system (5) is given as follows:

$$\dot{y} = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Ay, \tag{6}$$

so that the rotation in the unstable subspace is around the axis given by the line

$$\begin{bmatrix} -k_1 \\ -k_2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If a non-zero vector V in the neutral direction is added, then the point

$$\begin{bmatrix} -k_1 \\ -k_2 \\ 0 \end{bmatrix}$$

is no longer an equilibrium, and the vector field at this point is equal to

$$V = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix},$$

so that

$$\dot{y} = Ay + V. \tag{7}$$

The solution of the initial value problem X_0 for (7) is given by

$$y(t) = \begin{bmatrix} e^{mt} \cos(nt) & -e^{mt} \sin(nt) & 0 \\ e^{mt} \sin(nt) & e^{mt} \cos(nt) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v * t \end{bmatrix}, \tag{8}$$

$$x(t) = e^{mt} \begin{bmatrix} \cos(nt) & -\sin(nt) & 0 \\ \sin(nt) & \cos(nt) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) + k_1 \\ x_2(0) + k_2 \\ x_3(0) \end{bmatrix} + \begin{bmatrix} -k_1 \\ -k_2 \\ v * t \end{bmatrix}. \tag{9}$$

We wish to generate a ‘‘saddle-focus like’’ behavior which bounds motion, and so, a piecewise linear system is constructed as follows. A switching surface S will be a hyperplane oriented by its positive normal. S divides \mathbb{R}^3 into two connected components, if a point x lies in the component pointed to by the positive normal of S we will write $x > S$.

To illustrate we take our switching surface to be $x_3 = 0$ and define a flow on S by

$$\dot{x} = \begin{cases} Ax + W, & \text{if } x_1 < 0; \\ Ax - W, & \text{if } x_1 \geq 0. \end{cases} \tag{10}$$

The vector W is chosen in the plane S (hence in the unstable subspace of A), with sign such that the x_1 component of $p_1 := -A|_S^{-1}W$ is positive. This ensures that the flow on $S \cap \{x_1 < 0\}$ has an unstable focus at p_1 , while the flow on $S \cap \{x_1 \geq 0\}$ has an unstable focus at $-p_1$. Thus, there is no equilibria for the PWL flow on S . Next, we define the flow on $x > S$ and $x < S$

$$\dot{x} = \begin{cases} Ax + V + W, & \text{if } x < S, x_1 < 0; \\ Ax + V - W, & \text{if } x < S, x_1 \geq 0; \\ Ax - V + W, & \text{if } x > S, x_1 < 0; \\ Ax - V - W, & \text{if } x > S, x_1 \geq 0; \end{cases} \quad (11)$$

where we choose the direction of V so that the dynamical system defined by (11) $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ gives the motion of the points towards the switching plane S .

See Figure 1 where $W = k_1 a_1 + k_2 a_2$ and the plane S is given by $x_3 = \tau$. With a pair of systems $F_1(x), F_3(x)$ defined by parallel switching surfaces S_1, S_3 , respectively, similar to those described by (10) and (11) along with another switching surface S_2 transverse to the other two, it is possible to generate multiscroll attractors. For example, a double scroll attractor can be generated with the following Piecewise Linear System provided that $F_3(x)$ is correctly displaced:

$$\dot{x} = \begin{cases} F_1(x), & \text{if } x < S_2; \\ F_3(x), & \text{if } x \geq S_2. \end{cases} \quad (12)$$

Note that switching surfaces S_1 and S_3 are used to generate two “saddle-focus like” behavior, and switching surface S_2 is used to commute between these two “saddle-focus like” behavior (see Fig. 2). The surface S_2 is responsible for the stretching and folding behavior in the system in order to generate chaos. This switching surface is constrained to be transverse to the unstable manifold and neutral manifold. Note that the planes S_1, S_2 , and S_3 are not invariant under the flow defined by (12).

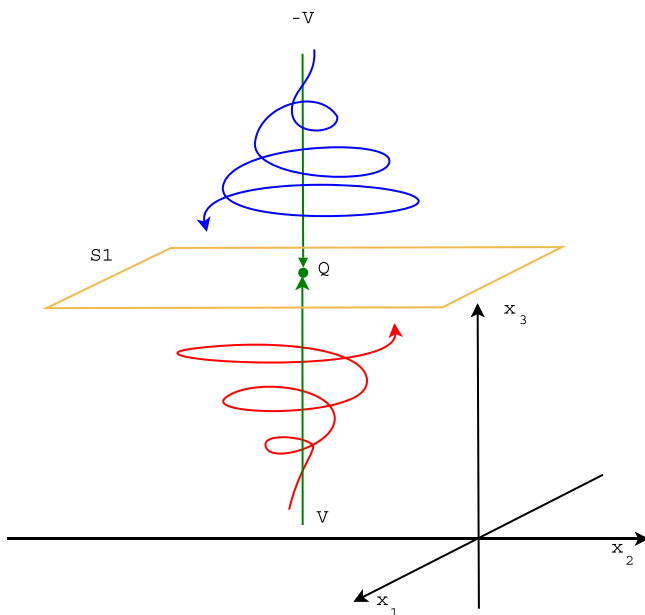


FIG. 1. A “saddle-focus like” locate at the point $Q = (-k_1, -k_2, \tau)$.

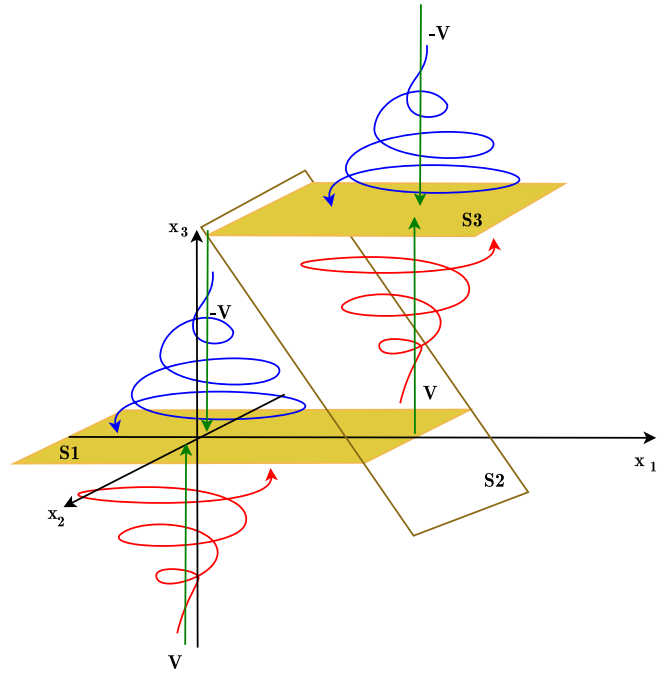


FIG. 2. A mechanism to generate a double-scroll chaotic attractors without equilibria.

It is possible to define a system with 3-scroll attractors, for example, as follows (see Fig. 3):

$$\dot{x} = \begin{cases} F_1(x), & \text{if } x < S_2; \\ F_2(x), & \text{if } S_2 \leq x < S_4; \\ F_3(x), & \text{if } x \geq S_4. \end{cases} \quad (13)$$

Such a system demonstrates the chaotic nature of the associated flow via symbolic dynamics. If we take small ϵ -neighborhoods of the surfaces S_1, S_3 , and S_5 and record a 1,

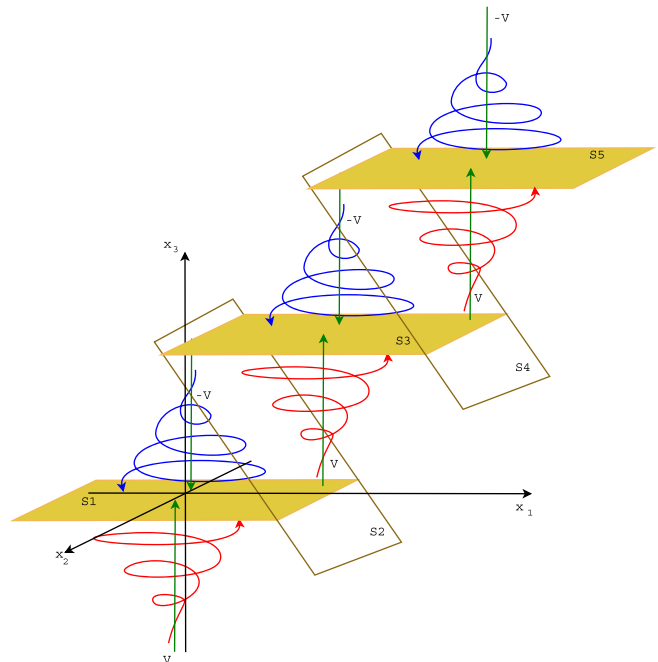


FIG. 3. A mechanism to generate a triple-scroll chaotic attractors without equilibria.

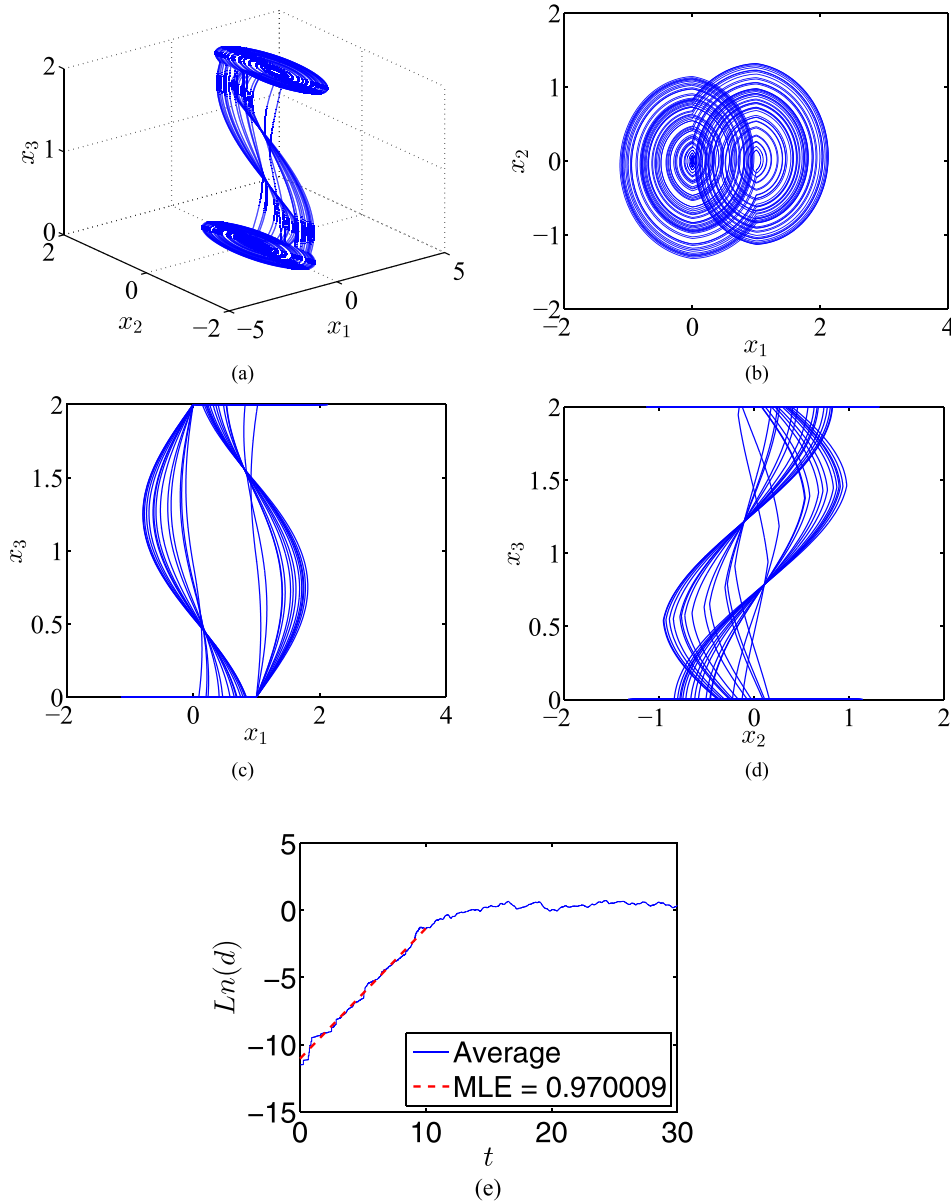


FIG. 4. Attractor of the system (15) with A and V given in (14) for the initial condition $(0, 0, 0)$ in (a) the space (x_1, x_2, x_3) and its projections onto the planes: (b) (x_1, x_2) , (c) (x_1, x_3) , and (d) (x_2, x_3) . In (e), the largest Lyapunov exponent.

3, or 5 each time a trajectory under the flow crosses into that neighborhood from outside that neighborhood we generate a shift on the space $\{1, 3, 5\}^N$. Both the symbol sequences 135, 131 and correspondingly 535 and 531 may occur, and the symbol sequences generated are complex, non-periodic and we call them “chaotic.” How disordered the system is in terms of entropy, for example, is a subject for another paper.

B. Numerical simulations

Example 1. Figure 4 shows a double scroll attractor which was obtained by using 4th order Runge Kutta (0.01 integration step) and considering the following matrix A and vector V :

$$A = \begin{bmatrix} 0.5 & -10 & 0 \\ 10 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \quad (14)$$

with PWL system (details can be seen in Appendix A)

$$\dot{x} = F_i(x), \quad i = 1, \dots, 12. \quad (15)$$

The switching surface are given by the planes $S_1 : x_3 = 0$, $S_2 : x_1 + x_3/2 = 1$ and $S_3 : x_3 = 2$. The vectors W_i , with $i = 1, \dots, 4$ are given in Table III (see Appendix A).

The resulting attractor with a double scroll is shown in the Figure 4(a) and its projections on the planes (x_1, x_2) , (x_1, x_3) , and (x_2, x_3) in Figures 4(b)–4(d), respectively. Its largest Lyapunov exponent calculated is $\lambda = 0.97$ (Figure 4(e)).

It can be extended to a triple scroll attractor by considering the system given by (13) with the additional surfaces $S_4 : x_1 + x_3/2 = 3$ and $S_5 : x_3 = 4$ and W_i , with $i = 1, \dots, 6$ given in Table III (see Appendix A).

The resulting attractor with a triple scroll is shown in the Figure 5(a) and its projections on the planes (x_1, x_2) , (x_1, x_3) and (x_2, x_3) in Figures 5(b)–5(d), respectively. Its largest Lyapunov exponent calculated is $\lambda = 1.056$ (Figure 5(e)).

Assigning the numbers 1, 3, and 5 to the regions where $x < S_2$, $S_2 \leq x < S_4$, and $x \geq S_4$, respectively, the symbol

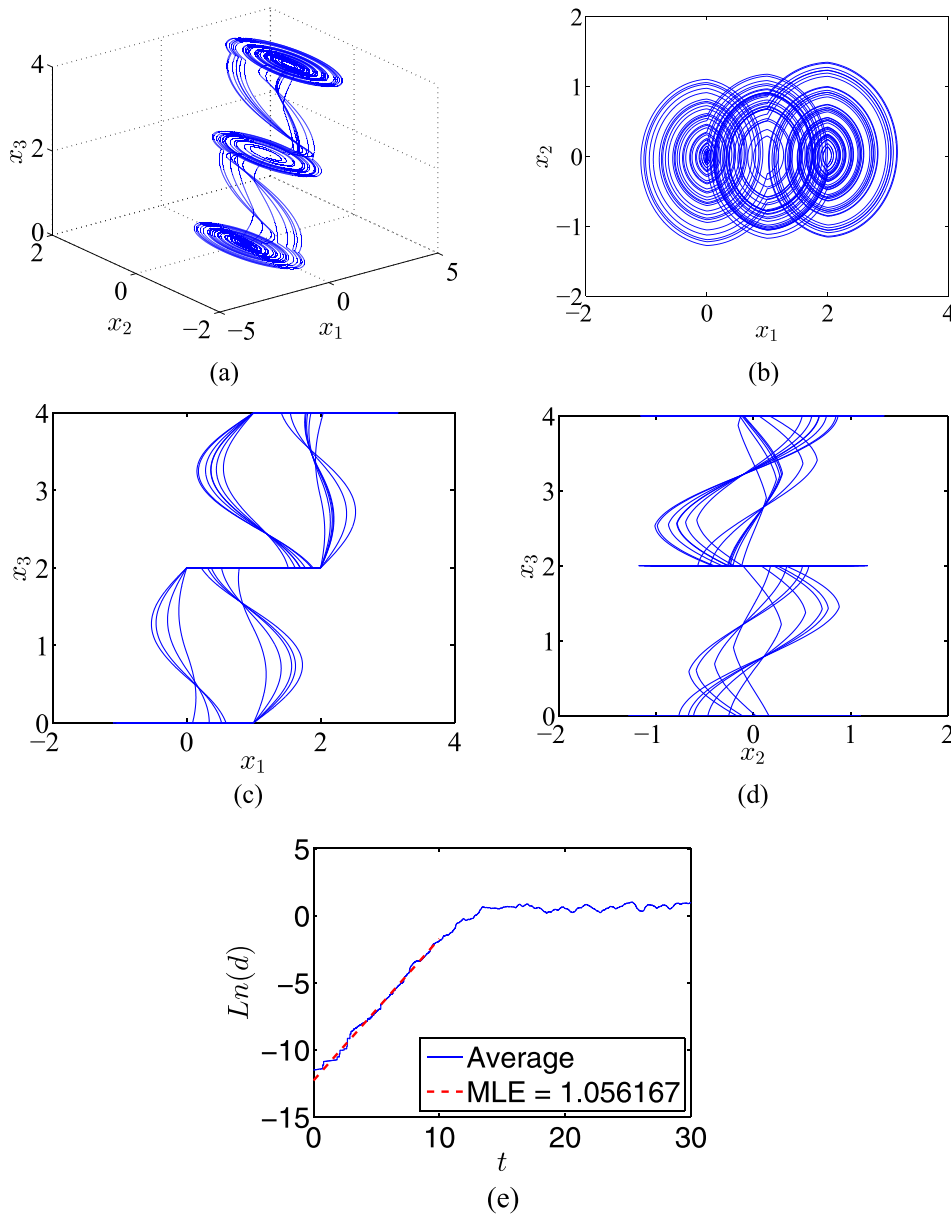


FIG. 5. Triple scroll attractor for the initial condition $(0, 0, 0)$ in (a) the space (x_1, x_2, x_3) and its projections onto the planes: (b) (x_1, x_2) , (c) (x_1, x_3) , and (d) (x_2, x_3) . In (e), the largest Lyapunov exponent.

sequences 135, 131, 535, and 531 may be produced. Table I shows three sequences for different close initial conditions.

Later, we show that this system passes the 0–1 Test for Chaos of Ref. 11, justifying our description of chaotic motion.

III. SMALL PERTURBATIONS OF THE ZERO EIGENVALUE: PWL DYNAMICAL SYSTEM $\dot{x} = Ax + B$ WITH MULTISCROLL ATTRACTOR AND INVERTIBLE A

Now we consider perturbing the real eigenvalue. Consider the matrix A_η , where

$$A = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & \eta \end{bmatrix}, \quad A = [a_1, a_2, a_3], \quad (16)$$

where a_1, a_2 , and a_3 are the column vectors of the matrix A and we suppose $m > 0$ and $n \neq 0$. The eigenvectors V associated with the eigenvalue $\lambda = \eta$ are given as follows:

$$V = (0, 0, v)^T, \quad (17)$$

with $v \neq 0$.

The column space of A equals the two-dimensional unstable subspace $\langle a_1, a_2 \rangle$. As before, we consider the vector field formed by adding a vector $k_1 a_1 + k_2 a_2$ in the span of the column vectors.

$$\dot{x} = Ax + k_1 a_1 + k_2 a_2. \quad (18)$$

TABLE I. Sequences produced by the system from example 1 starting from three close to zero initial conditions for a 50 s simulation with a RK4 of step = 0.01.

$x(0)$	Sequence
$(0, 0, 0, 0, 0)$	13135313135353531313535
$(0, 0, 0, 1, 0, 0)$	131313535313135313135
$(-0.1, 0, -0.1)$	13135353135313531353531

Using the matrix A given by (16), we have the following linear system:

$$\dot{x} = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} x_1 + k_1 \\ x_2 + k_2 \\ x_3 \end{bmatrix}. \tag{19}$$

As before the solution of the initial value problem is given by

$$x(t) = \begin{bmatrix} e^{mt} \cos(nt) & -e^{mt} \sin(nt) & 0 \\ e^{mt} \sin(nt) & e^{mt} \cos(nt) & 0 \\ 0 & 0 & e^{\eta t} \end{bmatrix} \times \begin{bmatrix} x_1(0) + k_1 \\ x_2(0) + k_2 \\ x_3(0) \end{bmatrix} + \begin{bmatrix} -k_1 \\ -k_2 \\ \frac{v}{\eta}(e^{\eta t} - 1) \end{bmatrix}. \tag{20}$$

Consider the equation $\dot{x} = \eta x + w$ which has solution $x(t) = e^{\eta t} \left[x(0) + \frac{w}{\eta} \right] - \frac{w}{\eta}$. Suppose that the sign of v is such

that the flow is directed toward S_1 in Figure 1. So, $\dot{x}_3 = \eta x_3 - |v|$ if $x_3(0) > 0$ and $\dot{x}_3 = \eta x_3 + |v|$ if $x_3(0) < 0$. If $\eta < 0$ and $x_3(0) > 0$ then $\lim_{t \rightarrow \infty} x_3(t) = \frac{|v|}{\eta}$, so that the flow in Figure 1 still intersects S_1 from an initial condition $x_3(0) > 0$. Similarly, if $x_3(0) < 0$ then the flow intersects S_1 . Thus, the topological structure of the flow is unchanged for small $|\eta|$, $\eta < 0$

However, if $\eta > 0$ then $\lim_{t \rightarrow \infty} x_3(t) \rightarrow \infty$ if $x_3(0) > \frac{|v|}{\eta}$ and $\lim_{t \rightarrow \infty} x_3(t) \rightarrow -\infty$ for $x_3(0) < -\frac{|v|}{\eta}$. So, in this case, there is a neighborhood of S_1 of points closer than $\frac{|v|}{\eta}$ to S_1 consisting of points attracted to S_1 . Otherwise, points are repelled from S_1 .

Consider now the solution where the sign of v is such that the flow is directed away from S_1 in Figure 1. So, $\dot{x}_3 = \eta x_3 + |v|$ if $x_3(0) > 0$ and $\dot{x}_3 = \eta x_3 - |v|$ if $x_3(0) < 0$.

Clearly, if $\eta > 0$ and $x_3(0) > 0$ then $\lim_{t \rightarrow \infty} x_3(t) = \infty$ and if $\eta > 0$ and $x_3(0) < 0$, then $\lim_{t \rightarrow \infty} x_3(t) = -\infty$. However, if $\eta < 0$ and $x_3(0) > 0$ then $\lim_{t \rightarrow \infty} x_3(t) = \frac{-|v|}{\eta}$ and if $\eta < 0$ and $x_3(0) < 0$ then $\lim_{t \rightarrow \infty} x_3(t) = \frac{|v|}{\eta}$. Thus, for small $|\eta|$ the multi-scroll attractors persist.

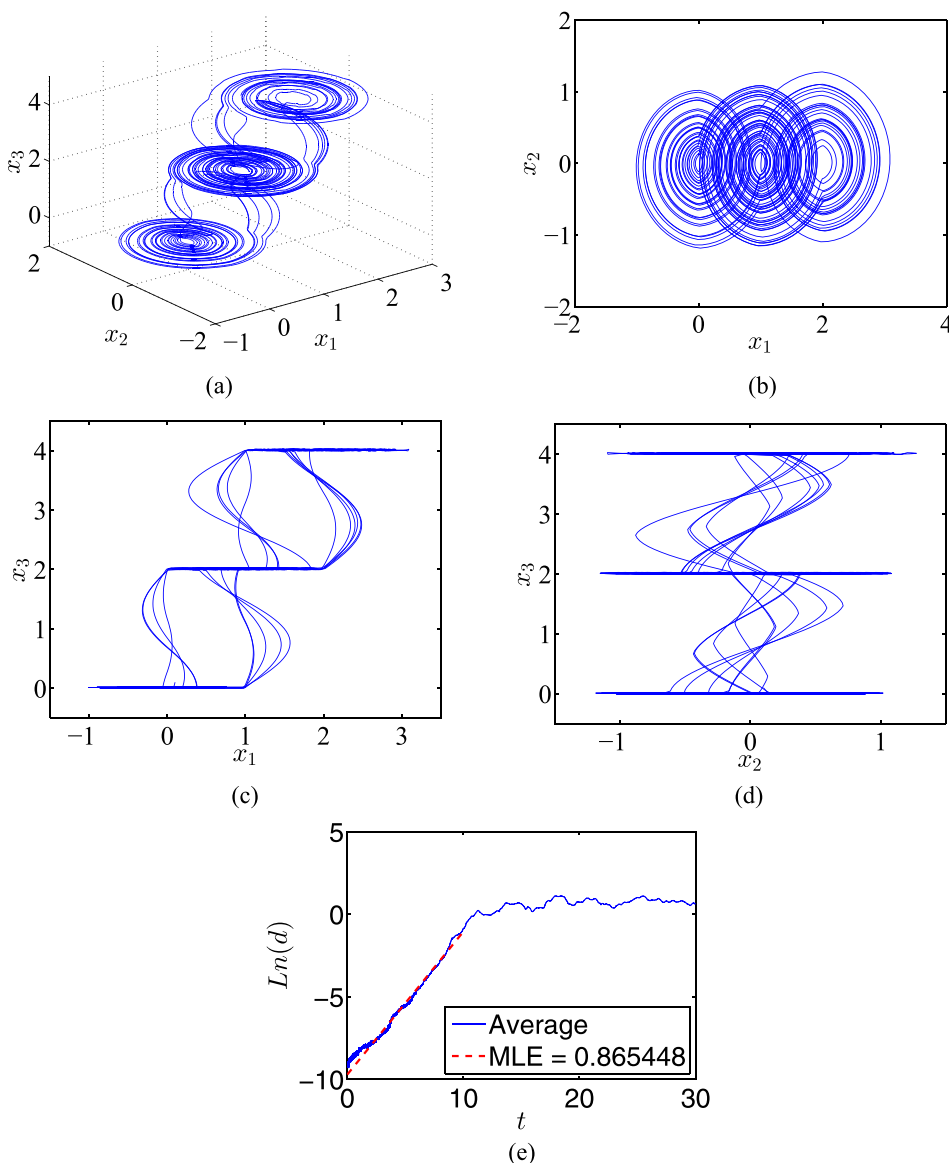


FIG. 6. Attractor of the system (21) with A given in (22) for the initial condition $(0.1, 0.1, 0.1)$ in (a) the space (x_1, x_2, x_3) and its projections onto the planes: (b) (x_1, x_2) , (c) (x_1, x_3) , and (d) (x_2, x_3) . In (e), the largest Lyapunov exponent.

A. Numerical simulations

Example 2. As an example of this construction with an invertible matrix, consider the system described by

$$\dot{x} = F_i(x), \quad i = 1, \dots, 18. \quad (21)$$

Details of the function (21) are given in Appendix B. The linear operator and the vector V are given as follows:

$$A = \begin{bmatrix} 0.5 & -10 & 0 \\ 10 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}. \quad (22)$$

The switching surfaces are given by the planes $S_1 : x_3 = 0, S_2 : x_1 + x_3/2 = 1, S_3 : x_3 = 2, S_4 : x_1 + x_3/2 = 3, S_5 : x_3 = 4$. The vectors W_i , with $i=1, \dots, 6$ are given in Table III (see Appendix A).

The resulting attractor obtained by using a 4th order Runge Kutta (0.01 integration step) is shown in Figure 6.

Assigning the numbers 1, 3, and 5 to the regions where $x < S_2, S_2 \leq x < S_4$, and $x \geq S_4$, respectively, the symbol sequences 135, 131, 535, and 531 may be produced. Table II shows three sequences generated from different close to zero initial conditions.

As for the previous system, we show in Sec. IV that this system passes the 0–1 Test for Chaos of Ref. 11, justifying our description of chaotic motion.

IV. DYNAMICS OF THE PROPOSED SYSTEMS

To test for chaotic dynamics in the systems we have investigated, we used the test algorithm proposed in Ref. 11. The input for the test is a one dimensional time series $\phi(n)$ which drives a two-dimensional system $PQ(\phi(n), c)$ as described in Ref. 11, namely,

$$p_c(n) = \sum_{j=1}^n \phi(j) \cos(jc),$$

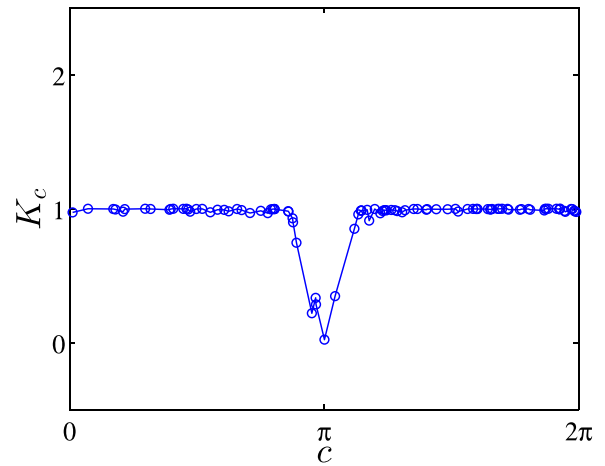
$$q_c(n) = \sum_{j=1}^n \phi(j) \sin(jc),$$

where $c \in (0, 2\pi)$ is a real parameter. The rate of growth of the variance of this system distinguishes between chaotic ($K=1$) and regular motion ($K=0$) as determined by a derived quantity K .

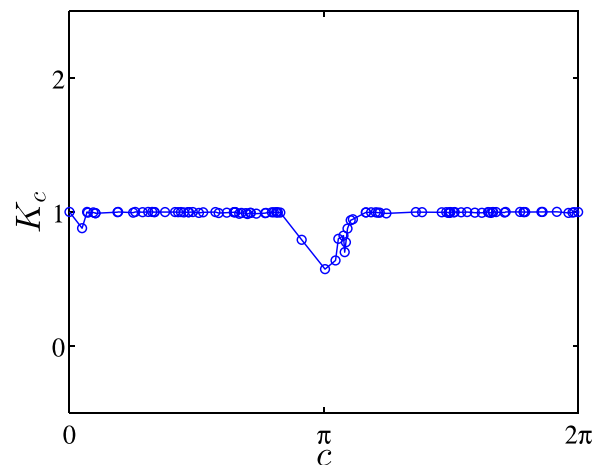
For both systems with a triple scroll attractor previously introduced, three-dimensional time series were generated by means of a RK4 integrator with a time step equal to 0.01, which was then sampled T time $\tau=0.25$ to get three-

TABLE II. Sequences produced by the system from example 2 starting from three close to zero initial conditions for a 50 s simulation with a RK4 of step = 0.01.

$x(0)$	Sequence
(0.1,0.1,0.1)	123232121232121232323
(0.0, 0.1, 0.0)	1212321232121212123
(-0.1, 0, -0.1)	123212121212321232321



(a)



(b)

FIG. 7. Asymptotic growth rates K_c calculated for the triple scroll attractor systems presented in Secs. II(a) and III(b).

dimensional time series of length $N=2000$. The one-dimensional time-series was given by $\phi \circ T^n(p_0)$ for initial conditions $p_0 = (x_1^0, x_2^0, x_3^0)$ and $\phi(x_1, x_2, x_3) = x_3$, so that we were observing the z -component of a trajectory under the time-T map of the flow.

The growth rates calculated were $K=0.9930$ and $K=0.9962$ for the first and second example, respectively. As described in Ref. 11, K is calculated as the median value of the asymptotic growth rates of a growth rate K_c for different values of the parameter c belonging to the two-dimensional driven system. In Figures 7(a) and 7(b), the K_c values are shown. Thus, our system, according to the 0–1 test, is chaotic.

TABLE III. Vectors W_i for $i = 1, \dots, 6$.

W_i	k_1	k_2
W_1	-0.1	0
W_2	0.1	0
W_3	-1.1	0
W_4	-0.9	0
W_5	-2.1	0
W_6	-1.9	0

V. CONCLUSION

In this paper, new classes of piecewise linear dynamical systems (with no equilibria) which present multiscroll attractors were introduced. The systems are very simple geometrically and stable to perturbation. The attractors generated by these classes have a similar chaotic behavior. Further investigation, via perhaps symbolic dynamics, is needed to quantify the disorder of the system.

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APPENDIX A: PWL DESCRIPTION OF A SYSTEM WITH DOUBLE SCROLL ATTRACTOR

System described by (15)

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{Ax} + V + W_1, & \text{if } \mathbf{x} < S_1, x_1 < 0; \\ \mathbf{Ax} + V + W_2, & \text{if } \mathbf{x} < S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} + W_1, & \text{if } \mathbf{x} \in S_1, x_1 < 0; \\ \mathbf{Ax} + W_2, & \text{if } \mathbf{x} \in S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} - V + W_1, & \text{if } \mathbf{x} > S_1, \mathbf{x} < S_2, x_1 < 0; \\ \mathbf{Ax} - V + W_2, & \text{if } \mathbf{x} > S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} + V + W_3, & \text{if } \mathbf{x} < S_3, \mathbf{x} \geq S_2, x_1 < 1; \\ \mathbf{Ax} + V + W_4, & \text{if } \mathbf{x} < S_3, \mathbf{x} \geq S_2, x_1 \geq 1; \\ \mathbf{Ax} + W_3, & \text{if } \mathbf{x} \in S_3, \mathbf{x} \geq S_2, x_1 < 1; \\ \mathbf{Ax} + W_4, & \text{if } \mathbf{x} \in S_3, x_1 \geq 1; \\ \mathbf{Ax} - V + W_3, & \text{if } \mathbf{x} > S_3, \mathbf{x} \geq S_2, x_1 < 1; \\ \mathbf{Ax} - V + W_4, & \text{if } \mathbf{x} > S_3, x_1 \geq 1. \end{cases} \quad (\text{A1})$$

The switching surface are given by the planes $S_1 : x_3 = 0$, $S_2 : x_1 + x_3/2 = 1$ and $S_3 : x_3 = 2$. The vectors W_i , with $i = 1, \dots, 4$ are given in Table III.

APPENDIX B: PWL DESCRIPTION OF A SYSTEM WITH TRIPLE SCROLL ATTRACTOR

System described by (21)

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{Ax} + V + W_1, & \text{if } \mathbf{x} < S_1, x_1 < 0; \\ \mathbf{Ax} + V + W_2, & \text{if } \mathbf{x} < S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} + W_1, & \text{if } \mathbf{x} \in S_1, x_1 < 0; \\ \mathbf{Ax} + W_2, & \text{if } \mathbf{x} \in S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} - V + W_1, & \text{if } \mathbf{x} > S_1, \mathbf{x} < S_2, x_1 < 0; \\ \mathbf{Ax} - V + W_2, & \text{if } \mathbf{x} > S_1, \mathbf{x} < S_2, x_1 \geq 0; \\ \mathbf{Ax} + V + W_3, & \text{if } \mathbf{x} < S_3, \mathbf{x} \geq S_2, x_1 < 1; \\ \mathbf{Ax} + V + W_4, & \text{if } \mathbf{x} < S_3, \mathbf{x} \geq S_2, \mathbf{x} < S_4, x_1 \geq 1; \\ \mathbf{Ax} + W_3, & \text{if } \mathbf{x} \in S_3, \mathbf{x} \geq S_2, x_1 < 1; \\ \mathbf{Ax} + W_4, & \text{if } \mathbf{x} \in S_3, \mathbf{x} < S_4, x_1 \geq 1; \\ \mathbf{Ax} - V + W_3, & \text{if } \mathbf{x} > S_3, \mathbf{x} \geq S_2, \mathbf{x} < S_4, x_1 < 1; \\ \mathbf{Ax} - V + W_4, & \text{if } \mathbf{x} > S_3, \mathbf{x} < S_4, x_1 \geq 1; \\ \mathbf{Ax} + V + W_5, & \text{if } \mathbf{x} < S_5, \mathbf{x} \geq S_4, x_1 < 2; \\ \mathbf{Ax} + V + W_6, & \text{if } \mathbf{x} < S_5, \mathbf{x} \geq S_4, x_1 \geq 2; \\ \mathbf{Ax} + W_5, & \text{if } \mathbf{x} \in S_5, \mathbf{x} \geq S_4, x_1 < 2; \\ \mathbf{Ax} + W_6, & \text{if } \mathbf{x} \in S_5, x_1 \geq 2; \\ \mathbf{Ax} - V + W_5, & \text{if } \mathbf{x} > S_5, \mathbf{x} \geq S_4, x_1 < 2; \\ \mathbf{Ax} - V + W_6, & \text{if } \mathbf{x} > S_5, x_1 \geq 2. \end{cases} \quad (\text{B1})$$

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