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## Article

### On the Emergence Of Chaos In Dynamical Networks

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We investigate how changes of specific topological features result on transitions among different bounded behaviors in dynamical networks. In particular, we focus on networks with identical dynamical systems, synchronized to a common equilibrium point, then a transition into chaotic behavior is observed as the number of nodes and the strength of their coupling changes. We analyze the network's transverse Lyapunov exponents (tLes) to derive conditions for the emergence of bounded complex behavior on different basic network models. We find that, for networks with a given number of nodes, chaotic behavior emerges when the coupling strength is within a specific bounded interval; this interval is reduced as the number of nodes increases. Furthermore, the endpoints the emergence interval depend on the coupling structure of network. We also find that networks with homogeneous connectivity, such as regular lattices and small-world networks are more conducive to the emergence of chaos than networks with heterogeneous connectivity like scale-free and star-connected graphs. Our results are illustrated with numerical simulations of the chaotic benchmark Lorenz systems, and to underline their potential applicability to real-world systems our results are used to establish conditions for the chaotic activation of a network of electrically coupled pancreatic  $\beta$ -cell models.

**Keywords:** Emergence, Dynamical networks, Coupled systems, Cell activation.

#### 1. Introduction

In recent year, complex networks has emerge as an attracted field of research, capturing the attention of different groups within the scientific community (Boccaletti et al. 2006, Newman et al. 2006, Wu 2007, Newman 2010). The main focus of concern has been network synchronization; that is, to determine conditions under which the trajectories of each node in the network evolve at unison (Strogatz 2001, Wang and Chen 2003, Belykh et al. 2005, Arenas et al. 2008, Lu and Chen 2009). Although, synchronization is a very important aspect of network dynamics, in many situations is necessary to solve the converse problem; that is, to determine under what conditions stable systems coupled with a give topology transits to regimes of, not necessarily synchronized, bounded oscillatory behavior. This issue has motivations in the oscillatory behavior of systems in nature, where for example, the interaction between biological units, like cells or individuals, result on transitions from a “switch-off” stable equilibrium to an “activated” oscillatory behavior (Smolen et al. 1993, Pinto et al. 1993, Widmaier et al. 2007). The emergence of dynamically complex behavior from simple stable systems is an observed phenomenon in the real-world, where association, collaboration, and interdependence allows for well-organized groups to achieve together much more that with individual efforts. Pioneering research in this direction include the works by Turing on Morphogenesis (Turing 1952) and by Smale (Smale 1976) on the emergence

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of oscillations on coupled cell models. More recently, the work by Pogromsky *et al.* focus on the emergence of stable oscillations as the result of a Poincaré-Andronov-Hopf bifurcation on diffusively coupled dynamical systems (Pogromsky and Nijmeijer 1999, 2001). The emergence of chaotic behavior in coupled maps was investigated in (Li *et al.* 2004), later Li *et al.* formalize these results showing that a network of of Logistic maps can satisfy the conditions for chaos in the sense of Li and Yorke (Ott 2002). The results in (Li *et al.* 2004) were extended to the case of dynamical networks where each node is a continuous-time multidimensional system by Zhang *et al.* (Zhang *et al.* 2006) and Yuan *et al.* (Yuan *et al.* 2008). In (Barajas-Ramírez and Femat 2008a,b) the transition to complex behavior was considered for small-world and scale-free topologies.

In this contribution, we are interested on determining conditions for the emergence of bounded complex behaviors on networks of stable dynamical systems, and relate these conditions to specific aspects of network topology. Thus, our problem can be stated as follows: Under what conditions a network of systems with a common stable equilibrium point, by virtue of their coupling, produces trajectories that evolve away from its synchronized solution but remain bounded? The answer to this question is not trivial. On the one side, if the network synchronizes its solution will be that of an stable equilibrium point. On the other hand, if the trajectories to move away from the common equilibrium point solution instability of the solutions is implied, which can easily result on unbounded trajectories. Therefore, the emergence of oscillatory bounded solutions, requires two complementary conditions: (1) the instability of the synchronization manifold, and at the same time, (2) its overall attractiveness. To determine what topological conditions satisfy these two requirements, we analyze the stability of the synchronized solution in terms of transverse Lyapunov exponents (tLes) (Ding and Yang 1997, Chen *et al.* 2003, Rangarajan and Ding 2002, Li and Chen 2003, Li 2005) and establish conditions, in terms of coupling strength and number of nodes, such that the solutions of the network remain ultimately bounded. More specifically, we determine intervals of coupling strength and number of nodes such that positive tLes are generated, resulting on local instability, while the trajectories are attracted to a ball containing the synchronized solution, within this ball the trajectories evolve away from synchrony following oscillatory, possibly chaotic, solutions.

The remainder of the paper is organized as follows: In Section 2, we derive the conditions for the emergence of complex behavior in terms of the network's tLes. In Section 3, the effect of changing the coupling strength and number of nodes on the emergence of bounded complex solutions for different network models is evaluated. We find that as the number of nodes or the coupling strength increases the emergence of bounded complex solutions is less likely. Additionally in Section 4, we show numerical evidence that for dynamical network with homogeneous connectivity the emergence of chaos is more likely that for networks with heterogeneous connectivity. We also illustrate the potential applicability of these results to real world systems with numerical simulations of the chaotic activation of a network of electrically coupled pancreatic  $\beta$  cell models. Finally, the paper is concluded with some comments and closing remarks.

## 2. Emergence of bounded solutions from an equilibrium point

Consider a set of dynamical systems, each one with parameters such that all its trajectories move towards a common asymptotically stable equilibrium point; if these dynamical systems are coupled in such a way that the resulting network synchronizes, all the trajectories of the network evolve towards the common equilibrium. For such a network, we investigate how changing the coupling strength and number of nodes forces its trajectories to evolve away from synchrony, but at the same time remaining bounded; oscillating in a possibly chaotic manner within a ball around its synchronized solution. In the following subsection, this problem is stated with more detail.

## 2.1 Problem Statement

The dynamical behavior of a network with  $N$  linearly and diffusively coupled identical nodes is described by the following state equations:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} (h(x_j) - h(x_i)), \text{ for } i = 1, \dots, N \quad (1)$$

where each node is an  $m$ -dimensional system  $x_i = [x_{i1}, x_{i2}, \dots, x_{im}]^T \in \mathbf{R}^m$ ; with  $f : \mathbf{R}^m \rightarrow \mathbf{R}^m$  a  $\mathcal{C}^1$  function which describes the dynamics of an isolated node ( $\dot{x} = f(x)$ ); and  $h : \mathbf{R}^m \rightarrow \mathbf{R}^m$  the output function for every node, which describes the inner connectivity between nodes. Every connection is assumed bidirectional and with a uniform coupling strength  $c \in \mathbf{R}$ . The topology of the network is captured by the connectivity matrix  $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N \times N}$  which is symmetric ( $a_{ij} = a_{ji} \geq 0, \forall i, \forall j, j \neq i$ ) and its diagonal entries are given by:  $a_{ii} = -\sum_{j=1, j \neq i}^N a_{ij} = -\sum_{j=1, j \neq i}^N a_{ji}$ , for  $i = 1, \dots, N$ , such that the sum by rows and by columns is null ( $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji} = 0, \forall i$ ). Then, if the network has no isolated nodes, it follows immediately from Gerschgorin's theorem that (Wu 2007): Zero is an eigenvalue of multiplicity one; and all its non-zero eigenvalues are strictly negative. Thus, the eigenvalue spectrum of  $\mathcal{A}$  can be ordered as follows:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_N \quad (2)$$

A dynamical network is said to (asymptotically) achieve complete synchronization if the trajectories of every node satisfy:

$$\lim_{t \rightarrow \infty} \|x_i(t) - \bar{x}(t)\| = 0, \text{ for } i = 1, \dots, N \quad (3)$$

where  $\bar{x} \in \mathbf{R}^m$  is the synchronized solution of the network, which describes a diagonal  $m$ -dimensional manifold in the  $mN$ -dimensional state space of the network, usually called the synchronization manifold, corresponding to:

$$x_j = x_i = \bar{x}, \text{ for } \forall i, \forall j \quad (4)$$

For the network in (1), by construction, the synchronization manifold corresponds to the dynamics of an isolated node ( $\dot{\bar{x}} = f(\bar{x})$ ) and it is associated with the eigenvalue  $\lambda_1 = 0$  of the connectivity matrix  $\mathcal{A}$ . The stability of the synchronized solution (4) can be established analyzing the transverse directions to the synchronization manifold; if all the transverse directions are contracting, the trajectories of the network will ultimately synchronize (Wang 2002, Li 2005, Wu 2007, Arenas et al. 2008). We are interested in an alternative situation, where the trajectories of the network move away from the stable equilibrium ( $\bar{x}$ ) and evolve along bounded, possibly chaotic, solutions that emerge for specific values of coupling strength and number of nodes. In the following subsection a tLes analysis will be used to establish conditions for the instability of the synchronization manifold in combination with a convergent systems framework to guarantee that the emergent trajectories remain bounded within a ball around the synchronized solution.

## 2.2 Synchronization analysis via tLes

The dynamical behavior of a  $m$ -dimensional system ( $\dot{x} = f(x) \in \mathbf{R}^m$ ) can be characterized in terms of its Lyapunov exponents (Ott 2002):

$$\eta_i = \lim_{t \rightarrow +\infty} \frac{1}{t} |Df(t, x_0) \varpi_i|, \text{ for } i = 1, \dots, m \quad (5)$$

where  $\eta_i$  is the  $i$ -th Lyapunov exponent along the direction  $\varpi_i$ ;  $Df(t, x_0) \in \mathbf{R}^{m \times m}$  is the Jacobian matrix of  $f : \mathbf{R}^m \rightarrow \mathbf{R}^m$  evaluated at a randomly selected initial condition  $x_0$ , and  $\{\varpi_1, \varpi_2, \dots, \varpi_m\}$  is a set of orthonormal vectors in the tangent space of the system. If a system has an unique asymptotically stable equilibrium point, all its Lyapunov exponents are strictly negative and its Lyapunov spectrum can be ordered as follows:

$$0 > \eta_1 \geq \eta_2 \geq \dots \geq \eta_m \quad (6)$$

In a similar manner, the dynamical behavior of a network can also be characterized in terms of Lyapunov exponents. Different bases ( $\varpi_i$ ) can be used to visualize the Lyapunov spectrum of a dynamical network, a particularly useful description is found in terms of the deviation of the network trajectories from the synchronization manifold. Along these directions the tLes are calculated linearizing the synchronization errors ( $\xi_i(t) = x_i(t) - \bar{x}(t)$ ) around the synchronized solution  $\bar{x}(t)$ , that is:  $\dot{\xi}_i(t) = Df(x(t))\xi_i(t) + c \sum_{j=1, j \neq i}^N a_{ij} \xi_j(t)$ , for  $i = 1, 2, \dots, N$ ; which in vector form becomes:

$$\dot{\mathcal{X}}(t) = Df(x(t))\mathcal{X}(t) + c\mathcal{X}(t)\mathcal{A}^\top \quad (7)$$

where  $\mathcal{X}(t) = [\xi_1, \xi_2, \dots, \xi_N] \in \mathbf{R}^{m \times N}$  and  $Df(x(t)) \in \mathbf{R}^{m \times m}$  is the Jacobian of  $f$ . Noting that, by construction, the connectivity matrix  $\mathcal{A}$  can be expressed as:  $\mathcal{A} = \Gamma\Lambda\Gamma^{-1}$ , where  $\Gamma = [\gamma_1, \gamma_2, \dots, \gamma_N] \in \mathbf{R}^{N \times N}$ ; and  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbf{R}^{N \times N}$ ; with  $\gamma_i$  the  $i$ -th eigenvector of  $\mathcal{A}$  and  $\lambda_i$  its corresponding eigenvalue. The variational equation (7) can be written as:

$$\dot{\nu}_i(t) = Df(x(t))\nu_i(t) + c\lambda_i\nu_i(t), \text{ for } i = 1, \dots, N \quad (8)$$

where  $\nu_i(t) = \mathcal{X}(t)\gamma_i \in \mathbf{R}^m$ .

In the variational equations (8), the synchronization manifold corresponds to the eigenvector  $\gamma_1 = [1, 1, \dots, 1] \in \mathbf{R}^N$ , while the remaining  $N - 1$  equations correspond to the directions transverse to the synchronization manifold. Then, applying the Lyapunov exponent definition in (5) to (8), the tLes ( $\mu_i(\lambda_k)$ ) for the entire network are given by:

$$\mu_i(\lambda_k) = \eta_i + c\lambda_k, \text{ for } i = 1, \dots, m; \text{ and } k = 2, \dots, N. \quad (9)$$

Synchronization is achieved in the sense of (3) if all the tLes are negative. From (2) and (6) this condition is equivalent to require that the largest tLe be negative. Notice that for attractive coupling ( $c > 0$ ), the largest tLe is  $\mu_1(\lambda_2) = \eta_1 + c\lambda_2$ . Then, for a network with attractively coupled nodes, the synchronization condition becomes the so-called  $\lambda_2$  criterion (Wang 2002, Li 2005):

$$|c| > \frac{\eta_1}{|\lambda_2|} \quad (10)$$

While, for repelling coupling ( $c < 0$ ), the largest tLe is  $\mu_1(\lambda_N) = \eta_1 - |c|\lambda_N$ . In this case, the

synchronization criterion has the form:

$$|c| < \frac{|\eta_1|}{|\lambda_N|} \quad (11)$$

Then, as the coupling strength increases ( $|c| \rightarrow \infty$ ) for attractive coupling the synchronization manifold becomes more and more stable, while for repelling coupling the stability margin of the synchronized solution is reduced. Considering that our objective is to have bounded trajectories that move away from the synchronization manifold, in what follows we consider only repelling coupling.

The instability of the synchronization manifold by itself is not sufficient to guarantee the existence of bounded solutions. In fact quite the opposite, since the trajectories will tend to become unbounded along the directions of the positive tLes. Then, an initial requirement for bounded complex solutions is that the sum of tLes be negative for every node in the network, while at the same time at least the largest tLe be positive, this can be express as follows (Barajas-Ramírez and Femat 2008a,b):

$$\mu_1(\lambda_N) > 0; \text{ and } \sum_{i=1}^m \mu_i(\lambda_k) < 0, \text{ for any } k \quad (12)$$

Observing that (9) inherits an order from (2) and (6), a limit on the number of positive tLes on the network can be establish requiring that  $\mu_{p_1}(\lambda_{p_2})$  be negative for two numbers:  $1 \leq p_1 \leq m$  and  $2 \leq p_2 \leq N$ . This condition can be expressed as follows:

$$\frac{|\eta_1|}{|\lambda_N|} < |c| < \frac{|\eta_{p_1}|}{|\lambda_{p_2}|}, \text{ with } c < 0 \quad (13)$$

At this point we will like to remark the following:

(i) Notice that for a given repelling coupling strength value  $c < 0$ , such that (11) is not satisfied, the synchronization manifold is unstable. If for that same value of  $c$  the condition (12) is satisfied, the trajectories of the network are attracted, at least locally to a ball around the synchronization manifold. Then, for at most  $p_1 p_2$  nodes we have a combination of positive and negative tLes, the solutions of these nodes will experience stretching and folding of their trajectories, a characteristic mechanism of chaotic dynamics. As such, when this combination occurs the behavior of the bounded solutions of the network around the synchronized solution will be assumed chaotic. This combination of factors are illustrated with numerical examples in the following Sections.

(ii) Notice that the conditions (12) and (13) are expressed in terms of the Lyapunov exponents of a node in isolation ( $\eta_i$ ) and the eigenvalues of the connectivity matrix of the network ( $\lambda_i$ ). Thus, changes in the topology of the network will directly determine the endpoints of the interval of values for emergence of bounded solutions. In the following section, we investigate how the topological features of different basic network models affect the emergence of bounded complex behaviors.

### 3. Effects of topology on emergence of bounded solutions

Intuitively, a diffusively coupled network with identical nodes, where every node has the same stable equilibrium, will generate oscillations only if its trajectories evolve away from its synchronized solution. According to the conditions (12) and (13), there are two possible ways achieve this: (I) Increasing the coupling strength, ( $c < 0$ ), beyond a threshold value that marks the initial

point of the interval in (13); or (II) changing the topology of the network such that the smallest eigenvalue of the connectivity matrix,  $(\lambda_N)$ , be large enough to make the initial point of the interval for emergence smaller than a given coupling strength. In what follows, the emergence of bounded complex solutions as the result of these changes is investigated for different basic network models.

Firstly, we analyze the case of globally connected networks, which are constructed linearly and diffusively coupling each node to all others in the network. The eigenvalue spectrum of the resulting connectivity matrix  $\mathcal{A}_{gc}$  is found to be

$$\lambda_{gc,1} = 0, \text{ and } \lambda_{gc,k} = -N, \text{ for } k = 2, \dots, N. \tag{14}$$

For this network model; the conditions (12) and (13) become:

$$\eta_1 + |c|N > 0; \text{ with } \sum_{i=1}^m \mu_i(\lambda_{gc,k}) < 0, \text{ for any } k; \text{ and} \tag{15}$$

$$\frac{|\eta_1|}{N} < |c| < \frac{|\eta_{p_1}|}{N}, \text{ with } c < 0 \tag{16}$$

Note that for globally coupled networks with a fixed coupling strength, as the number of nodes in the network increases the interval in (16) becomes smaller, while at the same time, the number of positive tLes increases. In fact, from (15) we have that all the tLes of the network become positive when  $N > \frac{|\eta_m|}{|c|}$ . Further, the sum of tLes becomes positive when  $N > \frac{\sum_{i=1}^m |h_i|}{m|c|}$ . A similar situation is found for a network with a fixed number of nodes as the coupling strength increases. Then, there are intervals of values for the number of nodes ( $[\underline{N}^*, \overline{N}^*]$ ) and coupling strength ( $[\underline{c}^*, \overline{c}^*]$ ) outside of which the network no longer satisfies the conditions for the emergence of bounded solutions. These are given by

$$\begin{aligned} \underline{N}^* &= \frac{|\eta_1|}{|c|}; \overline{N}^* = \min \left\{ \frac{|\eta_m|}{|c|}, \frac{\sum_{i=1}^m |h_i|}{m|c|} \right\}, \text{ and} \\ \underline{c}^* &= \frac{|\eta_1|}{N}; \overline{c}^* = \min \left\{ \frac{|\eta_m|}{N}, \frac{\sum_{i=1}^m |h_i|}{mN} \right\} \end{aligned} \tag{17}$$

Next, we consider star coupled networks, which are constructed connecting linearly and diffusively every node only to a single hub node in the network. For this network model the connectivity matrix  $\mathcal{A}_{sc}$  has the following eigenvalue spectrum:

$$\lambda_{sc,1} = 0, \lambda_{sc,k} = -1, \text{ for } k = 2, \dots, N - 1; \text{ and } \lambda_{sc,N} = -N \tag{18}$$

The conditions for emergence of bounded solutions, (12) and (13), for this network model become:

$$\eta_1 + |c|N > 0, \text{ with } \sum_{i=1}^m \mu_i(\lambda_{sf,k}) < 0, \text{ for any } k; \text{ and} \tag{19}$$

$$\frac{|\eta_1|}{|N|} < |c| < |\eta_{p_1}|, \text{ with } c < 0 \tag{20}$$

In this case, for a fixed coupling strength the increment on the number of nodes in the network does not affect the endpoint of the interval (20), and the number of positive tLes does not increase with the number of nodes. In fact, according to (18), only the  $m$  tLes associated with

the hub node ( $\eta_i + |c|N$ , for  $i = 1, \dots, m$ ) become positive for when  $N > \frac{|\eta_m|}{|c|}$ . Furthermore, the sum of Lyapunov exponents for the hub node violates condition (19) when  $N > \frac{\sum_{i=1}^m |h_i|}{m|c|}$ , and produces unbounded trajectories, even though the trajectories for the rest of the nodes remain bounded. For the complementary situation, where the number of nodes remains fixed and the coupling strength increases, the same transitions occur. Again, for this network model the intervals  $[\underline{c}^*, \bar{c}^*]$  and  $[\underline{N}^*, \bar{N}^*]$  are given by (17), with the significant difference that on star coupled networks bounded solutions are not chaotic, since the hub node becomes unbounded long before positive tLes are generated for the rest of the nodes in the network.

Finally, we consider the effects of small-world and scale-free topologies on the emergence of bounded solutions. In a similar way as for globally and star coupled networks, the form of conditions (12) and (13) for small-world and scale-free networks is determined by the eigenvalue spectrum of their respective connectivity matrices. In (Barajas-Ramírez and Femat 2008b), it was shown that: All the nonzero eigenvalues of a linearly and diffusively coupled small-world network, constructed with the Newman-Watts-Strogatz algorithm (Wang and Chen 2002a), grow proportionally with the number of nodes in the network. While for scale-free networks, constructed with the Barabási-Albert algorithm (Wang and Chen 2002b), the largest nonzero eigenvalue approaches  $-1$  independently from the number of nodes, and the smallest eigenvalue grows proportionally to the number of nodes in the network. Then, the conditions (12) and (13) for small-world networks have similar transitions as for globally coupled networks, while for scale-free networks the emergence of bounded solutions is similar to that of star coupled networks.

From the observations above, two scenarios for the emergence of bounded solutions can be identified: In the first, the number of nodes increase for a network with fixed coupling strength. On the second, the coupling strength increases for a fixed number of nodes. In both scenarios, the emergence of bounded solutions is restricted to the intervals  $[\underline{c}^*, \bar{c}^*]$  and  $[\underline{N}^*, \bar{N}^*]$ ; with the type of bounded behavior depending on the connectivity of the network. Thus, while for heterogeneous connectivity, like in star coupled or scale-free networks, the emergence of bounded behavior is restricted to the hub nodes; for globally coupled and small-worlds, their homogeneous connectivity allows for the emergence of chaotic behavior for the entire network. To illustrate these findings, a numerical study of the emergence of bounded solution on networks of the benchmark Lorenz system and of electrically coupled pancreatic  $\beta$  cells is presented in the following section.

## 4. Numerical Simulations

### 4.1 An illustrative example

Consider that each node in the network described by (1) is a Lorenz system, with the following dynamical description (Chen and Dong 1998):

$$\begin{bmatrix} \dot{\chi}(t) \\ \dot{\psi}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} a(\psi(t) - \chi(t)) \\ \rho\chi(t) - \psi(t) - \chi(t)\omega(t) \\ -b\omega(t) + \chi(t)\psi(t) \end{bmatrix} \quad (21)$$

where  $x(t) = [\chi(t), \psi(t), \omega(t)]^\top \in \mathbf{R}^3$  are the state variables of each node in isolation; and the parameters are set to  $a = 10$ ,  $b = \frac{8}{3}$ , and  $\rho = 0.5$ . For these parameter values the Lorenz system has a unique asymptotically stable equilibrium point at  $\bar{x} = [0, 0, 0]^\top$ . The Lyapunov exponents of (21) are estimated from its Jacobian matrix evaluated around  $\bar{x}$ , and are taken to be  $\eta_1 = -0.5$ ,  $\eta_2 = -3$  and  $\eta_3 = -12$ .

Figure 1 shows the intervals  $[\underline{c}^*, \bar{c}^*]$  and  $[\underline{N}^*, \bar{N}^*]$ , obtained using (17), for a network of stable Lorenz systems. The line with ‘ $\star$ ’ corresponds to the lower limits ( $\underline{N}^*$  and  $\underline{c}^*$ ), while the line with ‘o’ corresponds to the upper limits of the emergence interval ( $\bar{N}^*$  and  $\bar{c}^*$ ). For combinations



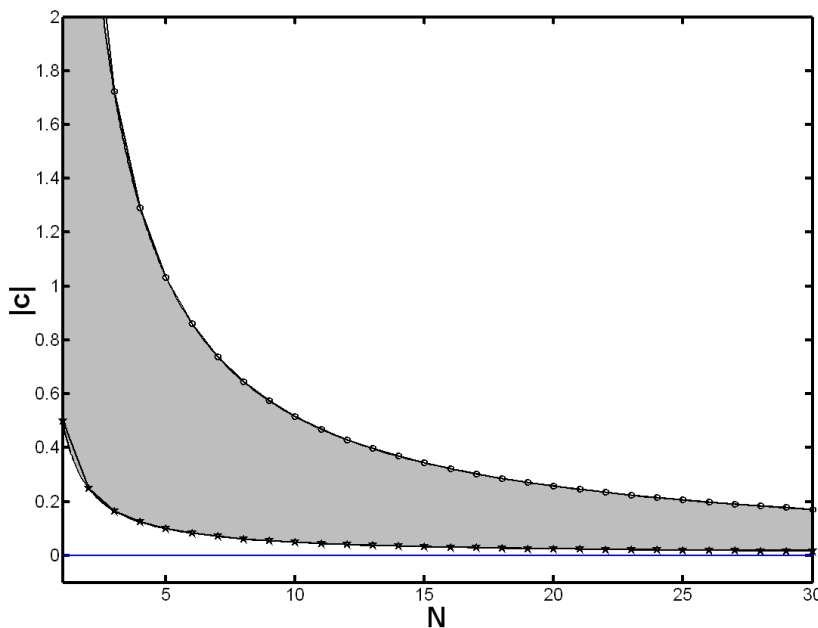


Figure 1. The shaded area corresponds to the intervals  $[c^*, \bar{c}^*]$  and  $[\underline{N}^*, \overline{N}^*]$ , obtained using (17), such that bounded behaviors emerge on networks of stable Lorenz systems.

$(N, c)$  below the line with ‘\*’ the condition for synchronization (11) is satisfied and the network synchronizes to the common equilibrium point. For values inside the shaded area, the conditions (12) and (13) are satisfied and bounded solutions emerge away from the equilibrium. While for combinations above the line with ‘o’ the trajectories of at least one node become unbounded.

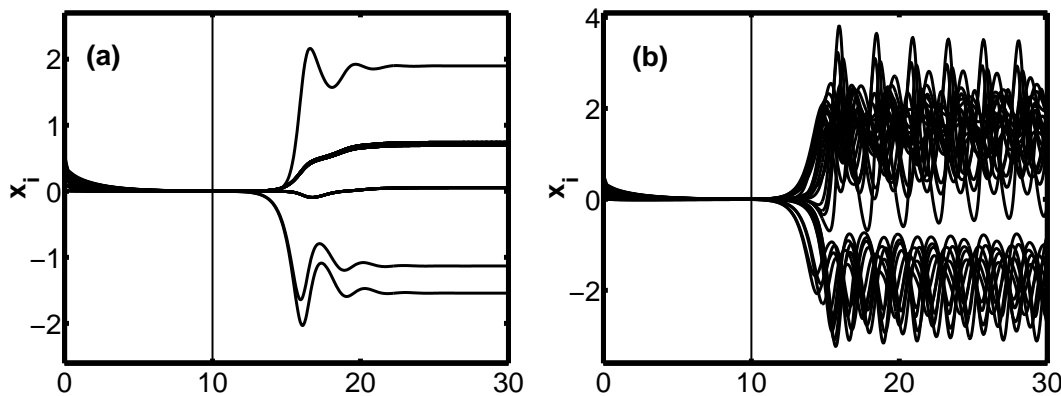


Figure 2. Trajectories of  $N = 10$  stable Lorenz systems connected into: (a) star coupled, and (b) globally coupled networks.

The type of bounded solutions generated in the network of stable Lorenz systems is determined by the connectivity of the network. As discussed above, the number of positive tLes generated as  $c$  or  $N$  increases is different for homogeneously than heterogeneously coupled networks. In Figure 2, we show the trajectories for globally and star coupled networks with a fixed number of nodes ( $N = 10$ ) with two coupling strengths, initially forming a combination below the shaded area ( $c = 0.5$ ), and then at given time ( $t = 10s$ ), the coupling strength is changed to a combination within the shaded area of the Figure 1 ( $c = 2.0$ ). The results on Figure 2(a) corresponds to the star coupled network, in this case emergence of bounded trajectories is concentrated on the hub node, which moves away from the equilibrium, but the rest of the nodes remain at the equilibrium. However, if the nodes are connected into a globally coupled network, all the nodes have positive tLes; then as shown in Figure 2(b), unsynchronized chaotic behavior emerges for

every node in the network.

#### 4.2 A networks of electronically coupled pancreatic $\beta$ cells

In this subsection, we illustrate the potential applicability of the results presented above to biological networks. In particular, we consider networks of pancreatic  $\beta$  cells. These networks are the insulin producing part of the Langerhans islets in the pancreas; as such, they play a fundamental part in the homeostasis of glucose in the body. The insulin liberation process in  $\beta$  cells is directly related to the electrical activity of their membrane. A particularly significant aspect of their operation is that in isolation  $\beta$  cells are usually neurons which are switch-off, that is, their insulin liberation electrical activity is inhibited, with their dynamical behavior that of a stable equilibrium associated to their electrical activity. However, when they are coupled together, within the islet, all the cells transits to their active state, that is, a state characterized by a spike-busting behavior (Smolen et al. 1993). The dynamic description of the electrical activity of an isolated  $\beta$  cell is given by (Pernarowski 1998):

$$\begin{aligned}\dot{v}(t) &= f(v(t)) - w(t) - k(d(t)) + I(t) \\ \dot{w}(t) &= \frac{1}{\tau}(w_{\infty}(v(t)) - w(t)) \\ \dot{d}(t) &= \varepsilon(h(v(t)) - d(t)).\end{aligned}\tag{22}$$

where  $v(t)$  refers to the membrane potential;  $w(t)$  refers to an ionic channel activation;  $d(t)$  refers to the glucose concentration of the cell, and  $I$  is a exogenous current input. The terms  $f(v(t))$ ,  $k(d(t))$ ,  $h(v(t))$  and  $w_{\infty}(v(t))$  that are used in above formulas are given by:  $f(v(t)) = f_3v(t)^3 + f_2v(t)^2 + f_1v(t)$ ,  $f_3 = -\frac{a}{3}$ ,  $f_2 = a\hat{v}$ ,  $f_1 = \frac{1}{\tau} - a(\hat{v}^2 - \eta^2)$ ,  $h(v(t)) = \beta(v(t) - v_{\beta})$ ,  $k(d(t)) = \bar{\tau}d(t)$ ,  $w_{\infty}(v(t)) = w_3v(t)^3 + w_2v(t)^2 + w_1v(t) + w_0$ ,  $w_3 = \bar{\tau} - \frac{a}{3}$ ,  $w_2 = a\hat{v}$ ,  $w_1 = \frac{1}{\tau} - a(\hat{v}^2 - \eta^2) - 3\bar{\tau}$ , and  $w_0 = -3\bar{\tau}$ . The parameter values for an inhibited  $\beta$  are:  $a = \frac{1}{4}$ ,  $\eta = \frac{3}{4}$ ,  $\hat{v} = \frac{3}{2}$ ,  $\beta = 4$ ,  $\varepsilon = 0.0025$ ,  $\bar{\tau} = 1$ ,  $I(t) = 0$ , and  $v_{\beta} = -1.300$ . For this set of values the  $\beta$  cell has an stable equilibrium point  $\bar{x} = [-1.0472, -1.1043, 1.0071]$ , and its Lyapunov exponents in its vicinity are found to be:  $\eta_1 = -0.0277$ ,  $\eta_2 = -0.3621$  and  $\eta_3 = -1.1233$ .

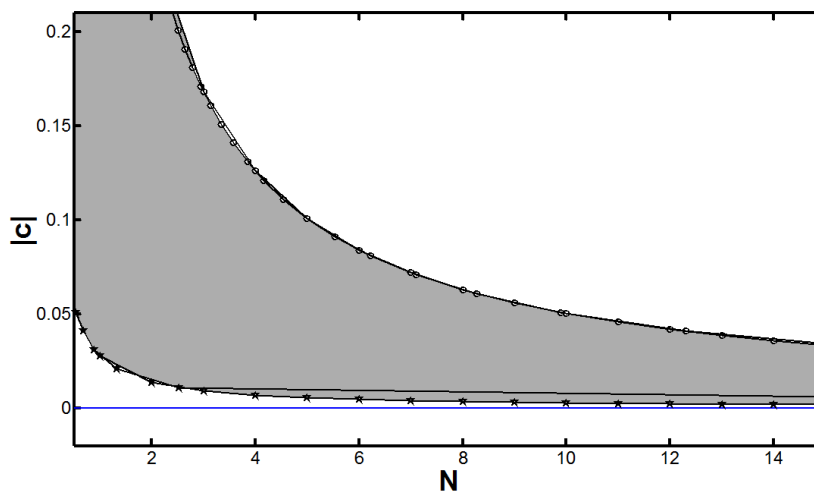


Figure 3. The shaded area corresponds to the intervals  $[c^*, \bar{c}^*]$  and  $[N^*, \bar{N}^*]$ , obtained using (17), such that bounded behaviors emerge on networks of electrically coupled inhibited pancreatic  $\beta$  cells.

We will assume that inside the Langerhans islet  $\beta$  cells are only coupled electrically, that is, they form a network coupled only through the variable  $v(t)$ . According to the conditions in (12) and (13), a linearly and diffusively coupled network of inhibited  $\beta$  cells produces bounded behaviors for combinations of  $N$  and  $c$  within the shaded area of Figure 3.

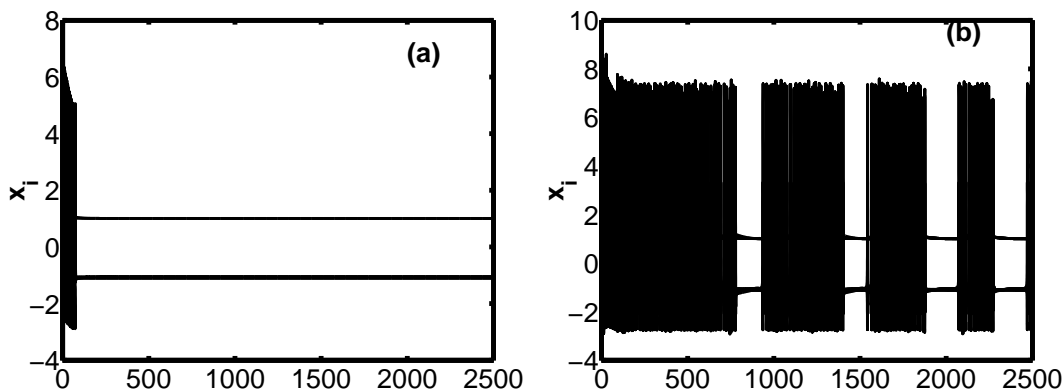


Figure 4. Trajectories of globally coupled networks of electrically connected inhibited  $\beta$  cells for the combinations: (a)  $N = 3$  with coupling strength  $c = 0.005$ , and (b)  $N = 8$  with coupling strength  $c = 0.05$ .

In Figure 4(a) the trajectories of a globally coupled network with three electrically connected inhibited  $\beta$  cells are shown for a coupling strength of  $c = 0.005$ . Notice that this combination  $(N, c)$  is below the shaded area of Figure 3, accordingly the trajectories of the networks eventually synchronized at the stable equilibrium point. As shown in Figure 4(b) for a combination of coupling strength and the number of  $\beta$  cells within the interval for emergence  $(N = 8, c = 0.05)$ , all the  $\beta$  cells in the network are activated with the characteristic spike-busting behavior on their membrane potential.

### 5. Closing Remarks

There are two basic conditions for the emergence of bounded solutions on a network of stable dynamical systems: Local instability of the synchronized solution and boundedness of the trajectories of the network. Although, these conditions were already identified in previous contributions, here we present them in terms of topological characteristics of the network. Namely, intervals of values  $[\underline{c}^*, \bar{c}^*]$  and  $[\underline{N}^*, \bar{N}^*]$  for the emergence of bounded complex trajectories. We analyze the tLes of the network to derive the convergence of the trajectories to a ball containing the unstable synchronized solution. Further, we establish a relation between the nature of the behavior around the synchronized solution with the connectivity of the network. This relation exists since the intervals for emergence are determined by the Lyapunov spectrum of an isolated node, and the eigenvalue spectrum of the connectivity matrix. In this sense, the connectivity of the network not only affects the endpoints of the intervals for emergence, but also the number of positive tLes generated by the coupling topology. As such, the number of nodes that have associated positive tLes with a negative overall sum are limited for heterogeneous connectivity, like star couple and scale-free networks, where the emergent bounded behavior is restricted to oscillation on the hub nodes. While for globally coupled and small-world networks, the homogeneous connectivity forces the entire network towards complex oscillations. Using these results, we describe the transition toward complex behavior on networks of stable Lorenz systems; and the activation of inhibited pancreatic  $\beta$  cells.

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