On the controllability of networks with nonidentical linear nodes

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The controllability of dynamical networks depends on both network structure and node dynamics. For networks of linearly coupled linear dynamical systems the controllability of the network can be determined using the well-known Kalman rank criterion. In the case of identical nodes the problem can be decomposed in local and structural contributions. However, for strictly different nodes an alternative approach is needed. We decomposed the controllability matrix into a structural component, which only depends on the networks structure and a dynamical component which includes the dynamical description of the nodes in the network. Using this approach we show that controllability of dynamical networks with strictly different linear nodes is dominated by the dynamical component. Therefore even a structurally uncontrollable network of different *n* dimensional nodes becomes controllable if the dynamics of it's nodes are properly chosen. Conversely, a structurally controllable network becomes uncontrollable for a given choice of the node's dynamics. Furthermore, as nodes are not identical, we can have nodes that are uncontrollable in isolation, while the entire network is controllable, in this sense the node's controllability is overwritten by the network even if the structure is uncontrollable. We illustrate our results using single-controller networks and extend our findings to conventional networks with large number of nodes.

Keywords: Network control, controllability, linear system.

1. Introduction

Complex networks can be used to model almost any large scale system, in this representation functional units are represented as nodes and their interactions as links. The structural complexity of a system is then described as a graph with features like the small-world and scale-free effects, sparsely connected nodes with high clustering coefficients, among others [Chen *et al.* (2015)]. Additionally to the structural complexity of the system there are different sources of complexity that can be considered while modelling, *e.g.*, one can consider the complexity in its node's dynamical evolution; the diverse nature of its nodes and links, or even mechanisms for adaptation that affect the network's structural evolution [Strogatz (2001)]. In particular, a dynamical network is a mathematical model where additionally to the structural complexity of the system, the dynamical complexity given by the evolution of its nodes is taken into consideration. As such, the dynamical analysis of its behaviours must include both: structural and dynamical complexity [Wang (2002)]. Furthermore, the main reason to investigate the dynamics of a system is to impose on it our desired objectives, that is to control it. Therefore, the first question one can ask is about the possibility of achieving such a control objective, or in other words, one asks if the dynamical network is controllable [Liu *et al.* (2011)].

Controllability is a central concept in control theory. A dynamical system is said to be controllable if a control input can be designed to take it from an initial state to a desired state in finite time [Rugh (1996)]. For a dynamical network, designing a control input for each node is a prohibitive

and unnecessary effort. It has been shown that by applying controllers to only a fraction of its nodes a dynamical network can be stabilized to its equilibrium. That is, a *virtual control* is applied to the uncontrolled nodes as the control actions travel through the network connections, in this way one is capable of controlling the entire network. This form of network control is usually referred to as pinning control [Li *et al.* (2004)]. From this perspective, the controllability of a dynamical network, does not only depends on the dynamical features of its nodes and the structure of the network, but also on the choice of where to apply the control inputs. Inspired by this realization on [Liu *et al.* (2011)] a matching algorithm was proposed to identify the minimum set of locations to control to direct the entire network to a desired state.

In [Sorrentino *et al.* (2007); Sorrentino (2007)] the concept of *pinning-controllability* of a network was coined to describe whether a dynamical network can be stabilized to an equilibrium point by controlling only a small fraction of its nodes. However, this is markedly different to the conventional meaning of controllability in control theory. In fact, one can see *pinning-controllability* as a stabilizability condition rather than actual controllability of the network.

In the case of large scale linear systems, the concept of *structural-controllability* was first introduced by Lin in 1974. A linear system is structurally controllable if the graph of (A, B) is spanned by a "Cactus" (where, a Cactus is a connected graph in which any two cycles have at most one node in common and any two graph cycles have no edge in common.) In other words, the graph of (A, B) contains only accessible nodes and not dilation [Lin (1974)].This simple idea was further developed by [Liu *et al.* (2011)], to provide a matching algorithm that identifies the minimum number of nodes that required a control action (a feedback loop) to ensure that a directed and weighted network is controllable, more specifically they show the network to be *structurally* controllable. Afterward, the concept of structural permeability was introduced in [Lo Iudice F. (2015)], where an algorithm to measure the structural propensity of networks to be controlled was developed. Additionally, Wang *et al.* proposed to optimize the controllability of the network by minimum structural perturbations [Wang *et al.* (2012)]. It is worth noticing that in these works the question is restricted to the structural component of the network. In fact, as remarked in [Cowan *et al.* (2012)], the above results consider only one dimensional integrator nodes leaving the contribution of the node dynamics out of consideration. In this paper, controllability of a dynamical network will always mean the classical concept of controllability, taking into consideration both aspects: the connection structure of the network and the dynamical description of its nodes.

Recently, Wang *et al.* investigated the controllability of MIMO networks with identical nodes, their findings show that both: structural and dynamical aspects must be taken into consideration to establish the controllability of a general structure networks Wang *et al.* (2015). In the context of multiagent systems the controllability problem has been addressed by many authors. For example, in the work by Tanner in 2004, it is shown that for a nearest-neighbours formation leader-follower, controllability was achievable if the eigenspectrum of the resulting Laplacian submatrix was dominated by the leaders contribution [Tanner (2004)]. These results were extended to multi-leader formations by Ji *et al.* and by Rahmani *et al.* considering almost symmetric partitions of the follower population amount the leaders [Ji & Egerstedt (2007); Rahmani *et al.* (2009)]. Additional works on the controllability of multiagent systems with multiple leaders with and without direct connection to all the followers on almost symmetric formations were considered in [Lou & Hong (2012); Zhang *et al.* (2011)]. The case of switching topologies was also considered in [Liu *et al.* (2008)]. Moreover, the formation controllability for identical high-dimension linear and time invariant agents was considered in [Cai & Zhong (2010)].

It is worth remaking that in the above works, all nodes are identical. Therefore an important

additional issue that needs consideration is the case of networks with different nodes. In this sense, the work of Xiang *et al.* investigates the controllability of networks with nonidentical nodes. However, the attention was restricted to a very particular type of non-identical nodes, which have intrinsic dynamic described by an identical matrix in each node multiplied by a different kinetic constant. Moreover, this matrix is the same one of the inner-couplings between the states of the network. That is, they are only different by a kinetic constant [Xiang *et al.* (2013)].

In this paper we investigate the controllability of weighted and directed networks with strictly different nodes. We restrict our attention to the case of linear nodes with linear couplings, as such the controllability of the network can be determined using the Kalman rank criterion. To show the contributions of the network structure and the node's dynamics we decomposed the controllability matrix into a structural and a dynamical components. We find that for dynamical networks with strictly different linear nodes controllability is dominated by the dynamical component. Then, regardless of the controllability of the structure of the network the dynamics of the nodes can be chosen as to determine the controllability of the network. That is, the network structure is uncontrollable even if the dynamics of its different n-dimensional nodes is properly chosen. Alternatively, if the network topology are controllable, the entire network becomes uncontrolable for a given choice of the node's dynamics. Moreover, since the node's dynamics are nonidentical, is possible to have situations where the node dynamics are not controllable isolated from the network, however, we show that is possible for the entire network to become controllable. In this way, the controllability of the node in isolation is overwritten by the network even in the case where the structure is uncontrollable. Our findings are extended to conventional single-controller networks with large number of nodes.

2. Preliminaries

Consider a controlled network of N identical linear system with weighted and directed connections, the dynamics of each node are given by:

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1, j \neq i}^{N} \mathscr{L}_{ij} \Gamma x_j(t) + \delta_i B u_i, \ i = 1, ..., N \tag{2.1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), ..., x_{in}(t)]^\top \in \mathbf{R}^n$ is the state vector of the i-th node; the system's matrix $A \in \mathbf{R}^{n \times n}$ describes the intrinsic dynamics of each linear node. $\Gamma \in \mathbf{R}^{n \times n}$ is a zero-one constant matrix indicating the inner-couplings between states, the outer-coupling matrix describes the connections between nodes $\mathscr{L} = \{\mathscr{L}_{ij}\} \in \mathbf{R}^{N \times N}$, and it is constructed as follows: the entry $\mathscr{L}_{ij} \neq 0$ if the j-th node receives information from the i-th node, otherwise $\mathscr{L}_{ij} = 0$. Since the network is directed $\mathscr{L}_{ij}$ is not necessarily identical to $\mathscr{L}_{ji}$. The control input to the i-th node is $u_i(t) \in \mathbf{R}^p$, with $B \in \mathbf{R}^{n \times p}$ the control input matrix, which is identical for every node.

Following the pinning control approach, we consider that only a small fraction $q = \lfloor \rho N \rfloor$ ($\rho \ll 1$) of nodes in the network are controlled. To indicate that the i-th node in (2.1) is subject to a control action we set $\delta_i = 1$, otherwise $\delta_i = 0$. Without lost of generality, we can reorder the node indexes such that the first q nodes of (2.1) are controlled, while the remaining $N - q$ nodes have no controller. Then, in vector form the dynamical network can be rewritten as:

$$\dot{X}(t) = (I_N \otimes A + \mathscr{L} \otimes \Gamma)X(t) + (\Delta \otimes B)U(t) \tag{2.2}$$

where $X(t) = [x_1(t)^\top, ..., x_N(t)^\top]^\top \in \mathbf{R}^{Nn}$ is the state vector of the entire network, and $U(t) = [u_1(t)^\top, ..., u_q(t)^\top, 0, ..., 0]^\top \in \mathbf{R}^{Np}$ is the network's control input. I_N is the N-dimensional identity matrix, $\otimes$ is

the Kronecker product, and $\Delta = \mathrm{Diag}([\underbrace{1,..,1}_{q},\underbrace{0,...,0}_{N-q}]) \in \mathbf{R}^{N\times N}$.

Defining $\mathscr{A}_0 = (I_N \otimes A + \mathscr{L} \otimes \Gamma)$ and $\mathscr{B} = \Delta \otimes B$, one verifies that the network in (2.2) is a linear system of the form:

$$\dot{X}(t) = \mathscr{A}_0 X(t) + \mathscr{B}U(t) \tag{2.3}$$

A classical result in control theory for linear time invariant systems is the so-called Kalman Controllability criterion, which can be expressed as follows:

Lemma 1. [Rugh (1996)] For a system in the form of (2.3) the following declarations are equivalent:

    I. System (2.3) is completely controlable.

    II. The controllability matrix

$$\mathscr{Q}_0 = [\mathscr{B}, \mathscr{A}_0\mathscr{B}, ..., \mathscr{A}_0^{Nn-1}\mathscr{B}] \tag{2.4}$$

    is of full rank, *i.e.* $\mathrm{Rank}(\mathscr{Q}_0) = Nn$.

    III. The relation

$$v^T \mathscr{A}_0 = \lambda v^\top \text{ implies } v^T \mathscr{A}_0 \neq 0^\top \tag{2.5}$$

    where v is a no-zero left-eigenvalue of the matrix $\mathscr{A}_0$ corresponding to the eigenvalue λ.

Consider now the case when the nodes have no dynamics ($A = 0_n$), the network becomes $\dot{X}(t) = (\mathscr{L} \otimes \Gamma)X(t) + (\Delta \otimes B)U(t)$ or equivalently

$$\dot{X}(t) = \mathscr{A}_\mathscr{L} X(t) + \mathscr{B}U(t) \tag{2.6}$$

where $\mathscr{A}_\mathscr{L} = \mathscr{L} \otimes \Gamma$ which following **Lemma 1**, is controllable if

$$\mathscr{Q}_\mathscr{L} = [\mathscr{B}, \mathscr{A}_\mathscr{L}\mathscr{B}, ..., \mathscr{A}_\mathscr{L}^{Nn-1}\mathscr{B}] \tag{2.7}$$

is full rank. In particular, for one-dimensional systems ($n = 1$, $\Gamma = 1$) the network (2.7) becomes the one considered in [Liu *et al.* (2011)], to identify the locations for control action using the matching algorithm. As such, using their proposed matching algorithm, given a ($\mathscr{L}$) network connection description one can identify a minimum set of locations to control ($\mathscr{B}$) that renders the dynamical network controllable. That is, it makes the network structurally controllable. By contrast in [Cowan et al. (2012)]where inclusion of first-order self dynamics is addressed; it is shown that structural controllability can be achieved with a single input, which should be attached to a spanning tree that start at input. Even though in Cowan *et al.* the contribution is in the sense of structural controllability, this brings up to the question whether the dynamic component predominates over the structural component of the network. Moreover, they assumed that the dimension of the state of each node is one.

Now lets consider the case where the nodes in the network are nonidentical. That is, the network is given by

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1, j\neq i}^{N} \mathscr{L}_{ij}\Gamma x_j(t) + \delta_i B u_i, \ i = 1,...,N \tag{2.8}$$

where $A_i \in \mathbf{R}^{n \times n}$ is the system's matrix of the i-th node. In vector form the network becomes

$$\dot{X}(t) = (\mathbb{A} + \mathscr{L} \otimes \Gamma)X(t) + \mathscr{B}U(t) \tag{2.9}$$

where $\mathbb{A} = \mathrm{Diag}([A_1, A_2, ..., A_N]) \in \mathbf{R}^{Nn \times Nn}$. As before, the controllability of the linear system in (2.9) is equivalent to requiring full rank for the matrix

$$\mathscr{Q} = [\mathscr{B}, \mathscr{A}\mathscr{B}, ..., \mathscr{A}^{Nn-1}\mathscr{B}] \tag{2.10}$$

where $\mathscr{A} = \mathbb{A} + \mathscr{L} \otimes \Gamma$.

In the work by Xiang *et al.* a very particular type of nonidentical node was considered, namely, $A_i = c_i\Gamma$. For these nearly identical nodes, one can write the node's dynamics as:

$$\mathbb{A} = \mathscr{C} \otimes \Gamma \tag{2.11}$$

where $\mathscr{C} = \mathrm{Diag}([c_1, c_2, ..., c_N]) \in \mathbf{R}^{N \times N}$ are kinetic constants [Xiang *et al.* (2013)]. It follows that the network of nonidentical nodes (2.9) can be rewritten as:

$$\dot{X}(t) = (\bar{\mathscr{L}} \otimes \Gamma)X(t) + \mathscr{B}U(t) \tag{2.12}$$

where $\bar{\mathscr{L}} = \mathscr{C} + \mathscr{L} \in \mathbf{R}^{N \times N}$. Letting v_1 and v_2 be the left eigenvectors of $\bar{\mathscr{L}}$ and Γ, respectively. The third part of **Lemma 1** can be used to establish the controllability of (2.12) in terms of two simplified conditions that must be satisfied simultaneously [Xiang *et al.* (2013)]:

   i. The pair (Γ, B) is completely controllable.

   ii. The left eigenvectors of $\bar{\mathscr{L}}$ have nonzero values in their first q entries.

Although this result simplifies the analysis by discomposing the controllability problem into a local and a global conditions (i and ii, above), it can only be applied to a very restricted type of networks. In the following section, we propose an alternative decomposition for the general case, *i.e.*, when $A_i \neq c_i\Gamma$. It is important to emphasize that the controllability addressed in the present paper is fundamentally different from the "structural controllability" and "pinning controllability".

3. Controllability of networks with strictly different nodes

The controllability matrix for a network of nonidentical linear nodes (2.8) becomes

$$\mathscr{Q} = [\mathscr{B}, (\mathbb{A} + \hat{\mathscr{L}})\mathscr{B}, (\mathbb{A} + \hat{\mathscr{L}})^2\mathscr{B}, ..., (\mathbb{A} + \hat{\mathscr{L}})^{Nn-1}\mathscr{B}]$$

where $\mathbb{A} = \mathrm{Diag}([A_1, A_2, ..., A_N]) \in \mathbf{R}^{Nn \times Nn}$ and $\hat{\mathscr{L}} = \mathscr{L} \otimes \Gamma \in \mathbf{R}^{Nn \times Nn}$.

The controllability matrix above can be readily decomposed into a structural and a dynamical component, such that

$$\mathscr{Q} = \mathscr{Q}_{Stru} + \mathscr{Q}_{Dyn} \tag{3.1}$$

where

$$\mathscr{Q}_{Stru} = [\mathscr{B}, \hat{\mathscr{L}}\mathscr{B}, \hat{\mathscr{L}}^2\mathscr{B}, ..., \hat{\mathscr{L}}^{Nn-1}\mathscr{B}], \text{ and} \tag{3.2}$$

$$\mathscr{Q}_{Dyn} = [0, \mathbb{A}\mathscr{B}, (\mathbb{A}^2 + \mathbb{A}\hat{\mathscr{L}} + \hat{\mathscr{L}}\mathbb{A})\mathscr{B}, ..., ((\mathbb{A} + \hat{\mathscr{L}})^{Nn-1} - \hat{\mathscr{L}}^{Nn-1})\mathscr{B}] \tag{3.3}$$

In general, the rank of $\mathscr{Q}$ is independent of the rank of this structural and dynamical components. That is, the interaction between $\mathscr{Q}_{Struc}$ and $\mathscr{Q}_{Dyn}$ can lead to a controllable network (2.8) even if the structural component is not full rank, and vice versa. To illustrate this point lets consider the following cases:
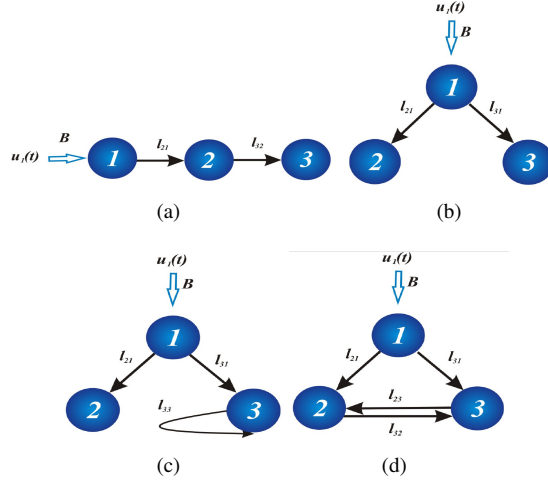
FIG. 1. Illustrative examples for the controllability of dynamical networks.

3.1 *Nodes without dynamics*

Let the nodes for each network described in Figure 1 be one-dimensional, with no dynamics ($A_1 = A_2 = A_3 = 0 \in \mathbf{R}$), and linearly coupled. Then, the controllability matrices for these networks are, respectively:

$$\mathcal{Q}_{(a)\mathscr{L}} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & 0 & Bl_{32}l_{21} \end{pmatrix}, \mathcal{Q}_{(b)\mathscr{L}} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & Bl_{31} & 0 \end{pmatrix},$$

$$\mathcal{Q}_{(c)\mathscr{L}} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & Bl_{31} & Bl_{33}l_{31} \end{pmatrix}, \mathcal{Q}_{(d)\mathscr{L}} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & Bl_{23}l_{31} \\ 0 & Bl_{31} & Bl_{32}l_{21} \end{pmatrix}.$$

As such, the networks (a) and (c) are structurally controllable for any nonzero choices of the connection strengths; while (b) is always structurally uncontrollable, and the controllability of (d) is dependent on choice of strength values, it is only controllable if the condition $l_{32}l_{21}^2 \neq l_{23}l_{31}^2$ is satisfied.

3.2 *Nodes with identical one-dimensional dynamics*

Considering identical one-dimensional node dynamics ($A_i = a \in \mathbf{R}, \forall i$), the corresponding controllability matrices for the networks in Figure 1 become:

$$\mathcal{Q}_{(a)} = \begin{pmatrix} B & aB & a^2B \\ 0 & Bl_{21} & 2aBl_{21} \\ 0 & 0 & Bl_{32}l_{21} \end{pmatrix}, \mathcal{Q}_{(b)} = \begin{pmatrix} B & aB & a^2B \\ 0 & Bl_{21} & 2aBl_{21} \\ 0 & Bl_{31} & 2aBl_{31} \end{pmatrix},$$

$$\mathcal{Q}_{(c)} = \begin{pmatrix} B & aB & a^2B \\ 0 & Bl_{21} & 2aBl_{21} \\ 0 & Bl_{31} & Bl_{31}(2a+l_{33}) \end{pmatrix}, \mathcal{Q}_{(d)} = \begin{pmatrix} B & aB & a^2B \\ 0 & Bl_{21} & B(2al_{21}+l_{23}l_{31}) \\ 0 & Bl_{31} & B(2al_{31}+l_{32}l_{21}) \end{pmatrix}$$

The conclusions of the previous case remain valid, the networks (a) and (c) are controllable, (b) is uncontrollable, and (d) is only controllable if $l_{32}l_{21}^2 \neq l_{23}l_{31}^2$.

3.3 *Nodes with nonidentical one-dimensional dynamics*

Consider that the nodes in the networks shown in Figure 1 are nonidentical ($A_i = a_i \in \mathbf{R}, \forall i$, with $a_1 \neq a_2 \neq a_3$) we have:

$$\begin{pmatrix} B & a_1 B & a_1^2 B \\ 0 & Bl_{21} & B(a_1 + a_2)l_{21} \\ 0 & 0 & Bl_{32}l_{21} \end{pmatrix}, \begin{pmatrix} B & a_1 B & a_1^2 B \\ 0 & Bl_{21} & B(a_1 + a_2)l_{21} \\ 0 & Bl_{31} & B(a_1 + a_3)l_{31} \end{pmatrix},$$

$$\begin{pmatrix} B & a_1 B & a_1^2 B \\ 0 & Bl_{21} & B(a_1 + a_2)l_{21} \\ 0 & Bl_{31} & B(a_1 + a_3 + l_{33})l_{31} \end{pmatrix}, \begin{pmatrix} B & a_1 B & a_1^2 B \\ 0 & Bl_{21} & B((a_1 + a_2)l_{21} + l_{23}l_{31}) \\ 0 & Bl_{31} & B((a_1 + a_3)l_{31} + l_{32}l_{21}) \end{pmatrix}$$

In the case of nonidentical node dynamics the conclusions change. The networks (a) and (c) are both controllable. With the network in (d) is controllable only if the new condition $(a_1 + a_3)l_{31}l_{21} + l_{32}l_{21}^2 - l_{23}l_{31}^2 \neq 0$ is satisfied. However, the most striking change occurs for network (b), which becomes controllable for nonidentical node dynamics ($a_2 \neq a_3$).

We remark that the controllability matrices of the networks shown in Figure 1 with nonidentical nodes can be easily decomposed as in (3.1) resulting on:

$$\mathcal{Q}_{(a)} = \mathcal{Q}_{(a)Stru} + \mathcal{Q}_{(a)Dyn} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & 0 & Bl_{32}l_{21} \end{pmatrix} + \begin{pmatrix} 0 & a_1 B & a_1^2 B \\ 0 & 0 & B(a_1 + a_2)l_{21} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{Q}_{(b)} = \mathcal{Q}_{(b)Stru} + \mathcal{Q}_{(b)Dyn} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & Bl_{31} & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 B & a_1^2 B \\ 0 & 0 & B(a_1 + a_2)l_{21} \\ 0 & 0 & B(a_1 + a_3)l_{31} \end{pmatrix},$$

$$\mathcal{Q}_{(c)} = \mathcal{Q}_{(c)Stru} + \mathcal{Q}_{(c)Dyn} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & 0 \\ 0 & Bl_{31} & Bl_{33}l_{31} \end{pmatrix} + \begin{pmatrix} 0 & a_1 B & a_1^2 B \\ 0 & 0 & B(a_1 + a_2)l_{21} \\ 0 & 0 & B(a_1 + a_3)l_{31} \end{pmatrix},$$

$$\mathcal{Q}_{(d)} = \mathcal{Q}_{(d)Stru} + \mathcal{Q}_{(d)Dyn} = \begin{pmatrix} B & 0 & 0 \\ 0 & Bl_{21} & Bl_{23}l_{31} \\ 0 & Bl_{31} & Bl_{32}l_{21} \end{pmatrix} + \begin{pmatrix} 0 & a_1 B & a_1^2 B \\ 0 & 0 & B(a_1 + a_2)l_{21} \\ 0 & 0 & B(a_1 + a_3)l_{31} \end{pmatrix}$$

It is easy to verify that the structural components of the controllability matrix for the network (b) in Figure 1 is not full rank (Rank($\mathcal{Q}_{(b)Stru}$) = 2), nonetheless the controllability of the dynamical network is full rank (Rank($\mathcal{Q}_{(b)}$) = 3). This is due to the contribution of dynamical component of the controllability matrix (Rank($\mathcal{Q}_{(b)Dyn}$) = 2). As the rank of the matrix sum must satisfy:

$$\text{Rank}(\mathcal{Q}_{(b)}) \leqslant \text{Rank}(\mathcal{Q}_{(b)Stru}) + \text{Rank}(\mathcal{Q}_{(b)Dyn}) \tag{3.4}$$

3.4 *Nodes with nonidentical n-dimensional dynamics*

Now we consider the case of higher dimensions ($n > 1$). For simplicity let $n = 2$, then we have

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \Gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

It is easy to verify that if the condition $b_1^2 a_{i3} - b_2^2 a_{i2} \neq b_1 b_2 (a_{i1} - a_{i4})$ is satisfied, then the i-th node is controllable, that is, the pair (A_i, B) is controllable.

In what follows we let $l_{ij} = 1$ if the j-th node receives information from the i-th node, zero otherwise; and $\gamma_1 = \gamma_2 = b_1 = b_2 = 1$.

3.4.1 *All pairs are controllable.* Let the node dynamics be given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then all pairs (A_i, B) are controllable. For the networks in Figure 1 the controllability matrices are:

$$\mathcal{Q}_{(a)} = \mathcal{Q}_{(a)Stru} + \mathcal{Q}_{(a)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & -2 & -9 & -18 \\ 0 & 0 & 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 4 & 14 & 41 \\ 0 & 0 & 0 & 6 & 18 & 42 \end{pmatrix}$$

$$\mathcal{Q}_{(b)} = \mathcal{Q}_{(b)Stru} + \mathcal{Q}_{(b)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & -2 & -9 & -18 \\ 0 & 0 & 4 & 8 & 10 & 6 \\ 0 & 0 & 4 & 13 & 36 & 93 \\ 0 & 0 & 4 & 11 & 28 & 69 \end{pmatrix}$$

$$\mathcal{Q}_{(c)} = \mathcal{Q}_{(c)Stru} + \mathcal{Q}_{(c)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & -2 & -9 & -18 \\ 0 & 0 & 4 & 8 & 10 & 6 \\ 0 & 0 & 4 & 20 & 78 & 280 \\ 0 & 0 & 4 & 17 & 60 & 205 \end{pmatrix}$$

$$\mathcal{Q}_{(d)} = \mathcal{Q}_{(d)Stru} + \mathcal{Q}_{(d)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 2 & 6 & 25 \\ 0 & 0 & 4 & 14 & 39 & 106 \\ 0 & 0 & 4 & 17 & 57 & 175 \\ 0 & 0 & 4 & 17 & 54 & 157 \end{pmatrix}$$

Although the structural components in (b) and (d) are not full rank ($\text{Rank}(\mathcal{Q}_{(b)Stru}) = \text{Rank}(\mathcal{Q}_{(d)Stru}) = 2$) all the networks in Figure 1 are controllable ($\text{Rank}(\mathcal{Q}_{(a)}) = \text{Rank}(\mathcal{Q}_{(b)}) = \text{Rank}(\mathcal{Q}_{(c)}) = \text{Rank}(\mathcal{Q}_{(d)}) = 6$). Furthermore, by construction the rank of the structural component can not be larger that N ($\mathcal{Q}_{Stru} \leqslant N$), therefore the contribution of the dynamical component ($\text{Rank}(\mathcal{Q}_{Dyn}) \leqslant Nn - 1$) dominates the rank of the controllability matrix of the entire network.

3.4.2 *A pair without control input is uncontrollable.* Consider that the node's dynamics are given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

In this case, the node that receives the control action is controllable with B the same as pair (A_3, B), however, pair (A_2, B) is uncontrollable. The results show that all networks in Figure 1 remain controllable ($\text{Rank}(\mathcal{Q}_a) = \text{Rank}(\mathcal{Q}_b) = \text{Rank}(\mathcal{Q}_c) = \text{Rank}(\mathcal{Q}_d) = 6$). It is worth remarking that although the node A_2 is uncontrollable with B in isolation, the entire dynamical network is controllable.

3.4.3 *The node with control input is uncontrollable.* Finally, consider the case of nonidentical linear nodes where only the node with the control input is uncontrollable. That is, the node's dynamics are given by:

$$A_1 = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

In this case the pairs (A_2, B) and (A_3, B) are controllable, while the pair (A_1, B) is uncontrollable. The controllability matrices for the network in Figure 1 are given by:

$$\mathcal{Q}_{(a)} = \mathcal{Q}_{(a)Stru} + \mathcal{Q}_{(a)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 & -4 \\ 0 & 0 & 2 & 2 & 0 & -4 \\ 0 & 0 & 0 & 3 & 9 & 23 \\ 0 & 0 & 0 & 4 & 9 & 18 \end{pmatrix}$$

$$\mathcal{Q}_{(b)} = \mathcal{Q}_{(b)Stru} + \mathcal{Q}_{(b)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 & -4 \\ 0 & 0 & 2 & 2 & 0 & -4 \\ 0 & 0 & 3 & 7 & 17 & 41 \\ 0 & 0 & 2 & 5 & 12 & 29 \end{pmatrix}$$

$$\mathcal{Q}_{(c)} = \mathcal{Q}_{(c)Stru} + \mathcal{Q}_{(c)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 & -4 \\ 0 & 0 & 2 & 2 & 0 & -4 \\ 0 & 0 & 3 & 13 & 47 & 163 \\ 0 & 0 & 2 & 9 & 33 & 115 \end{pmatrix}$$

$$\mathcal{Q}_{(d)} = \mathcal{Q}_{(d)Stru} + \mathcal{Q}_{(d)Dyn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 19 \\ 0 & 0 & 2 & 6 & 18 & 52 \\ 0 & 0 & 3 & 10 & 32 & 94 \\ 0 & 0 & 2 & 9 & 27 & 79 \end{pmatrix}$$

All the networks in Figure 1 are uncontrollable $(\text{Rank}(\mathcal{Q}_a) = \text{Rank}(\mathcal{Q}_b) = \text{Rank}(\mathcal{Q}_c) = \text{Rank}(\mathcal{Q}_d) = 5)$. Even in the cases where the network structure is controllable, *e.g.*, $\text{Rank}(\mathcal{Q}_{(a)Stru}) = \text{Rank}(\mathcal{Q}_{(c)Struc}) = 3$, the entire network is not controllable. Again, the controllability of the entire dynamical network is determine by the contribution of the dynamical component.

In the following Section we investigated the controllability of a couple of well-known directed network configurations with a single controller.

4. Controllability of typical networks with strictly different nodes.

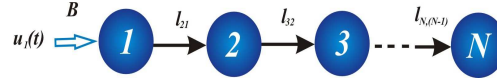### 4.1 *Directed chain of n-dimensional nodes*

FIG. 2. A directed chain network with a single controller

Consider a directed chain network with a single controller input. Naturally, we assume that the external control input is at node 1, as shown in Figure 2. The nodes in the network are nonidentical *n*-dimensional linear systems, such that $\mathbb{A} = \text{Diag}(A_1, A_2, ..., A_N) \in \mathbf{R}^{Nn \times Nn}$. For simplicity, let the inter-coupling matrix be $\Gamma = I_n$, with $B = [b_1, b_2, ..., b_n]^\top \in \mathbf{R}^n$ and $b_i = 1\ \forall i$. Since the outer-coupling matrix for the network in Figure 2 is:

$$\mathscr{L}_{Chain} = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ l_{2,1} & 0 & \cdots & & 0 \\ 0 & l_{3,2} & 0 & & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \cdots & 0 & l_{N,(N-1)} & 0 \end{pmatrix}$$

The controllability matrix for the directed chain network is readily found to be:

$$\mathscr{Q}_{Chain} = \mathscr{Q}_{Chain,Stru} + \mathscr{Q}_{Chain,Dyn} = \begin{pmatrix} B & 0 & \cdots & & 0 & 0 & \cdots & 0 \\ 0 & Bl_{21} & \cdots & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & Bl_{N,N-1}\cdots l_{21} & 0 & \cdots & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 & BA_1 & BA_1^2 & \cdots & BA_1^{Nn-2} & BA_1^{Nn-1} \\ 0 & 0 & B(A_1+A_2)l_{21} & \cdots & B(A_1+A_2)^{Nn-2}l_{21} & B(A_1+A_2)^{Nn-1}l_{21} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B(A_1+\ldots+A_N)l_{N,N-1}\cdots l_{21} & B(A_1+\ldots+A_N)^{Nn-1}l_{N,N-1}\cdots l_{21} \end{pmatrix}$$

We have that $\text{Rank}(\mathscr{Q}_{Chain,Stru}) = N$, that is the network structure is controllable. However, the controllability of the overall dynamical network is determine by the contribution of the dynamical component, since its rank larger, in fact is at most $\text{Rank}(\mathscr{Q}_{Chain,Dyn}) \leqslant Nn - 1$. From the above observations we have the following result:

Theorem 1.   Although the directed chain network in Figure 2 is structurally controllable, it is possible to choose strictly different node dynamics $A_1, A_2, ..., A_N$, such that the entire dynamical network becomes uncontrollable.

Proof: Since the contribution of the structural component to the rank condition is N. We need to choose A_1 such that, BA_1 is a null matrix, this reduces the rank of $\mathcal{Q}_{Chain,Dyn}$ by one. Next, $n-1$ of the matrix sums $A_1 + A_2$, and $A_1 + A_2 + ... + A_N$ are force to be null. For this chose of node dynamics one has that $\text{Rank}(\mathcal{Q}_{Chain,Dyn}) \leqslant N(n-1) - 1$. Then, if follows that $\text{Rank}(\mathcal{Q}_{Chain}) \leqslant Nn - 1$, that is, the controllability matrix for the dynamical network is not full rank. ***Q.E.D.***

Theorem 2.   The directed chain network of n-dimensional linear systems in Figure 2 can become uncontrollable even if the isolated pairs (A_i, B) with $i = 2, 3, ..., N$ are controllable.

Proof: In the same sense than the previous proof, for uncontrollability of the entire dynamical network we requires that BA_1 be a null matrix, this is equivalent to requiring the pair (A_1, B) uncontrollable when $A_1 B = BA_1$. In other words, $[B, A_1 B, A_1^2 B, ..., A_1^{n-1} B]$ not be of full rank, which is simply to verify when $A_1 B = 0$. However, the requirement that $n-1$ of the matrix sums $A_1 + A_2$, and $A_1 + A_2 + ... + A_N$ being null, does not involve B, therefore is possible to satisfy this restriction even if the node dynamics are controllable. ***Q.E.D.***

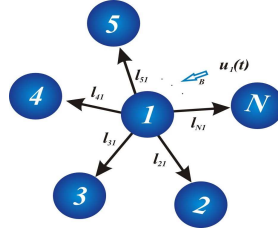## 4.2 *Directed star of strictly different nodes*

FIG. 3. A directed star network with a single controller

Consider a directed star network with a single controller input. Naturally, we assume that the external control input is at the central node, as shown in Figure 3. As before, he nodes in the network are nonidentical n-dimensional linear systems ($\mathbb{A} = \text{Diag}(A_1, A_2, ..., A_N) \in \mathbf{R}^{Nn \times Nn}$); the inter-coupling matrix is $\Gamma = I_n$, with $B = [b_1, b_2, ..., b_n]^\top \in \mathbf{R}^n$ and $b_i = 1 \; \forall i$. Then, the outer-coupling matrix for the network in Figure 3 is:

$$\mathscr{L}_{Star} = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ l_{2,1} & 0 & \cdots & & 0 \\ 3,1 & 0 & 0 & & 0 \\ \vdots & 0 & \ddots & & \vdots \\ l_{N,1} & \cdots & 0 & 0 & 0 \end{pmatrix}$$

The controllability matrix for the directed star network is:

$$\mathcal{Q}_{Star} = \mathcal{Q}_{Star,Stru} + \mathcal{Q}_{Star,Dyn} =$$

$$
\begin{pmatrix}
B & 0 & \cdots & 0 \\
0 & Bl_{21} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & Bl_{n1} & 0 & 0
\end{pmatrix}
+
\begin{pmatrix}
0 & BA_1 & BA_1^2 & \cdots & BA_1^{Nm-1} \\
0 & 0 & B(A_1+A_2)l_{21} & \cdots & B(A_1+A_N)^{Nm-1}l_{21} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & B(A_1+A_N)l_{N1} & \cdots & B(A_1+A_N)^{Nm-1}l_{N,1}
\end{pmatrix}
$$

For this network we have that $\text{Rank}(\mathcal{Q}_{Star,Stru}) = 2$, that is the network structure is uncontrollable. However, since the controllability of the overall dynamical network is determine by the contribution of the dynamical component, is possible to chose the node's dynamics such that the dynamical network becomes controllable. That is:

Theorem 3. Although the directed start network in Figure 3 has the structure component uncontrollable, it is possible to chose strictly different node dynamics $A_1, A_2, ..., A_N$, such that the entire dynamical network becomes controllable.

Proof: Since the contribution of the structural component to the rank condition is 2. We need to choose A_1 such that, BA_1 is different that zero, requiring additionally that the matrix sums $A_1 + A_j$, for $j = 2, ..., N$ also be different than zero. We have that $\text{Rank}(\mathcal{Q}_{Star,Dyn}) = Nn - 1$. Then, if follows that $\text{Rank}(\mathcal{Q}_{Star}) \leqslant Nn + 1$, that is, the controllability matrix for the dynamical network can be full rank, even if the rank of the structural component is 2. **Q.E.D.**

Theorem 4. The directed star network of n-dimensional linear systems in Figure 3 can become controllable even if an isolated pair (A_i, B) with $i \neq 1$ is uncontrollable.

Proof: Following the same reasoning as in the prevous proofs. For the directed star of nonidentical n-dimensional linear systems, we require that BA_1 be a non zero matrix, this is equivalent to requiring the pair (A_1, B) controllable when $A_1B = BA_1$. However, the requirement that the matrix sums $A_1 + A_2$, and $A_1 + A_3$, ..., $A_1 + A_N$ be different that zero, can be relaxed. That is, we can have a pair (A_i, B) with $i \neq 1$ uncontrollable, which results in a reduction of the rank of dynamical component by one $(\text{Rank}(\mathcal{Q}_{Star,Dyn}) = Nn - 2)$ and still satisfy the rank conditions since $(\mathcal{Q}_{Star}) \leqslant Nn$. **Q.E.D.**

5. Concluding Remarks

The controllability of dynamical networks with strictly different n-dimensional linear systems is dominated by the dynamical component of the controllability matrix. That is, although the structural aspects of the network are significant and provide effective guidelines for the design of pinning control strategies, our results show that the controllability of the dynamical network depends on the choice of local node dynamics. As such, network of strictly different nodes, with the structural component controllable can become uncontrollable; conversely, a network with structural component uncontrollable becomes controllable, depending on the contributions of its different node dynamics. Moreover, our results show that is not necessary for every node in the network be controllable in isolation to have a controllable network: Additionally, even if some pairs are controllable is possible to have an uncontrollable dynamical network.

We restrict our attention to conventional single controller networks of strictly different

n-dimensional linear nodes. In particular, for a directed chain network, which by construction the structural component is controllable, we have that a choice of strictly different local dynamics can be made such that the dynamical network becomes uncontrollable. Even if the nodes without control input are controllable. Similarly, we investigate a directed star network, which by construction the structural component is uncontrollable, in this case we showed that there is a choice of strictly different node dynamics such that the dynamical network becomes controllable. In addition to this, we found that even if a node without control input was uncontrollable the directed star dynamical network can still be controllable.

At first glance, to find that the controllability of a dynamical network of nonidentical nodes is dominated by the dynamical component seem discouraging. Since in recent year the structural controllability have been used to establish the set of matching node where control actions need to be applied to the network in a pinning control scheme. However, one must realize that it opens a door for the possibility of controllability for networks with uncontrollable structure. In fact, our results show that even in the case of a single control input, is possible for the dynamical network to be controllable if their nodes are strictly different. This seems to indicate that, in the case of networks with strictly different nodes, far less control actions are require to make a dynamical network controllable that does indicated a matching algorithm based only in its structure.

6. Acknowledgments

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