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Modification of Mikhailov stability criterion for fractional commensurate order systems

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Abstract

In this paper we present the modification of the Mikhailov stability criterion for linear fractional commensurate order systems. The modification consists in determining the appropriate measure for the total argument change depending on the highest fractional order $\alpha_n = n\alpha$ of the system and not only on the integer n as stated in the literature. The validity of the result is illustrated by means of several examples.

Keywords: Fractional commensurate systems, stability, Mikhailov criterion

1. Introduction and problem formulation

The use of fractional calculus for modelling physical systems has attracted increasing attention in last decades [1, 2, 3]. In particular, the application of fractional calculus in control theory has recently become an active area, where one of the most fundamental problems is the stability analysis of fractional order systems, see, for instance, [4, 5, 6, 7].

It is a well-known result, firstly proved by Matignon [8] and then generalized by Bonnet and Partington [9], that a linear fractional order system is asymptotically stable if and only if all the roots of the characteristic pseudo-polynomial associated to the system lie in the open left half of the complex plane. However, the application of such a result demands to determine the roots distribution of pseudo-polynomials which, in the general case, is a difficult task. Therefore, it is desirable to have some methods of determining the

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stability without solving the roots of pseudo-polynomials. In this sense, some classical methods for the stability of linear integer order systems have been extended to the stability analysis of fractional order systems see, for instance, [6] for a Nyquist type result and the recent paper [10] for a Routh-Hurwitz type criterion.

Among other frequency criteria, a Mikhailov type stability criterion for a class of linear fractional order systems with delays was presented in [11]. Since the class of linear fractional commensurate order systems is a particular case of the class of delay systems considered in [11] it then appears that the Mikhailov result presented in [11] can be applied to investigate the stability of a linear fractional commensurate order system whose characteristic pseudo-polynomial is of the form

$$p(s) = a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_1 s^{\alpha_1} + a_0,$$

where $a_k, k = 1, 2, \ldots, n$, are real with $a_n \neq 0$ and $\alpha_k = k\alpha, k = 1, 2, \ldots, n$, for some $\alpha \in \mathbb{R}^+$. In fact, note that $p(s)$ is directly obtained from the quasipolynomial (4) in [11] by making all the delays equal to zero.

For a general description of roots distribution of pseudo-polynomials within the structure of Riemann surfaces, see the fundamental works of Matignon [8] and Bonnet and Partington [9]. Based on the results of [8] and [9], we here adopt the following definition of stability:

Definition 1. *The pseudo-polynomial $p(s)$ is said to be stable if all its roots lie in $\mathbb{C}_-$, the open left half of the complex plane.*

The main idea behind the classical Mikhailov criterion is to substitute $s = i\omega$ in $p(s)$ to get $p(i\omega) = u(\omega) + iv(\omega)$ and then measure the total variation of the argument of the function $p(i\omega)$ as ω increases from 0 to ∞. The corresponding plot of $p(i\omega)$ in the complex plane is the so-called Mikhailov curve. For the pseudo-polynomial $p(s)$ the Mikhailov type result (see Theorem 2 in [11] and also Theorem 9.3 in [13]) looks like follows:

Theorem 1. *[11] The pseudo-polynomial $p(s)$, with $0 < \alpha \leq 1$, is stable if and only if*

$$\Delta \arg p(i\omega)|_0^\infty = n\left(\frac{\pi}{2}\right),$$

which means that the plot of $p(i\omega)$ with ω increasing from 0 to $+\infty$ runs in the positive direction by n quadrants of the complex plane, missing the origin of this plane.

One immediately observes that the theorem's condition coincides with that of the classical Mikhailov condition for integer order systems which, in principle, is not expected due to the fractional nature of $p(s)$. However, when one plots the Mikhailov curve of some particular pseudo-polynomials of the form of $p(s)$ one realizes the following (as it will be shown later in the examples section of the paper):

1. The condition of the theorem is not correct, and
2. The Mikhailov curve does not always runs in the positive (counter-clockwise) direction.

These two main issues motivate us to search for the modified Mikhailov stability criterion for the pseudo-polynomial $p(s)$.We will show that the right condition for the Mikhailov stability criterion is $\alpha_n\left(\frac{\pi}{2}\right)$ and not $n\left(\frac{\pi}{2}\right)$ as stated in above theorem, a result that to the best of our knowledge has not been reported in the literature. We here present such a result not only when $0 < \alpha < 1$ but also when $1 \leq \alpha < 2$. The result is derived by following the main ideas of the Mikhailov criterion as viewed by Popov [12] and exploiting the fact that the stability of $p(s)$ can be determined by means of the root distribution of the integer order polynomial

$$\tilde{p}(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0, \tag{1}$$

in the region

$$M_\alpha = \left\{\lambda \in \mathbb{C} : \frac{\alpha\pi}{2} < \arg(\lambda) \leq \pi \text{ or } -\pi < \arg(\lambda) < -\frac{\alpha\pi}{2}\right\},$$

of the complex $\lambda-$plane, as it is stated by the following result:

Lemma 1. *[8] The pseudo-polynomial $p(s)$ is stable if and only if $0 < \alpha < 2$ and the n roots $\lambda_j, j = 1, 2, \ldots, n,$ of the polynomial $\tilde{p}(\lambda)$ lie in M_α.*

2. Main Results

Before to establish the Mikhailov criterion we present some necessary conditions for stability of pseudo-polynomials. For integer order polynomials, it is well-known that a necessary condition for Hurwitz polynomials, i.e. polynomials with all roots lying in $\mathbb{C}_-$, is to have all its coefficients non zero and of the same sign, either all positive or all negative. Such necessary condition is known as the Stodola stability criterion for polynomials see [14]. The following Lemma shows that the Stodola criterion holds for $p(s)$ when $1 \leq \alpha < 2$.

Lemma 2. *Let $1 \leq \alpha < 2$. If $p(s)$ is stable then all its coefficients are non zero and have the same sign.*

Proof. If $p(s)$ is stable then all roots of $\tilde{p}(\lambda)$ lie in M_α. Since $M_\alpha \subseteq \mathbb{C}_-$, it then follows that $\tilde{p}(\lambda)$ is a Hurwitz polynomial and, therefore, the result follows from the Stodola criterion. ■

Unfortunately, the Stodola criterion does not hold for $p(s)$ when $0 < \alpha < 1$ but, however, we are still able to establish a simple necessary condition for stability.

Lemma 3. *Let $0 < \alpha < 1$. If $p(s)$ is stable then the coefficients a_n and a_0 are non zero and have the same sign.*

Proof. Clearly, $a_n \neq 0$. If $p(s)$ is stable then all roots of $\tilde{p}(\lambda)$ lie in M_α. Since $M_\alpha \nsubseteq \mathbb{C}_-$, $\tilde{p}(\lambda)$ may has complex conjugated roots with positive real parts but, in particular, neither positive real nor zero roots. The no existence of zero roots of $\tilde{p}(\lambda)$ implies that $a_0 \neq 0$; otherwise we can write $\tilde{p}(\lambda) = \lambda^m \tilde{q}(\lambda)$, where $m < n$ and $\tilde{q}(\lambda)$ a certain polynomial of degree $n - m$ with a nonzero constant coefficient, which, in turn, it will imply the existence of a zero root of $\tilde{p}(\lambda)$. Let us suppose that $a_n > 0$ and $a_0 < 0$. For λ real we have

$$\tilde{p}(0) = a_0 < 0 \text{ and } \lim_{\lambda \to \infty} \tilde{p}(\lambda) = +\infty.$$

Then, the continuity of $\tilde{p}(\lambda)$ w.r.t. λ implies that $\tilde{p}(\lambda)$ has at least one positive real root which contradicts the stability of $p(s)$. Now, suppose that $a_n < 0$ and $a_0 > 0$. For λ real we have

$$\tilde{p}(0) = a_0 > 0 \text{ and } \lim_{\lambda \to \infty} \tilde{p}(\lambda) = -\infty$$

and, therefore, $\tilde{p}(\lambda)$ has at least one positive real root which again contradicts the stability of $p(s)$. Thus, we conclude that the coefficients a_n and a_0 are non zero and have the same sign as desired. ■

Remark 1. *It is important to note that the Lemma 3 is equivalent to Lemma 9.3 in [13] but here derived from a different line of arguments independently of the Mikhailov criterion.*

We now address the Mikhailov stability criterion. Firstly, we observe that from the Lemmas 2 and 3 one can assume without loss of generality that:

1) If $1 \leq \alpha < 2$ then all the coefficients of $p(s)$ are positive, and
2) If $0 < \alpha < 1$ then the coefficients a_n and a_0 are positive.

Hence, in any case, one has that $p(0) = a_0 > 0$ and, therefore, the Mikhailov curve always starts on the positive real axis, as occurs in the polynomial case. Now, consider the polynomials

$$\tilde{p}_0(\lambda) = (\lambda - \lambda_0) \text{ and } \tilde{p}_1(\lambda) = (\lambda - \zeta_1)\left(\lambda - \bar{\zeta}_1\right),$$

where $\lambda_0 \in \mathbb{R}, \lambda_0 \neq 0, \zeta_1 = \rho_1 e^{\phi_1 i}$ and $\bar{\zeta}_1 = \rho_1 e^{-\phi_1 i}$, with $\rho_1 > 0$ and $\phi_1 \in (0, \pi)$. For $0 < \alpha < 2$ and $r \in [0, \infty)$ let $\theta_0(r) = \arg\left(re^{\frac{\alpha\pi}{2}i} - \lambda_0\right), \theta_1(r) = \arg\left(re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{\phi_1 i}\right)$ and $\theta_2(r) = \arg\left(re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{-\phi_1 i}\right)$. Then

$$\begin{aligned}
\theta_0(r) &= \arg\left(\tilde{p}_0(re^{\frac{\alpha\pi}{2}i})\right), \\
\theta(r) &= \theta_1(r) + \theta_2(r) = \arg\left(\tilde{p}_1(re^{\frac{\alpha\pi}{2}i})\right).
\end{aligned}$$

The following two Lemmas characterize the total change in $\theta_0(r), \theta_1(r), \theta_2(r)$ and $\theta(r)$ as r varies from 0 to ∞, and it will be essential for deriving the Mikhailov stability result.

Lemma 4. *For $\theta_0(r)$ the following hold:*

1. *If $\lambda_0 < 0$ then $\theta_0(r)$ is an* increasing function *of r in the interval* $[0, \infty)$ *and*
$$\Delta\theta_0(r)|_0^\infty = \frac{\alpha\pi}{2}. \tag{2}$$
2. *If $\lambda_0 > 0$ then $\theta_0(r)$ is a* decreasing function *of r in the interval* $[0, \infty)$ *and*
$$\Delta\theta_0(r)|_0^\infty = \frac{\alpha\pi}{2} - \pi. \tag{3}$$

Proof. For the sake of brevity we will prove the statement for the case when $0 < \alpha < 1$, see Figure 1. The proof of the case $1 \leq \alpha < 2$ follows the same line of arguments.

1. $\lambda_0 < 0$. We have $\theta_0(0) = \arg(-\lambda_0) = 0$. It is geometrically clear that $\theta_0(r)$ is an *increasing function* (the vector $v = re^{\frac{\alpha\pi}{2}i} - \lambda_0$ rotates counter-clockwise) of r in the interval $[0, \infty)$ and $\theta_0(r) \to \alpha\frac{\pi}{2}$ when $r \to \infty$, which leads to (2).

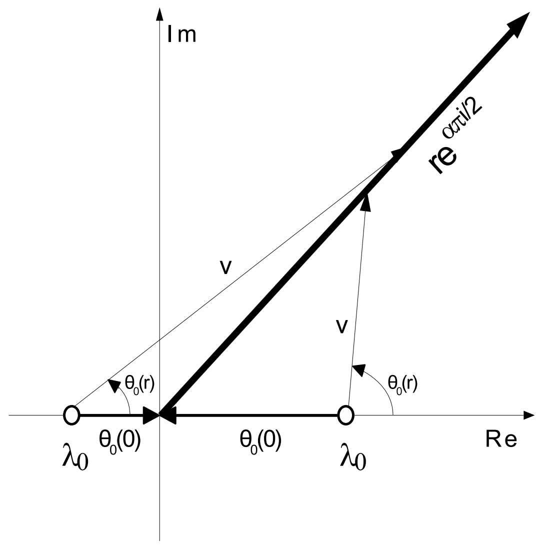

Figure 1: Negative and positive real roots when $0 < \alpha < 1$.

2. $\lambda_0 > 0$. We have $\theta_0(0) = \arg(-\lambda_0) = \pi$. It is geometrically clear that $\theta_0(r)$ is a *decreasing function* (the vector $v = re^{\frac{\alpha\pi}{2}i} - \lambda_0$ rotates clockwise) of r in the interval $[0, \infty)$ and $\theta_0(r) \to \alpha\frac{\pi}{2}$ when $r \to \infty$, which leads to (3).

■

Lemma 5. *For $\theta_1(r), \theta_2(r)$ and $\theta(r)$ the following hold:*

1. *If $\frac{\alpha\pi}{2} < \phi_1 < \pi$ then $\theta_1(r)$ is an* increasing function *and $\theta_2(r)$ is a* decreasing function *when $0 < \alpha < 1$, while that $\theta_1(r)$ and $\theta_2(r)$ are both* increasing functions *of r in the interval $[0, \infty)$ when $1 \le \alpha < 2$, and*

$$\Delta\theta_1(r)|_0^\infty = \frac{\alpha\pi}{2} - (-\pi + \phi_1), \Delta\theta_2(r)|_0^\infty = \frac{\alpha\pi}{2} - (\pi - \phi_1), \qquad (4)$$

which implies

$$\Delta\theta(r)|_0^\infty = 2\left(\frac{\alpha\pi}{2}\right). \qquad (5)$$

2. *If* $0 < \phi_1 < \frac{\alpha\pi}{2}$ *then* $\theta_1(r)$ *and* $\theta_2(r)$ *are both* decreasing functions *when* $0 < \alpha < 1$ *while that* $\theta_1(r)$ *is a* decreasing function *and* $\theta_2(r)$ *is an* increasing function *of* r *in the interval* $[0,\infty)$ *when* $1 \leq \alpha < 2$, *and*

$$\Delta\theta_1(r)|_0^\infty = \frac{\alpha\pi}{2} - (\pi + \phi_1), \Delta\theta_2(r)|_0^\infty = \frac{\alpha\pi}{2} - (\pi - \phi_1), \quad (6)$$

which implies

$$\Delta\theta(r)|_0^\infty = 2\left(\frac{\alpha\pi}{2} - \pi\right). \quad (7)$$

Proof. We prove the result by considering the following four cases:

1a) $\frac{\alpha\pi}{2} < \phi_1 < \pi$ and $0 < \alpha < 1$, see Figure 2. In this case, we have $\theta_1(0) = \arg\left(-\rho_1 e^{\phi_1 i}\right) = -\pi + \phi_1$. It is geometrically clear that $\theta_1(r)$ increases (the vector $v_1 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{\phi_1 i}$ rotates counter-clockwise) as r increases and $\theta_1(r) \to \frac{\alpha\pi}{2}$ as $r \to \infty$, which leads to (4). On the other hand, we have $\theta_2(0) = \arg\left(-\rho_1 e^{-\phi_1 i}\right) = \pi - \phi_1$ and $\theta_2(r)$ decreases (the vector $v_2 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{-\phi_1 i}$ rotates clockwise) as r increases and $\theta_2(r) \to \frac{\alpha\pi}{2}$ when $r \to \infty$, that yields at the expression in (4). Hence, the total change of $\theta(r)$ when r varies from 0 to ∞ is given by (5).

1b) $\frac{\alpha\pi}{2} < \phi_1 < \pi$ and $1 \leq \alpha < 2$, see Figure 3. By similar geometry, it can be seen that the expressions given in (4) and (5) hold, but, however, in this case both $\theta_1(r)$ and $\theta_2(r)$ increase (the vectors $v_1 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{\phi_1 i}$ and $v_2 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{-\phi_1 i}$ rotate counter-clockwise) as r increases from 0 to ∞.

2a) $0 < \phi_1 < \frac{\alpha\pi}{2}$ and $0 < \alpha < 1$, see Figure 4. In this case, we have $\theta_1(0) = \arg\left(-\rho_1 e^{\phi_1 i}\right) = \pi + \phi_1$. It is geometrically clear that $\theta_1(r)$ decrease (the vector $v_1 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{\phi_1 i}$ rotates clockwise) as r increases and $\theta_1(r) \to \frac{\alpha\pi}{2}$ when $r \to \infty$, which leads to (6). On the other hand, we have $\theta_2(0) = \arg\left(-\rho_1 e^{-\phi_1 i}\right) = \pi - \phi_1$. Geometrically, it can be seen that $\theta_2(r)$ decreases (the vector $v_2 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{-\phi_1 i}$ rotates clockwise) as r increases and $\theta_2(r) \to \frac{\alpha\pi}{2}$ when $r \to \infty$, that leads to (6). Therefore, the total change of $\theta(r)$ when r varies from 0 to ∞ is given by (7).

2b) $0 < \phi_1 < \frac{\alpha\pi}{2}$ and $1 \leq \alpha < 2$, see Figure 5. Again, by similar geometry, one can arrive at the expressions in (6) and (7) but, however, in this case, $\theta_1(r)$ decreases (the vector $v_1 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{\phi_1 i}$ rotates clockwise) while $\theta_2(r)$ increases (the vector $v_2 = re^{\frac{\alpha\pi}{2}i} - \rho_1 e^{-\phi_1 i}$ rotates counter-clockwise) as r increases from 0 to ∞.

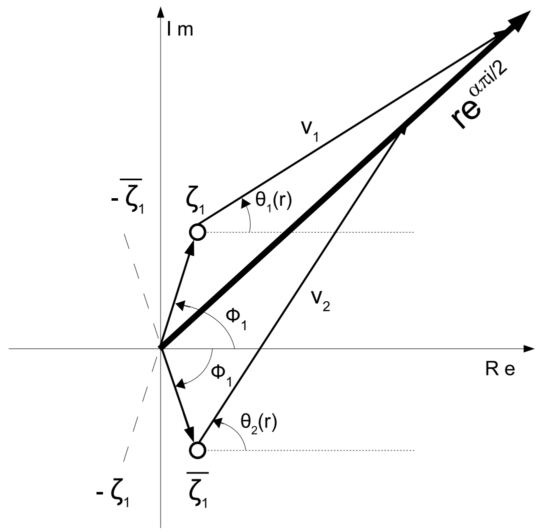

Figure 2: Stable complex conjugated roots when $0 < \alpha < 1$

■

Theorem 2. *The pseudo-polynomial $p(s)$ is stable if and only if*

$$\Delta \arg p(i\omega)|_0^\infty = \alpha_n \left(\frac{\pi}{2}\right) = \alpha \left(\frac{n\pi}{2}\right), \tag{8}$$

where $0 < \alpha < 2$.

Proof. We consider only pseudo-polynomials $p(s)$ satisfying the Lemmas 2 and 3 since if the necessary conditions are not satisfied then there is no significance in investigating their stability. Additionally, as we concern with stability one can assume without loss of generality that $p(s)$ does not have any pure imaginary roots.

By virtue of the transformation, $\lambda = s^\alpha$, a point $s = i\omega = \omega e^{\frac{\pi}{2}i}$ in the complex $s-$plane is transformed to the complex $\lambda-$plane as $\lambda = \omega^\alpha e^{\frac{\alpha\pi}{2}i}$. Hence, it holds

$$p(i\omega) = \tilde{p}\left(\omega^\alpha e^{\frac{\alpha\pi}{2}i}\right). \tag{9}$$

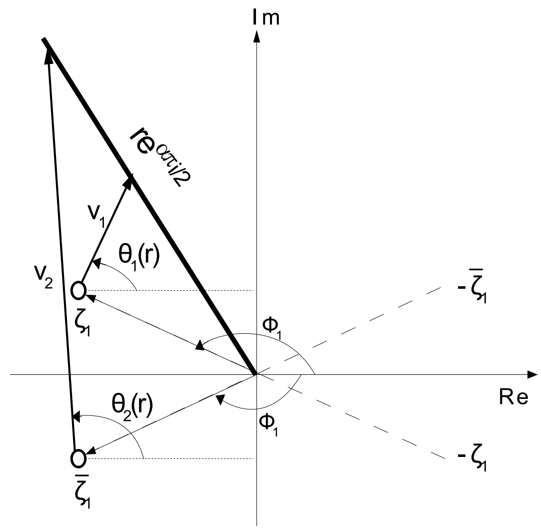

Figure 3: Stable complex conjugated roots when $1 \leq \alpha < 2$.

From this equality follows that the problem of measuring the total variation of the argument of $p(i\omega)$ when ω varies from 0 to ∞ is equivalent to the problem of measuring the total variation of the argument of $\tilde{p}\left(re^{\frac{\alpha\pi}{2}i}\right)$ when r varies from 0 to ∞. The integer order polynomial $\tilde{p}(\lambda)$ has n different roots in the complex $\lambda-$plane and it can be factorized into a product of the form

$$\tilde{p}(\lambda) = a_n\left(\lambda - \lambda_1\right)\cdots\left(\lambda - \lambda_m\right)\left(\lambda - \zeta_1\right)\left(\lambda - \bar{\zeta}_1\right)\cdots\left(\lambda - \zeta_l\right)\left(\lambda - \bar{\zeta}_l\right),$$

where $\lambda_j, j = 1, \ldots, m$ are real roots and $\zeta_j, \bar{\zeta}_j, j = 1, \ldots, l$, are complex conjugated roots, with $n = m + 2l$. Then, we have

$$\begin{aligned} \arg\tilde{p}\left(re^{\frac{\alpha\pi}{2}i}\right) &= \arg\left(a_n\right) + \sum_{j=1}^{m}\arg\left(re^{\frac{\alpha\pi}{2}i} - \lambda_j\right) \\ &\quad + \sum_{j=1}^{l}\arg\left(re^{\frac{\alpha\pi}{2}i} - \zeta_j\right) + \sum_{j=1}^{l}\left(re^{\frac{\alpha\pi}{2}i} - \bar{\zeta}_j\right). \end{aligned} \tag{10}$$

The necessary conditions assure that $\tilde{p}(\lambda)$ does not have neither zero nor positive real roots. On the other hand, the assumption that $p(s)$ does not

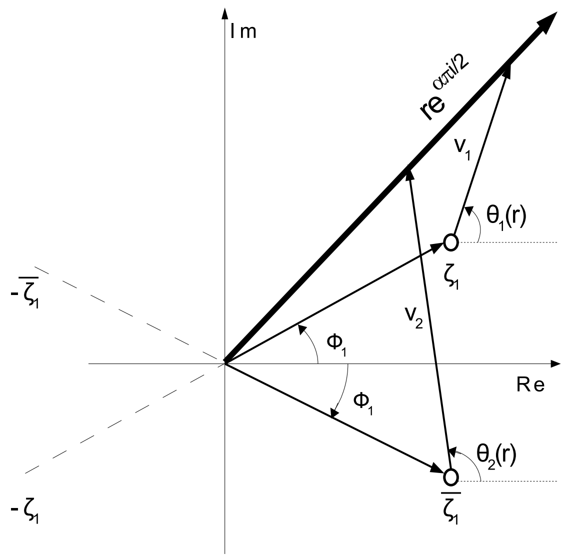

Figure 4: Unstable complex conjugated roots when $0 < \alpha < 1$.

have any pure imaginary roots implies that $\tilde{p}(\lambda)$ does not have a pair of complex conjugated roots lying in ∂M_α. Let p be the number of roots of $\tilde{p}(\lambda)$ lying in $(M_\alpha \cup \partial M_\alpha) \setminus \mathbb{C}$. Then, there are $n-p$ roots of $\tilde{p}(\lambda)$ in the region M_α. From the Lemmas 4 and 5 follow that each root (real or complex) lying in M_α contributes $\frac{\alpha\pi}{2}$ to the total argument change of $\tilde{p}\left(re^{\frac{\alpha\pi}{2}i}\right)$ and that any complex root lying in $(M_\alpha \cup \partial M_\alpha) \setminus \mathbb{C}$ provides a total argument change in $\tilde{p}\left(re^{\frac{\alpha\pi}{2}i}\right)$ of $\frac{\alpha\pi}{2} - \pi$. Hence, it follows from (9) and (10) that

$$\begin{aligned} \Delta \arg p(i\omega)\big|_0^\infty &= \Delta \arg \tilde{p}\left(re^{\frac{\alpha\pi}{2}i}\right)\Big|_0^\infty \\ &= (n-p)\left(\frac{\alpha\pi}{2}\right) + p\left(\frac{\alpha\pi}{2} - \pi\right). \end{aligned} \tag{11}$$

For stability of $p(s)$ is necessary and sufficient that $\tilde{p}(\lambda)$ has all its roots in M_α, i.e., in the formula (11) we should have $p=0$ which leads to (8) and the proof of the result. ■

It is very important to note that the Mikhailov curve provides information of the roots distribution of $p(s)$ with respect to the imaginary axis when the Mikhailov stability criterion is not satisfied, as occurs in the polynomial case.

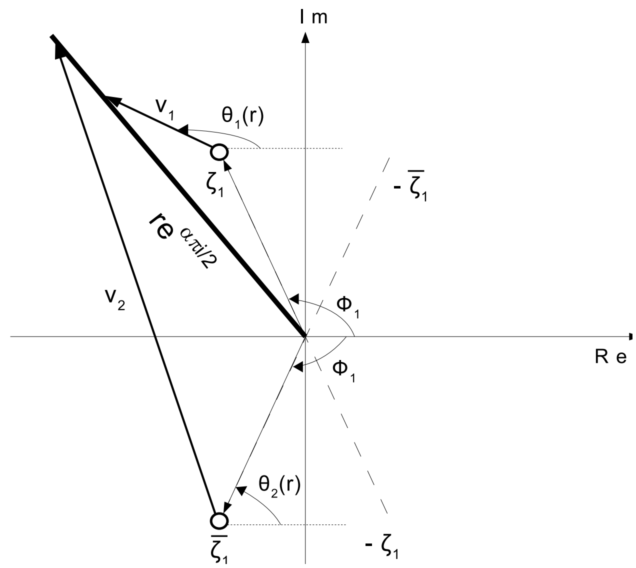

Figure 5: Unstable complex conjugated roots when $1 \leq \alpha < 2$.

Indeed, if $p(s)$ has a pair of pure imaginary roots $s = \pm i\omega_0$ then $p(i\omega_0) = 0$. Graphically, this is equivalent to the Mikhailov plot of $p(i\omega)$ passes through the origin of coordinates at the point $\omega = \omega_0$. Therefore, from this and the formula (11) follow that the Mikhailov curve allows us to determine the number of roots of $p(s)$ having negative or positive real parts, and also if there are purely imaginary roots and, if so, their value.

Finally, note that the Theorem 2 does not say anything about the behavior of the Mikhailov curve since in general one cannot determine the direction (positive or negative) the curve will run for a stable $p(s)$. In fact, it follows from the Lemma 5 that for stable complex conjugated roots with $0 < \alpha < 1$, $\theta_1(r)$ increases but $\theta_2(r)$ decreases and, therefore, $\theta(r) = \theta_1(r) + \theta_2(r)$ may increase and/or decrease as r goes from 0 to ∞. As a consequence, to have stable complex conjugated roots not necessarily implies that the Mikhailov curve will run in the positive direction. On the other hand as it is also established in Lemma 5, when $1 \leq \alpha < 2$, for any pair of stable complex conjugated roots one has that both $\theta_1(r)$ and $\theta_2(r)$ increases and, therefore, $\theta(r) = \theta_1(r) + \theta_2(r)$ increases when r goes from 0 to ∞. Thus, when

$1 \leq \alpha < 2$, a stable pseudo-polynomial $p(s)$ always has a Mikhailov curve that runs in the positive direction.

3. Illustrative examples

Example 1. *Let us consider the pseudo-polynomial*

$$p(s) = s^{2/8} - 2s^{1/8} + 3. \tag{12}$$

We have that $\alpha_n = 2\left(\frac{1}{8}\right)$, *i.e.* $n = 2$ *and* $\alpha = \frac{1}{8}$. *As the necessary condition in Lemma 3 is satisfied for (12) we then proceed to plot the Mikhailov curve, see Figure 6, in order to investigate its stability. As it can be seen from the Figure 6, the Mikhailov curve runs in the negative direction and the total argument change asymptotically approaches to* $\frac{\pi}{8}$ *that is equal to* $\alpha_n\left(\frac{\pi}{2}\right)$ *and, therefore, from Theorem 2 follows that (12) is stable, a result that can be also verified by means of the roots calculation.*

Note that this is in fact a counterexample to the Theorem 1 (Theorem 2 in [11]) since (12) is stable but neither the condition $n\left(\frac{\pi}{2}\right) = \pi$ *nor the positive direction of the Mikhailov curve hold.*

Indeed, as it can be seen from Figure 6, the argument of $p(i\omega)$ *firstly is negative and decreasing then increases passing through zero and* $\frac{\pi}{8}$ *to finally decreases again by asymptotically approaching to* $\frac{\pi}{8}$. *It is very important to note that this could never happen if* $p(s)$ *were a polynomial.*

Example 2. *Consider the pseudo-polynomial*

$$p(s) = 0.8s^{2.2} + 0.5s^{0.9} + 1 \tag{13}$$

which has been widely studied in the literature, see for instance [15]. We have that $\alpha_n = 22\left(\frac{1}{10}\right) = 2.2$ *and, therefore,* $n = 22$ *and* $\alpha = \frac{1}{10}$. *Firstly, we observe that the necessary condition in Lemma 3 is satisfied. The Mikhailov plot of (13) is shown in Figure 7. As it can be seen from the Figure, the total argument change asymptotically approaches to* $\frac{11}{10}\pi$ *which is equal to* $\alpha_n\left(\frac{\pi}{2}\right)$. *Hence, from the Theorem 2 one concludes that (13) is stable as it is well-known by means of roots calculation, see [15]. Note that, in this case, the curve runs in the positive direction.*

Example 3. *Consider the following pseudo-polynomial studied in [5]:*

$$p(s) = s - 2s^{1/2} + 1.25. \tag{14}$$

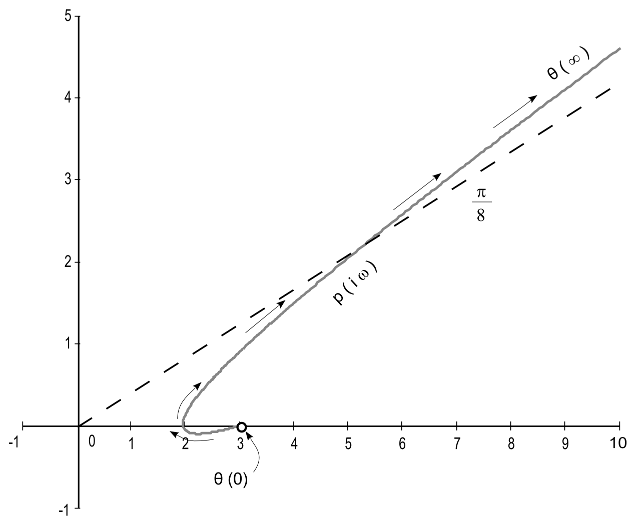

Figure 6: Mikhailov curve for (12)

We have that $\alpha_n = 2\left(\frac{1}{2}\right) = 1$ and, therefore, $n = 2$ and $\alpha = \frac{1}{2}$. Clearly, the necessary condition in Lemma 3 is satisfied. The Mikhailov curve of (14) is plotted in Figure 8 from which is obtained that the curve runs in the negative direction and the total argument change asymptotically approaches to $-\frac{3\pi}{2}$. It follows from Theorem 2 that (14) is unstable.

Now, we illustrate that the Mikhailov curve allows us to determine the number of unstable roots. Firstly, we observe that (14) has no pure imaginary roots since the curve does not pass through the origin. Then, from the formula (11) one directly obtain $p = 2$ and thus (14) has two complex conjugated roots with positive real part, a result verified in [5] by means of roots calculation.

Example 4. *In this example we illustrate the main results for pseudo-polynomials with $1 \leq \alpha < 2$. Firstly, we illustrate the potential of the Stodola type criterion given in Lemma 2. To this aim, consider the following two pseudo-polynomials $p_1(s) = s^{\frac{8}{3}} + 1$ and $p_2(s) = s^{\frac{8}{3}} + 2s^{\frac{4}{3}} - 1$. Since, in these cases, we have that $\alpha = \frac{4}{3}$ and for $p_1(s)$ there is a zero coefficient (there is not a term of $s^{\frac{4}{3}}$) while that for $p_2(s)$ the coefficients have not the same sign, it then directly follows from Lemma 2 that $p_1(s)$ and $p_2(s)$*

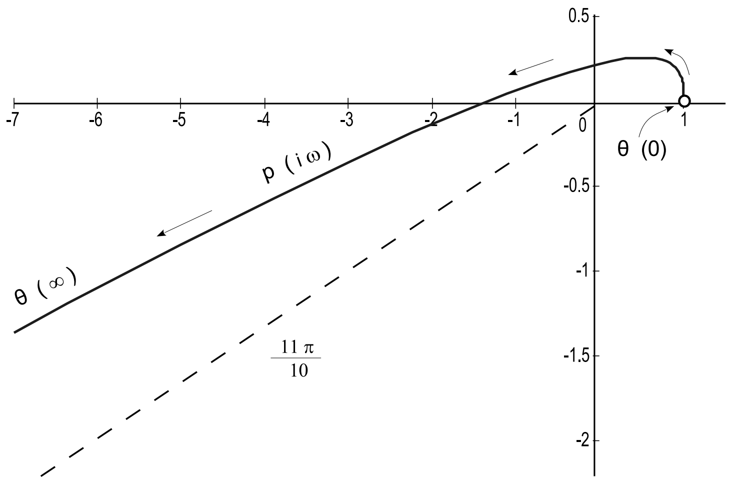

Figure 7: Mikhailov curve for (13)

are unstable. Of course, in both cases, such conclusions can be verified by numerical calculation of the roots.

To illustrate the Mikhailov stability criterion let us consider the following pseudo-polynomial:

$$p(s) = s^{2\sqrt{2}} + 4s^{\sqrt{2}} + 8. \tag{15}$$

We have $\alpha_n = 2\sqrt{2}$ *that leads to* $n = 2$ *and* $1 \leq \alpha = \sqrt{2} < 2$. *For (15) the necessary conditions given in Lemma 2 hold. The Mikhailov curve of (15) is plotted in Figure 9 from which one gets that the total argument change asymptotically approaches to* $\sqrt{2}\pi$ *being equal to* $\alpha_n\left(\frac{\pi}{2}\right)$ *and hence (15) is stable by Theorem 2. Note that the Mikhailov curve runs in the positive direction as it is established by our results.*

4. Conclusions

In this paper, we addressed the Mikhailov stability criterion for general pseudo-polynomials of commensurate order. Firstly, we derived some necessary conditions for stability of pseudo-polynomials which are the coun-

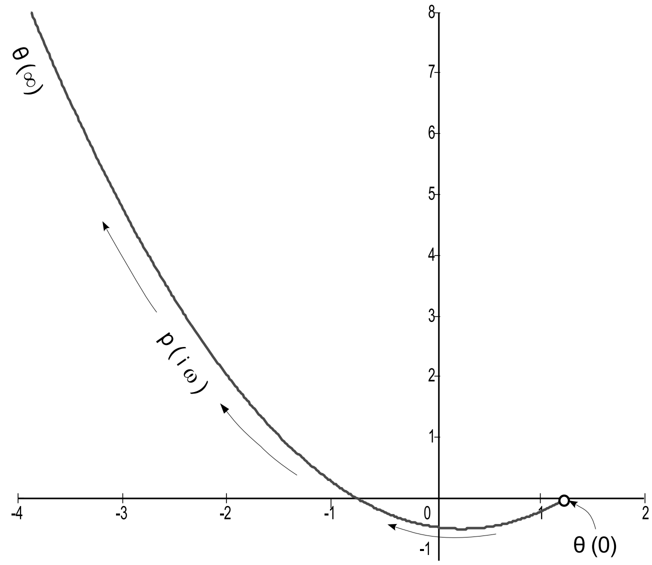

Figure 8: Mikhailov curve for (14)

terpart of the well-known Stodola stability criterion for polynomials. As demonstrated, the Stodola stability criterion remains valid for $1 \le \alpha < 2$ but not when $0 < \alpha < 1$. In this latter case, a necessary condition for stability in terms of the coefficients a_n and a_0 is obtained. The necessary conditions are very important since allow us to easily determine unstable pseudo-polynomials by simple inspection.

The main result of this paper is the modification of the Mikhailov stability criterion. The modification consists in determining the appropriate measure for the total argument change of $p(i\omega)$, when ω varies in the interval $[0, \infty)$. Thus, our main result in Theorem 2 established that such a measure is $\alpha_n \left(\frac{\pi}{2}\right)$, where $\alpha_n = n\alpha$ is the highest fractional degree of the pseudo-polynomial, and not $n \left(\frac{\pi}{2}\right)$ as it has been reported in the literature. Moreover, it is demonstrated that this Theorem is true not only in the case when $0 < \alpha < 1$ but also when $1 \le \alpha < 2$. Additionally, it is shown that from the Mikhailov curve the number of roots having negative or positive real parts as well as the existence of purely imaginary roots can be determined.

Another important result is that the Mikhailov curve need not necessarily

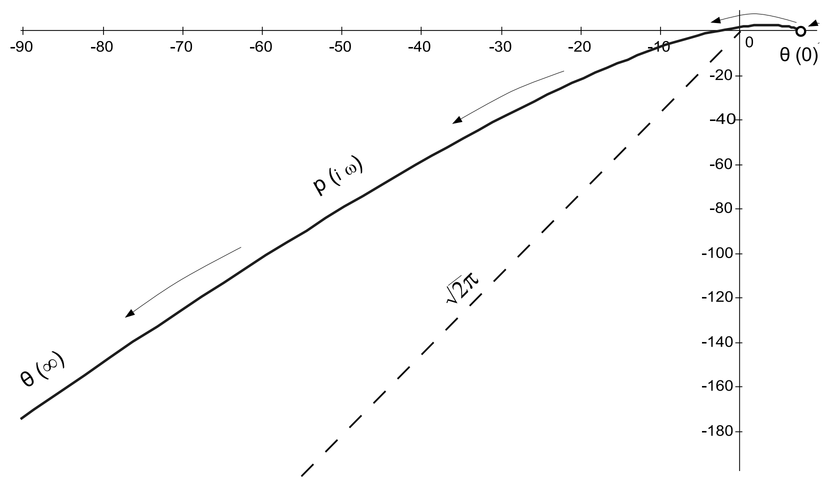

Figure 9: Mikhailov curve for (15)

always rotate in the positive (counterclockwise) direction for a stable pseudo-polynomial. In other words, the so-called *monotonic phase increase property* for stable integer order polynomials is not satisfied, in general, for stable pseudo-polynomials. It is only in the case when $1 \leq \alpha < 2$ that such a property holds for stable pseudo-polynomials.

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