A family of multimodal dynamic maps

E. Campos-Cantón[a,b*], R. Femat[b], A.N. Pisarchik[c]

[a]Departamento de Físico Matemáticas,

Universidad Autónoma de San Luis Potosí,

Alvaro Obregón 64, 78000, San Luis Potosí, S.L.P., Mexico.

[b]División de Matemáticas Aplicadas,

Instituto Potosino de Investigación Científica y Tecnológica A.C.,

Camino a la Presa San José 2055, Col. Lomas 4 sección,

78216, San Luis Potosí, S.L.P., Mexico.

[c]Centro de Investigaciones en Optica,

Loma del Bosque 115 Col. Lomas del Campestre,

37150, Leon, Gto., Mexico.

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Abstract

We introduce a family of multimodal logistic maps with a single parameter. The maps domain is partitioned in subdomains according

*Corresponding author: eric.campos@ipicyt.edu.mx, tel. (444) 8 34 20 00

to the maximal number of modals to be generated and each subdomain contains one logistic map. The number of members of a family is equal to the maximal number of modals. Bifurcation diagrams and basins of attraction of the fixed points are constructed for the family of chaotic logitic maps.

Keywords: Chaos, logistic map, bifurcations, multimodal maps.

1 Introduction

There are several mature topics in nonlinear science. The study of chaotic maps is one of these, where many considerable results have opened paths on chaos theory and impacted interdisciplinary areas; the classical result of *period three implies chaos* by Li and Yorke [1] is very important since it has had significant implication in understanding population dynamics [2]. As a consequence of maturity [3, 4], there is an active new area regarding applications on technology, such as dc-dc converters [5, 6] and chaotic cryptography (e.g., for image encryption [7, 8, 9, 10, 11, 12]), the latter with a high potential in secure communication research [13].

The basic ideas for studying chaos can be understood by analyzing one-dimensional maps, i.e. the iterative maps of a single variable. Many systems dynamics can be described by one-dimensional unimodal maps, e.g. the logistic and tent maps. These systems have been extensively studied [1] and implemented experimentally [14, 15]. A unimodal map is a continuous one-dimensional function $\mathbb{R} \to \mathbb{R}$ with a single critical point c_0 monotonically increasing on one side of c_0 and decreasing on the other. The dynamics of the system is governed by the function

$$x_{n+1} = f(x_n, \beta), \tag{1}$$

where x_n is the system state after n iterations and β is the bifurcation parameter. The time series $\{x_0, x_1, x_2, ...\}$ of Eq. 1 corresponds to a map orbit starting from initial condition x_0.

In this work, we present a simple generation of multimodal chaotic maps based on the logistic map, i.e. we define a family of maps whose domain is partitioned according to the maximal number of modals to be generated. Chaotic code generators have received a considerable attention due to engineering applications, especially for potential applications in mobile communication systems. Several approaches for the pseudo-random number code design exist using unimodal chaotic maps. Multimodal chaotic maps are expected to be useful for increasing security of chaotic communication systems. We also show that the family of chaotic maps with a maximal number of modals includes chaotic maps with less modals than that number.

The rest of the paper is organized as follows. In Section 2 we discuss the logistic map scaled to very little or very big intervals. Section 3 is devoted to the generation of multimodal chaotic maps based on the logistic map. A numerical example is given in Section 4. Finally, Section 5 represents main conclusions of this work.

2 Scaled chaotic maps

Here we introduce a class of scaled logistic maps based on the bi-parametric equation proposed by Verhulst in 1838 [16]

$$\frac{dN}{dt} = \alpha \left(1 - \frac{N}{\gamma}\right) N, \tag{2}$$

where $N(t)$ is the state of the system at time t, α is the intrinsic growth rate, and γ is the carrying on capacity. We then construct a bi-parametric family of logistic maps $f_{\alpha,\gamma} : I \to I$ on the interval $I = [a, b] \subset \mathbb{R}$ as

$$f_{\alpha,\gamma}(x) = \alpha(1 - x/\gamma)x, \tag{3}$$

where α and γ are the system parameters, and the interval I is determined by the roots of the system Eq. 3. Thus, the length of the interval I is given by the parameter $\gamma \in \mathbb{R}$, with $a = 0$ and $b = \gamma$. The map given by Eq. 3 is unimodal with critical point $c_0 = \gamma/2 \in I$; $f_{\alpha,\gamma}$ monotone increases on the left of c_0 and monotone decreases on the right of c_0. The maximal value of the parameter ($\alpha = 4$) is determined when $f_{\alpha,\gamma}$ maps $[a, b] \to [a, b]$. The orbits $\Phi_{\alpha,\gamma}(x_0) = \{x_j : x_j = f^j_{\alpha,\gamma}(x_0), j \in \mathbb{N}\}$ can be obtained for any $\gamma \in \mathbb{R}$, the *length of interval* I the capacity parameter is defined only by γ. Notice that for $\gamma = 1$ Eq. 3 is the classical logistic map with $I_L = [0, 1]$. If $\gamma > 1$ the closed intervals $I_g > I_L$; otherwise, this implies the closed intervals $I_s < I_L$. The relevance of γ is to yield small or large closed intervals I, where the chaotic map lies. Hence, above arguments provide us with the bases for multimodal maps by considering a family of small chaotic maps $f_{\alpha,\gamma} : I_s \to I_s$ and the closed interval I_L, note that the interval $I_s$is chosen arbitrarily. That is, if $2 \leq \|I_L\|/\|I_s\| = k \in \mathbb{N}$, it is possible to define a k-modal chaotic map by moving the map $f_{\alpha,\gamma}$, k times, as will be illustrated in the next Section.

3 Multimodal chaotic maps

We are interested in constructing a monoparametric family $\mathcal{F}$ of maps f_β capable of displaying an arbitrary number of modals on the interval $I = [a, b] \subset \mathbb{R}$, to do so we will give the definition of a multimodal map as a generalization of the unimodal map [17]. A multimodal map $f : I \to I$, is a smooth map with a finite number of critical points, all of them local maximum or local minimum. This definition can be found in [18], where the multimodal maps are constructed by polynomials with negative Schwarzian derivative and without attracting periodic orbits. The theory of multimodals maps is studied in [19]. Here, we are interested in the definition given in [20] for a particular type of multimodal maps.

Definition 1. A map f_β is k-modal, if it is continuous, has k critical points denoted by $c_0,\ c_1\ \ldots\ c_{k-1}$ in I, monotone increases on the left of each c_i and monotonically decreases on the right of each c_i, ($i = \{0, 1, 2, \ldots, k\}$).

We say that f is a k-modal map if it can be written as a composition of k unimodal maps $f_1, f_2, \ldots, f_k$ with the following propesties:

- $f_i : I_i \to I$ has an unique critical point (a maximum);
- $f(c_i) = f(c_j)$, for $i \neq j$;
- $\bigcup_{i=1}^{k} I_i = I$.

Definition 2. A monoparametric family $\mathcal{F}$ is an ensemble of k modal maps f_β determined by parameter β, where each map reproduces a different m-modal map ($m = \{1, 2, \ldots, k\}$).

The orbits $\Phi_\beta(x_0) = \{x_n : x_n = f_\beta^n(x_0),\ n \in \mathbb{N}\}$ are derived for each value of the parameter $\beta \in J \subset \mathbb{R}$, where the interval J is closed, from zero to a value defined by $f_\beta(c_i) = b$, $(i = \{1, 2, \ldots, m\})$.

Next, a quadratic map f_β is introduced such that m modals exist at a sub-interval $I = [a, b] \subset I_L$. The number k defines the maximal number of modals in the family $\mathcal{F}$, the interval I_L is partitioned between k subintervals $\hat{I}_j = [d_j, d_{j+1}) \subset I_L$ of equal length, so that f_β is a piecewise function formed by k unimodal maps Q_j. This implies that the first unimodal map $Q_0 = f_\gamma$ has the carrying on capacity $\gamma = 1/k$ and the quadratic map f_β is defined in the first interval $\hat{I}_0 = [d_0 = a, d_1)$ by the unimodal map Q_0. The quadratic map f_β is given in others intervals $\hat{I}_1 = [d_1, d_2)$, $\hat{I}_2 = [d_2, d_3), \ldots, \hat{I}_{k-1} = [d_{k-1}, d_k = b]$ as a result of moving the unimodal map Q_0 into them. The quadratic map f_β has its critical points c_i such that $f_\beta(c_i) = 1/k$. Since $f_\beta : I_L \to [0, 1/k]$, one more condition is required so that the *image* of I_L under (f_β) is equal to I_L, then it needs to be multiplied by k in order to obtain $f_\beta : I_L \to I_L$. Under the above arguments, the bifurcation parameter becomes $\beta = \beta(\alpha, k, \gamma)$.

The parameterized family $\mathcal{F}$ of maps f_β is defined by the following piecewise function

$$f_\beta(x) = \beta(d_{r+1} - x)(x - d_r), \text{ for } x \in [d_r, d_{r+1}), \tag{4}$$

where $d_r = r/k$ $(r = \{0, 1, 2, \ldots, k-1\})$. Note that $\beta \in J = [0, \alpha k/\gamma]$. The last interval is considered a closed interval in Eq. 4. Notice that $I = \cup_{r=0}^{k-1}[d_r, d_{r+1})$ and $\cap_{r=0}^{k-1}[d_r, d_{r+1}) = \emptyset$. For example, let $k = 2$ be the desired value for the number of modals. Hence, a bimodal map is derived, where I_L is divided in two subintervals $\hat{I}_0 = [0, 0.5)$ and $\hat{I}_1 = [0.5, 1]$, which determine the carrying on capacity $\gamma = 0.5$. Then, $r = 0, 1$ and the family $\mathcal{F}$ contains two members:

(1) One bimodal map obeying $\beta = \alpha(k-r)/\gamma = (4)(2)/0.5 = 16$, with $r = 0$, such that

$$f_{16}(x) = \begin{cases} 16(\frac{1}{2} - x)x & \text{for } \; x \in [0, 0.5); \\ 16(1-x)(x - \frac{1}{2}) & \text{for } \; x \in [0.5, 1]. \end{cases}$$

(2) One unimodal map obeying $\beta = \alpha(k-r)/\gamma = (4)(1)/0.5 = 8$, with $r = 1$, such that

$$f_8(x) = \begin{cases} 8(\frac{1}{2} - x)x & \text{for } \; x \in [0, 0.5); \\ 8(1-x)(x - \frac{1}{2}) & \text{for } \; x \in [0.5, 1]. \end{cases}$$

The fixed points of the k-modal map are given by the following equation

$$x^* = \frac{\beta(d_{r+1} + d_r) - 1 \mp \sqrt{(\beta(d_{r+1} + d_r) - 1)^2 - 4\beta^2 d_{r+1} d_r}}{2\beta} \tag{5}$$

with

$$\beta \geq k(2r + 1 + 2\sqrt{r^2 + r}) \Rightarrow \beta \in J = [0, 16].$$

Theorem. A k-modal chaotic map implies m-modal chaotic maps with $0 < m < k$ and $k, m \in \mathbb{N}$.

Proof. The proof can be sketched as follows. A k-modal chaotic map has k critical points given by $c_r = (d_r + d_{r+1})/2 = (2r+1)/(2k)$ with $r \in \{0, 1, 2, \ldots k-1\}$. Moreover, the maximum value L occurs at these critical points

$$f_\beta(c_r) = \beta(d_{r+1} - c_r)(c_r - d_r) = L. \tag{6}$$

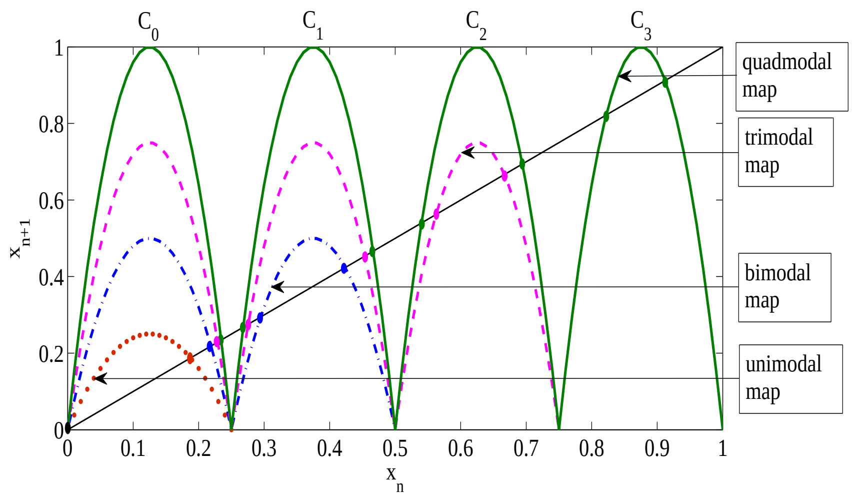

Fig. 1: Family $\mathcal{F}$ of quad-, tri-, bi-, and unimodal logistic maps. Note that the interval $I_L = [0, 1]$ is mapped by every f_β to the corresponding interval $\cup_{r=0}^{m-1}[d_r, d_{r+1}]$.

The bifurcation parameter β is responsible for the maximum values of L. If L is restricted to take only the discrete values d_{r+1}, then $f_\beta : [0, d_{r+1}] \to [0, d_{r+1}]$, remember that $d_{r+1} \leq 1$. From Eq. 6, $\beta = \alpha(r + 1)k$. So, it is possible to build m-modal chaotic maps considering $\beta = \alpha mk$ for a k-modal chaotic map f_β. Note, for the logistic map the results hold only for $\alpha = 4$.

4 Numerical example

Without loss of generality, we arbitrarily consider a particular case of a quad-modal chaotic map, i.e. $k = 4$. The monoparametric family $\mathcal{F}$ of multimodal chaotic maps f_β can be described as

$$f_\beta(x) = \beta \begin{cases} (1/4 - x)x, & \text{for } x \in [0, 1/4); \\ (1/2 - x)(x - 1/4), & \text{for } x \in [1/4, 1/2); \\ (3/4 - x)(x - 1/2), & \text{for } x \in [1/2, 3/4); \\ (1 - x)(x - 3/4), & \text{for } x \in [3/4, 1], \end{cases} \tag{7}$$

where $\beta \in J = [0, 64]$, this interval is determined by $k = 4$, $\gamma = 0.25$, and $\alpha = 4$ (the required values to map the function f into itself). Then, $r = 0, 1, 2, 3$ and the family $\mathcal{F}$ consists of the following four members:

(1) The quadmodal map f_{64} for $r = 0$.

(2) The trimodal map f_{48} for $r = 1$.

(3) The bimodal map f_{32} for $r = 2$.

(4) The unimodal map f_{16} for $r = 3$.

Figure 1 shows the family $\mathcal{F}$ of the multimodal logistic chaotic maps: $f_{64} : [0, 1] \to [0, 1]$, $f_{48} : [0, 1] \to [0, 0.75]$, $f_{32} : [0, 1] \to [0, 0.5]$, and $f_{16} : [0, 1] \to [0, 0.25]$. The critical points are located at $c_0 = 0.125$, $c_1 = 0.375$, $c_2 = 0.625$, and $c_3 = 0.875$. The eight fixed points for the quadmodal chaotic map are {0, 0.2344, 0.2681, 0.4663, 0.5402, 0.6941, 0.8223, 0.9121}. As one goes to the next map, the number of fixed points decreases by two. The trimodal chaotic map has six fixed points: {0, 0.2292, 0.2756, 0.4536, 0.5625, 0.6667}, the bimodal chaotic map has four fixed points: {0, 0.2188, 0.2950, 0.4238}, and the unimodal chaotic map has only two fixed points: {0, 0.1875}.

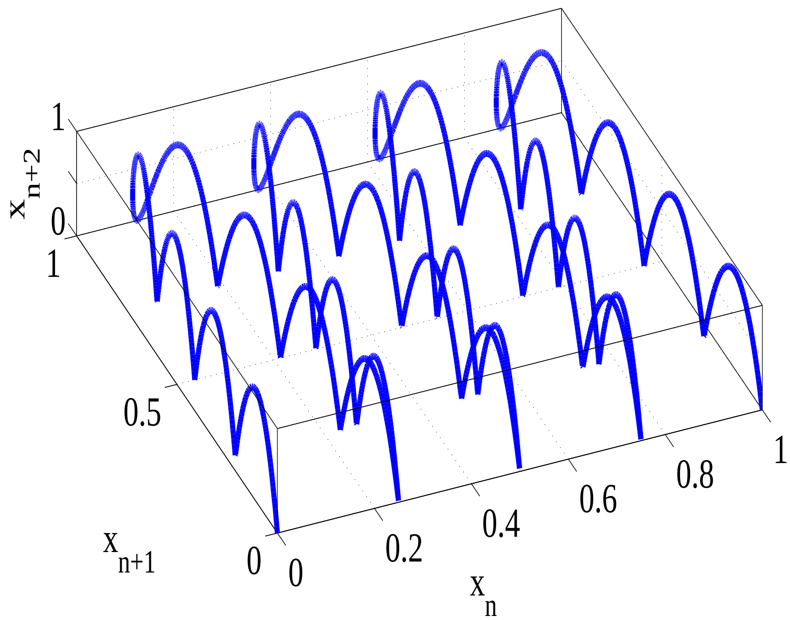

Fig. 2: Three-dimensional phase diagrams showing stretching-and-folding structure of the quadmodal chaotic map. Note that the projection on plane $(x_n, x_{n+1}) \in I_L \times I_L$ corresponds to the quadmodal map shown in Fig. 1.

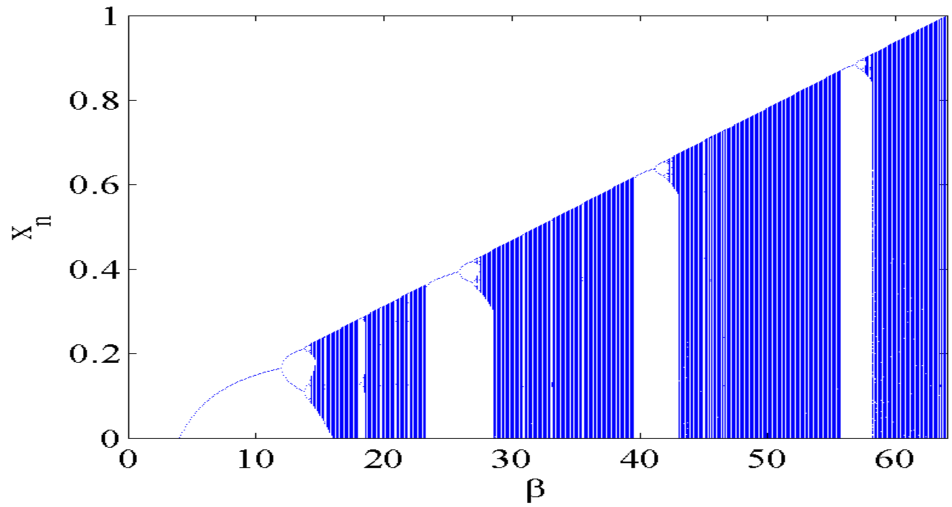

Fig. 3: Bifurcation diagram for the quadmodal logistic map in Eq. 7.

Figure 2 illustrates the stretching and folding structure of the quadmodal chaotic map f_{64} in a three-dimensional phase space.

Then, beyond $\beta\ = 16$ the dynamics continues being chaotic until a new fixed point appears at $\beta\ = 12+8\sqrt{2}$ cascade of period-doubling bifurcations $\beta\ = 12+8\sqrt{2}$, $20+8\sqrt{6}$, and $28+16\sqrt{3}$, as seen on Fig. 3.

Notice that the monoparametric family $\mathcal{F}$ is generated by the discrete values of $\beta = 16, 32, 48, 64$ with fixed $\alpha = 4$. However, since α is the bifurcation parameter for the logistic map, it should be allowed to take any value in the interval [0,4], meaning that β would be able to vary in an interval $J \subset \mathbb{R}$ to produce a family $\mathcal{F}_\beta$ that contains $\mathcal{F}$. The bifurcation diagram of the quadmodal logistic map in terms of β is shown in Fig. 3. To illustrate our meaning, let us consider the first equation of f_β in Eq. 7 given by

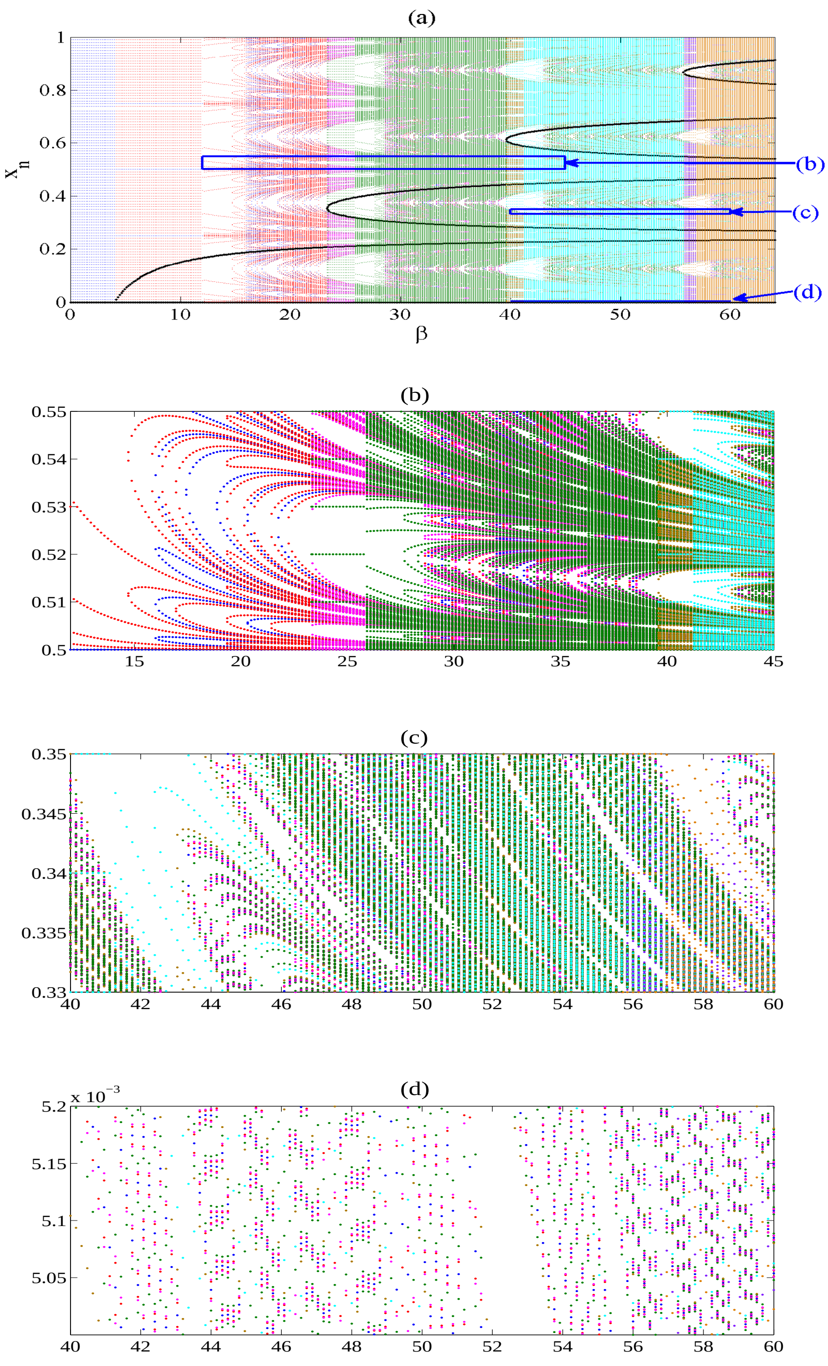

Fig. 4: Basins of attraction of each fixed point of the system given by Eq. 7. (b), (c), and (d) stand for the close up indicated in (a).

$\phi_{\hat{I}_0}(x) = \beta(1/4-x)x$. As the parameter β varies from 0 to 16, the bifurcation diagram of map $\phi_{\hat{I}_0}$ looks like that of the standard logistic map when the bifurcation parameter α varies from 0 to 4 (Fig. 3), i.e. the control parameter β is scaled by four. When $0 < \beta < 4$, the dynamics settles down at zero independently of the initial condition, meanwhile for $4 < \beta < 12$ the dynamics will stabilize on the value of $(\beta - 1)/\beta$. When $12 < \beta < 4 + 4\sqrt{6}$, the map is periodic of the period two. The cascade of the period-doubling bifurcations starts at $\beta = 4 + 4\sqrt{6}$ and terminates in chaos at $\beta \approx 14.28$. Then, beyond $\beta = 16$ the dynamics continues being chaotic until a new fixed point appears at $\beta = 12 + 8\sqrt{2}$ giving rise to a new cascade of period-doubling bifurcations and the mechanism repeats itself after a chaotic interval at $\beta = 12 + 8\sqrt{2}$, $20 + 8\sqrt{6}, 28 + 16\sqrt{3}$, as seen on Fig. 3.

In our example, the quadmodal map has eight fixed points and every fixed points has its own basin of attraction, entirely disconnected from the basins of attraction of the other fixed points. Figure 4 (a) shows the global basin of attraction of the eight fixed points consists of intricate nested of basin of attraction of each fixed point. If we increase the number of modal then the number of fixed point increase and this intricate nested of basins of attraction. Different basins of attraction appear as we explore higher values of β. Figure 4 (a) shows the global basin of attraction, while Figs. 4 (b)-(d) display different enlarged regions of Fig. 4 (a). Following a color scheme, we plot all points leading to the first fixed point in blue, to the second fixed point in red, to the third, fourth, fifth, and so on until eighth fixed point in pink, green, brown, light blue, purple, and orange, respectively.

5 Conclusions

Exploiting the versatility of multimodal logistic maps, we proposed a family of chaotic logistic maps which incorporates the carrying on capacity parameter γ, logistic map parameter α, and the number of modals k in a single parameter β, that is the bifurcation parameter for the family. This parameter determines the family $\mathcal{F}$ for β, which takes specific discrete values. Once a k-modal chaotic map is defined, it is possible to generate a less than k-modal map just by controlling the bifurcation parameter β. The multimodal logistic maps display very interesting stretching-and-folding structures and their bifurcation diagrams reproduce scaled bifurcations of the classical logistic map. The basins of attraction of each fixed point of the family were constructed in terms of the bifurcation parameter. We believe the family of chaotic maps will in the future enrich many present chaos applications, in particular, chaotic cryptography and radars. Research in this direction has already begun and results are expected to be reported fairly soon.

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References

[1] T-Y. Li, J.A. Yorke, Period three implies chaos, The American Mathematical Monthly 82 (10) (1975) 985–992.

[2] R.M. May, Simple mathematical models with very complicated dynamics, Nature 261 (5560) (1976) 459–467.

[3] S.N. Elaydi, Discrete Chaos, Chapman & Hall/CRC, 2000.

[4] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, The American Mathematical Monthly 99 (1992) 332-334.

[5] S. Banerjee, P. Rahjan, C. Grebogi, Bifurcations in two-dimensional piecewise smooth maps theory and applications in switching circuits, IEEE Trans. Circ. Syst. I 47 (2000) 633-643.

[6] J.H.B. Deane, Chaos in a current-mode controlled boost dc-dc converter, IEEE Trans. Circ. Syst. I 39 (1992) 680-683.

[7] A.N. Pisarchik, N.J. Flores-Carmona, M. Carpio-Valadez, Encryption and decryption of images with chaotic map lattices, Chaos: Interdiscipl J Nonlinear Sci 16 (3) (2006) 033118.

[8] A.N. Pisarchik, M. Zanin, Image encryption with chaotically coupled chaotic maps, Physica D 237 (2008) 2638–2648.

[9] V. Patidar, N.K. Pareek, K.K. Sud, A new substitution–diffusion based image cipher using chaotic standard and logistic maps, Commun Nonlinear Sci Numer Simulat 14 (2009)3056–3075.

[10] Y. Tang, Z. Wang, J. Fang, Image encryption using chaotic coupled map lattices with time-varying delays, Commun in Nonlinear Sci Numer Simulat 15 (2010) 2456–2468.

[11] H. Yang, K.W.Wong, X. Liao, W. Zhang, P. Wei, A fast image encryption and authentication scheme based on chaotic maps, Commun Nonlinear Sci Numer Simulat 15 (2010) 3507–3517.

[12] J.W. Yoon, H. Kim, An image encryption scheme with a pseudorandom permutation based on chaotic maps, Commun Nonlinear Sci Numer Simulat 15 (2010) 3998–4006.

[13] A.N. Pisarchik, F.R. Ruiz-Oliveras, Optical chaotic communication using generalized and complete synchronization, IEEE J Quant Electron 46 (3) (2010) 279–284.

[14] M. Suneel, Electronic circuit realization of the logistic map, Sadhana 31 (Part 1) (2006) 69-78.

[15] I. Campos-Cantón, E. Campos-Cantón, J. S. Murguía, H. C. Rosu, A simple electronic circuit realization of the tent map, Chaos, Solitons & Fractals 42 (1) (2009) 12–16.

[16] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, Correspondance Mathématique et Physique 10 (1838) 113–121.

[17] D. Arroyo, G. Alvarez, J. M. Amigó, Estimation of the control parameter from symbolic sequences: Unimodal maps with variable critical point, Chaos 19 (2009) 023125.

[18] N. Mihalache, Two counterexamples in rational and interval dynamics, arxiv:0810.1474v1 (2008) 1–49.

[19] W. de Melo, S. van Strien, One-dimentional dynamics, vol. 25 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1993.

[20] D. Smania, Phase space universality for multimodal maps, Bull. Braz. Math. Soc. New Series 36 (2) (2005) 225–274.