Research Article

Open Problems Related to the Hurwitz Stability of Polynomials Segments

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In the framework of robust stability analysis of linear systems, the development of techniques and methods that help to obtain necessary and sufficient conditions to determine stability of convex combinations of polynomials is paramount. In this paper, knowing that Hurwitz polynomials set is not a convex set, a brief overview of some results and open problems concerning the stability of the convex combinations of Hurwitz polynomials is then provided.

1. Introduction

The design of control systems arises from the need of human beings to manipulate real plants with some degree of confidence and accuracy. However, this design requires mathematical models which are often very complex if all the dynamics that involve a real plant are taken into account. Thus, a complex model requires a complex controller design. Robust control theory intends to analyze a complex model by studying its linear approximation and always assuming that this approximation will incur with some degree of modeling errors. This error is regarded as uncertainty which is modeled and bounded to determine stability conditions of a system and control laws are obtained.

Problems that arise from robust control theory containing one uncertain parameter, such as gain or constants time, and the system stability should be determined for the entire uncertainty range. The study of criteria for deciding whether a convex combination of polynomials is stable can be applied

to resolution of a wide variety of uncertainty systems. The problem to be raised is as follows: find conditions about the stable polynomials $p_0(t)$ and $p_1(t)$ such that the convex combination described by $p(t, \lambda) = \lambda P_0(t) + (1 - \lambda)P_1(t)$ is stable (Hurwitz), for all $\lambda \in [0, 1]$.

The first result, where necessary and sufficient conditions were presented, is known as the Bialas Theorem [1–3]. A different approach in terms of the frequency domain is known as the *Segment Lemma* and it was found by Bhattacharyya et al. [4, 5]. This result shows that the stability of $p(t, \lambda)$ is equivalent to certain conditions that must be met by even and odd part of the polynomials $p_0(t)$ and $p_1(t)$. In complex polynomials case, Bose developed a method to determine the stability of the segment of complex polynomials [6]. In regard to sufficient conditions, in [7, 8], the well-known Rantzer's conditions were obtained.

Based on the above criteria computer algorithms have been developed to resolve this problem. The Segment Lemma has been used to design an algorithm in [9]. On this same

path, a method to verify the stability of convex combinations of polynomials in a finite number of calculus is obtained in [10]. In [11] an algorithm to determine the stability of segment of complex polynomials is proposed; it is based on the work of Bose [12]. In the present paper, we will discuss some of these criteria and propose some open problems in these topics. Applications of Hurwitz stability criterion to Engineering problems are huge, for example, for the stability of wave propagation in strain gradient [13] or mixture [14].

The paper is organized as follows. In Section 2, the problem statement is presented. Some results with necessary and sufficient condition on stability of segment of Hurwitz polynomials and its open problems are presented in Section 3. In Section 4, some open problems with sufficient condition and using Rantzer's conditions are postulated. In Section 5, a related problem with segments of stable polynomials is the calculating of the Minimum Left Extreme, which was studied by Bialas. In Section 6 some results and open problems related to Hadamardized Hurwitz polynomials are postulated. Finally, in Section 7, some concluding remarks are provided.

2. Problem Statement

In this section some definitions and one motivation example in the framework of the stability of polynomials are given.

Definition 1. A polynomial with real coefficients, $f(t) = b_0 t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$, is Hurwitz if all its roots have negative real part.

Definition 2. Suppose that $p_0(t)$ and $p_1(t)$ are polynomials (real or complex) of degree n. Let

$$P(t, \lambda) = \lambda p_0(t) + (1 - \lambda) p_1(t), \tag{1}$$

and consider the following one parameter family of polynomials:

$$[p_0(t), p_1(t)] = P(t, \lambda): \quad \lambda \in [0, 1]. \tag{2}$$

This family will be referred to as a segment of polynomials. We shall say that the segment is stable if and only if every polynomial on the segment is stable.

The following example shows that a segment of polynomials is not necessarily a stable segment although the two extremes are Hurwitz polynomials.

Example 3. Let $P_0(t) = t^3 + 6t^2 + 11t + 6 = (t+1)(t+2)(t+3)$ and $P_1(t) = 6t^2 + 5t + 216$ be Hurwitz; nevertheless

$$\lambda P_0(t) + (1 - \lambda) P_1(t)$$
$$= \lambda \left(t^3 + 6t^2 + 11t + 6t \right) + (1 - \lambda) \left(6t^2 + 5t + 216 \right) \tag{3}$$

is not Hurwitz for all $\lambda \geq 0$:

   (i) for $\lambda \in [0, (5 - \sqrt{2})/7)$ it is Hurwitz;

   (ii) for $\lambda \in [(5 - \sqrt{2})/7, (5 + \sqrt{2})/7]$ it is not Hurwitz;

   (iii) for $\lambda \in ((5 + \sqrt{2})/7, 1]$ it is Hurwitz.

Now, there are several criteria to determine the stability of the segment of polynomials; we present some of them below.

3. Necessary and Sufficient Conditions

In this section, we present two of the most known necessary and sufficient conditions on the stability of segment of Hurwitz polynomials.

3.1. The Bialas Theorem

Theorem 4 (see [2]). *If p_0 is Hurwitz and $\deg(p_0) > \deg(p_1)$, then $P(t, \lambda) = \lambda p_0(t) + (1-\lambda) p_1(t)$ is Hurwitz for all $\lambda \in [0, 1]$, if and only if the matrix $H^{-1}(p_0) H(p_1)$ has eigenvalues not in $(-\infty, 0)$, where $H^{-1}(p_0)$ and $H(p_1)$ are matrices Hurwitz of p_0 and p_1, respectively.*

Open Problem 1. Is it possible to find necessary and sufficient conditions in a similar way of the Bialas Theorem for $\deg(p_0) = \deg(p_1)$ case?

In order to establish the Segment Lemma we give some definitions.

Given a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, we define

$$p^{\text{even}}(s) = a_0 + a_2 t^2 + \cdots,$$
$$p^{\text{odd}}(s) = a_1 t + a_3 t^3 + \cdots, \tag{4}$$

then $p(t) = p^{\text{even}}(t) + p^{\text{odd}}(t)$ and $p(i\omega)$ can be written as $p(i\omega) = p^e(\omega) + i\omega p^o(\omega)$, where

$$p^e(\omega) = a_0 - a_2 \omega^2 + a_4 \omega^4 + \cdots,$$
$$p^o(\omega) = a_1 - a_3 \omega^2 + a_5 \omega^4 + \cdots. \tag{5}$$

In the Bialas Theorem, it is necessary that the two extremes of the segment of polynomials have different degree. Now we present the Segment Lemma, where the two extremes of the segment of polynomials have equal degrees.

3.2. The Segment Lemma

Lemma 5 (see [4, 5]). *Let $p_1(t)$ and $p_2(t)$ be n-degree Hurwitz polynomials with leading coefficients of the same sign. Then the segment of polynomials $[p_1(t), p_2(t)]$ is Hurwitz, if and only if there are not $\omega > 0$ that satisfies the three conditions:*

   (1) $p_1^e(\omega) p_2^o(\omega) - p_2^e(\omega) p_1^o(\omega) = 0$,

   (2) $p_1^e(\omega) p_2^e(\omega) \leq 0$,

   (3) $p_1^o(\omega) p_2^o(\omega) \leq 0$.

Open Problem 2. Is it possible to get a result, based on the approach of the Segment Lemma, to be applicable when polynomials p_0 and p_1 have different degrees?

After we have discussed two necessary and sufficient conditions, now in the next section, we expose two sufficient conditions: Rantzer's conditions and one approach of matrix inequalities.

4. Sufficient Conditions

In this section, we present two sufficient conditions for the stability of segment that are known in the literature: Rantzer's conditions and one interesting approach with matrix inequalities.

4.1. Rantzer's Conditions. Supposing that $p_0(t)$ is a Hurwitz polynomial and $P_1(t)$ is semistable (their roots have real part 0), then the segment of polynomials $[p_0(t), p_1(t)]$ consists of Hurwitz polynomials if you have one of the following four conditions.

(i) The difference $d = p_1 - p_0$ satisfies

$$\frac{\partial \arg (d (i\omega))}{\partial \omega} < 0, \quad \omega \in \{w > 0 \mid d (iw) \neq 0\}. \quad (6)$$

(ii) Each of the polynomials p_0 and p_1 has at least one root in $\mathbb{C}^-$ and

$$\frac{\partial \arg (d (i\omega))}{\partial \omega} < \left| \frac{\sin (2 \arg [d (i\omega)])}{2\omega} \right|, \quad (7)$$
$$\omega \in \{w > 0 \mid d (iw) \neq 0\}.$$

(iii) Each of the polynomials p_0 and p_1 has at least one root in $\mathbb{C}^-$ and

$$\frac{\partial \arg (d (i\omega))}{\partial \omega} \leq 0, \quad \omega \in \{w > 0 \mid d (iw) \neq 0\}. \quad (8)$$

(iv) Each of the polynomials p_0 and p_1 has at least two roots in $\mathbb{C}^-$ and

$$\frac{\partial \arg (d (i\omega))}{\partial \omega} \leq \left| \frac{\sin (2 \arg [d (i\omega)])}{2\omega} \right|, \quad (9)$$
$$\omega \in \{w > 0 \mid d (iw) \neq 0\}.$$

The previous conditions are known as Rantzer's conditions; see [8].

Open Problem 3. Is there a mechanism to describe Rantzer's conditions in terms of the polynomial coefficients?

Open Problem 4. What should we add to Rantzer's conditions in order to obtain necessary and sufficient conditions?

In the next subsection, we present an interesting sufficient condition for checking the stability of segment of polynomials based on matrix inequalities.

4.2. A Matrix Inequality. Given a polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, we define the matrix $E_{(n,n)} \in \mathcal{M}_{(n+1)\times(n+1)}$ as

$$E_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}. \quad (10)$$

Theorem 6. *Let $p_1 = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1}$ be a polynomial of degree n. If the polynomials $p_0(t)$ and $p_1(t)$ are Hurwitz and vector $c = (c_1, c_2, \ldots, c_{n+1})^T \geq 0$ satisfies the system of linear inequalities,*

$$E_{(n,n)}c \gneq 0, \quad (11)$$

then $\lambda p_0(t) + (1 - \lambda) p_1(t)$ is Hurwitz for all $\lambda \in [0, 1]$. Here the symbol ≥ 0 (≤ 0) means that all components of a given vector are ≥ 0 (≤ 0) and the symbol $\gneq 0$ means that all components of a given vector are ≥ 0 but there is at least one component which is >0. See [15, 16].

Open Problem 5. With these approach matrix inequalities, what should we add to obtain necessary and sufficient conditions?

4.2.1. Case for Grade $p_1 = n - 1$. A similar case is obtained if $\deg(p_1(t)) = n - 1$. In this case, the matrix $E_{(n,n-1)} \in \mathcal{M}_{(n+1)\times n}$ is defined by

$$E_{(n,n-1)}$$
$$= \begin{pmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}, \quad (12)$$

and the corresponding inequality is

$$E_{(n,n-1)}c \gneq 0. \quad (13)$$

There is a relation between segments and rays of polynomials, which let us give applications in control theory. We explain it in the next two subsections.

4.3. Relationship between Rays and Segment of Polynomials. If $p_0(t) + kp_1(t)$ is Hurwitz for all $k \geq 0$, then $(1/(1 + k))p_0(t) + (k/(1 + k))p_1(t)$ is Hurwitz for all $k \geq 0$, from which if p_1 is Hurwitz, then, the stability of ray $p_0(t) + kp_1(t)$ is equivalent to the stability of the segment $[p_0(t), p_1(t)]$. An analysis in terms of ray polynomials was made in [15].

For $\deg(p_1(t)) = n$, $D_{(n,n)} \in \mathcal{M}_{n \times (n+1)}$ is the matrix:

$$D_{(n,n)}$$
$$= \begin{pmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-2} & a_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & a_n & -a_{n-1} \end{pmatrix}; \quad (14)$$

for $\deg(p_1(t)) = n-1$, $D_{(n,n-1)} \in \mathcal{M}_{n \times n}$ is the matrix:

$$D_{(n,n-1)}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & -a_n & a_{n-1} \end{pmatrix}; \quad (15)$$

for $\deg(p_1(t)) = n-2$, $D_{(n,n-2)} \in \mathcal{M}_{(n-1) \times (n-1)}$ is the matrix:

$$D_{(n,n-2)}$$
$$= \begin{pmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \cdots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & -a_n & a_{n-1} \end{pmatrix}. \quad (16)$$

Rewriting the results shown in [15, 16] in terms of segment polynomials yields the following result.

Theorem 7. *Consider the Hurwitz polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$. If $p_1(t)$ is Hurwitz with $\deg(p_1(t)) = n, n-1$ or $n-2$ and the vector of coefficients c satisfies the system of linear inequalities*

$$Dc \gneqq 0; \quad (17)$$

then the polynomial $\lambda p_0(t) + (1-\lambda) p_1(t)$ is Hurwitz for all $\lambda \in [0,1]$; where the matrix D is determined by the degree of $p_1(t)$ and is one of the matrices $D_{(n,n)}, D_{(n,n-1)},$ or $D_{(n,n-2)}$.

Example 8. Consider the polynomial Hurwitz $p_0(t) = t^3 + (1/12)t^2 + 5t + 1/12$. The vector of coefficients c of polynomial

$p_1(t) = t^3 + 37t^2 + (11/2)t + 1$ is solution of system of linear inequalities (11):

$$E_{(3,3)}c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & \dfrac{1}{12} & -1 & 0 \\ 0 & -\dfrac{1}{12} & 5 & -\dfrac{1}{12} \\ 0 & 0 & 0 & \dfrac{1}{12} \end{pmatrix} \begin{pmatrix} 1 \\ 37 \\ \dfrac{11}{2} \\ 1 \end{pmatrix}$$
$$\qquad (18)$$
$$= \begin{pmatrix} 1 \\ 8 \\ \dfrac{17}{2} \\ \dfrac{1}{12} \end{pmatrix};$$

therefore, the segment polynomial $[p_0, p_1]$ is Hurwitz. However, c is not a solution of (17):

$$D_{(3,3)}c = \begin{pmatrix} \dfrac{1}{2} & -1 & 0 & 0 \\ -\dfrac{1}{2} & 5 & -\dfrac{1}{2} & 1 \\ 0 & 0 & \dfrac{1}{2} & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 37 \\ \dfrac{11}{2} \\ 1 \end{pmatrix}$$
$$\qquad (19)$$
$$= \begin{pmatrix} -\dfrac{73}{2} \\ \dfrac{731}{4} \\ -\dfrac{9}{4} \end{pmatrix},$$

and Rantzer's conditions are not verified: to the difference $d = p_1 - p_0 = (73/2)t^2 + (1/12)t + 1/12$ has

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} = \frac{1/14 + (73/4)\,\omega^2}{\left[1/12 - (73/2)\,\omega^2\right]^2 + (1/12)\,\omega^2}$$
$$> 0,$$
$$\frac{\sin(2 \arg[d(i\omega)])}{2\omega} \qquad (20)$$
$$= \frac{1/14 - (73/4)\,\omega^2}{\left[1/12 - (73/2)\,\omega^2\right]^2 + (1/12)\,\omega^2}.$$

Open Problem 6. Will there be a stable segment of polynomial that satisfies Rantzer's conditions but does not satisfy (11) or (17)?

The importance of the segments and rays of Hurwitz polynomials with the design of stabilizing controls is pointed out in the next subsection.

4.4. Applications to Control Systems. Consider the system $\dot{x} = Ax + bu$, where

$$
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{bmatrix},
$$

$$
b = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}.
$$

(21)

For the uncontrolled system $\dot{x} = Ax$, the characteristic polynomial is $p_0(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_n$. If into the controlled system $\dot{x} = Ax + bu$ we choose a control u as $u(x) = -kc^T x = -k(c_n, c_{n-1}, \ldots, c_1)x$, where $k > 0$ and $c \in \mathbb{R}^n$, then the controlled system takes the form:

$$
\dot{x} = \left(A - kbc^T \right) x,
$$

(22)

and its characteristic polynomial is

$$
p(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_n \\
+ k \left[c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n \right],
$$

(23)

that is,

$$
p(t) = p_0(t) + kp_1(t),
$$

(24)

where $p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$.

Example 9. Consider the system

$$
\dot{x} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -\dfrac{1}{32} & -2
\end{pmatrix} x + \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} (-2k, -4k, -2k) x.
$$

(25)

In this case, the matrix E is given by

$$
E = \begin{pmatrix}
2 & -1 & 0 \\
-1 & \dfrac{3}{2} & -2 \\
0 & 0 & 1
\end{pmatrix},
$$

(26)

then we have the system

$$
\begin{pmatrix}
2 & -1 & 0 \\
-1 & \dfrac{3}{2} & -2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
4 \\
2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
2
\end{pmatrix}.
$$

(27)

Therefore, the system is stable $\forall k \geq 0$. But this cannot be verified with the results presented by Aguirre et al. in [15].

$$
D = \begin{pmatrix}
1 & 0 & 0 \\
-\dfrac{3}{2} & 2 & -1 \\
0 & -1 & \dfrac{3}{2}
\end{pmatrix},
$$

(28)

$$
\begin{pmatrix}
1 & 0 & 0 \\
-\dfrac{3}{2} & 2 & -1 \\
0 & -1 & \dfrac{3}{2}
\end{pmatrix}
\begin{pmatrix}
2 \\
4 \\
2
\end{pmatrix} = \begin{pmatrix}
2 \\
3 \\
-1
\end{pmatrix}.
$$

The Minimum Left Extreme is a subject very related to the study of segment of Hurwitz polynomials; hence in the next sections we present some ideas about the Minimum Left Extreme in order to propose some open problems.

5. The Minimum Left Extreme

A related problem with segment of Hurwitz polynomials is the calculating of the Minimum Left Extreme, which was studied by Bialas. We begin with some definitions.

Let $p_0(t)$ be a Hurwitz polynomial that generates a family of polynomials given as follows:

$$
p(t) = p_0(t) + kp_1(t),
$$

(29)

where $k \in \mathbb{R}$ and $p_1(t) = \sum_{i=1}^{n} c_i t^{n-i}$ is a polynomial with coefficient vector $c = (c_1, c_2, \ldots, c_n)^T \geq 0$. If $E_{(n,n-1)}$ is the matrix defined in the previous section and c is a solution of the system of linear inequalities $E_{(n,n-1)} c \gneqq 0$, then $p_0(t) + kp_1(t)$ is a Hurwitz polynomial $\forall k \geq 0$.

What is the value of $k_{\min} < 0$, so that $p_0(t) + kp_1(t)$ is Hurwitz $\forall k \geq k_{\min}$? From Bialas (see [2]),

$$
k_{\min} = \frac{1}{\lambda_{\min}^{-} \left[-H^{-1}(p_0) H(p_1) \right]},
$$

(30)

where $H(p_0)$ and $H(p_1)$ are Hurwitz matrices of p_0 and p_1, respectively, and $\lambda_{\min}^{-}[-H^{-1}(p_0)H(p_1)]$ is the smallest negative eigenvalue of the matrix

$$
-H^{-1}(p_0) H(p_1).
$$

(31)

5.1. Estimation Technique Proposed. Consider the polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ whereby the following matrix is defined:

$$
Z_{(n,n-1)}
$$

$$
= \begin{pmatrix}
a_1 & -2 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & -2a_1 & 2 & \cdots & 0 & 0 \\
0 & 0 & a_3 & -2a_2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & -2a_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & a_n
\end{pmatrix},
$$

(32)

with $Z^i_{(n,n-1)}$ the ith row of the matrix $Z_{(n\cdot n-1)}$ and $a = (a_1, a_2, \ldots, a_n)^T$.

Theorem 10. *Let $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ and $E_{(n,n-1)}$ be a Hurwitz polynomial and a matrix defined above, respectively. If the vector $c = (c_1, c_2, \ldots, c_n)^T \geq 0$ is a solution to the system of linear inequalities $E_{(n,n-1)}c \gneq 0$ and $p_1(t) = \sum_{i=1}^n c_i t^{n-i}$, then $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > k_0$, where $k_0 = \max_{i=1,\ldots,n}(-(Z^i_{(n,n-1)}a/E^i_{(n,n-1)}c))$; that is, $k_{\min} \leq k_0$.*

Example 11. Given $p_0(t) = t^3 + 7t^2 + 14t + 8$ and $p_1(t) = t^2 + 4t + 6$, we have that

$$Z_{(3,2)}a = \begin{pmatrix} 7 & -2 & 0 \\ 0 & 14 & -14 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 7 \\ 14 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix},$$

$$E_{(3,2)}c = \begin{pmatrix} 7 & -1 & 0 \\ -8 & 14 & -7 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 48 \end{pmatrix},$$

$$k_0 = \max\left(-\frac{21}{3}, -\frac{84}{6}, -\frac{64}{48}\right) = -\frac{1}{43},$$

$$H(p_0) = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 14 & 0 \\ 0 & 7 & 8 \end{pmatrix},$$

$$H(p_1) = \begin{pmatrix} 1 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 6 \end{pmatrix},$$

$$H^{-1}(p_0)H(p_1) = \begin{pmatrix} \dfrac{7}{45} & \dfrac{26}{45} & 0 \\ -\dfrac{1}{90} & \dfrac{11}{45} & 0 \\ \dfrac{7}{720} & -\dfrac{4}{45} & \dfrac{3}{4} \end{pmatrix}.$$

Then $\sigma(-H^{-1}(p_0)H(p_1)) = \{-1/34, -1/15 \pm (1/15)i\}$. We have that $\lambda_{\min} = -1/34$, $k_{\min} = -1/43$.

In the above example, the two approaches lead to $p_0(t) + kp_1(t)$ being Hurwitz $\forall k > -1/43$.

Example 12. Given $p_0(t) = t^3 + 7t^2 + 14t + 8$ and $p_1(t) = 26t^2 + 137t + 90$, we have that

$$Z_{(3,2)}a = \begin{pmatrix} 7 & -2 & 0 \\ 0 & 14 & -14 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 7 \\ 14 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix},$$

$$E_{(3,2)}c = \begin{pmatrix} 7 & -1 & 0 \\ -8 & 14 & -7 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 26 \\ 137 \\ 90 \end{pmatrix}$$

$$= \begin{pmatrix} 45 \\ 1080 \\ 720 \end{pmatrix},$$

$$k_0 = \max\left(-\frac{21}{45}, -\frac{84}{1080}, -\frac{64}{720}\right)$$

$$= -\frac{7}{90} = -0.07778,$$

$$H(p_0) = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 14 & 0 \\ 0 & 7 & 8 \end{pmatrix},$$

$$H(p_1) = \begin{pmatrix} 26 & 90 & 0 \\ 0 & 137 & 0 \\ 0 & 26 & 90 \end{pmatrix},$$

$$H^{-1}(p_0)H(p_1) = \begin{pmatrix} \dfrac{182}{45} & \dfrac{82}{45} & 0 \\ -\dfrac{13}{45} & \dfrac{869}{90} & 0 \\ \dfrac{91}{360} & -\dfrac{3743}{720} & \dfrac{45}{4} \end{pmatrix}.$$

Then $\sigma(-H^{-1}(p_0)H(p_1)) = \{-11.25, -4.1399, -9.5601\}$. Finally we have $\lambda_{\min} = -11.25$, $k_{\min} = -0.088889$.

Now, in Example 12 with the approach proposed in Theorem 10, we obtain that $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -0.07778$ and applying the Bialas approach we obtain that $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -0.088889$. That is, $k_{\min} < k_0$.

Example 13. Consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{pmatrix} x$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (-90k, -137k, -26)\, x.$$

Here $p_0(t) = t^3 + 7t^2 + 14t + 8$ and $p_1(t) = 26t^2 + 137t + 90$. Then it holds

$$k_0 = \max\left(-\frac{21}{45}, -\frac{84}{1080}, -\frac{64}{720}\right) = -\frac{7}{90} = -0.07778 \tag{36}$$

$$> -1.$$

Therefore, in addition, we can ensure that the system is stable for all $k > 0$ (the control u is a stabilizing control) and we can also say more, the system is stable for all $k > -0.07778$ because $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -0.07778$.

Open Problem 7. Describe the cases when $k_0 = k_{\min}$.

Remark 14. An advantage of our approach, in relation to the Bialas approach, is that we can estimate the Minimum Left Extreme when the polynomials have the same degree. Given the polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, the matrices $Z_{(n,n)}$ and $E_{(n,n)}$ are defined by

$$Z_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 & -2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & -2a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -2a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix},$$

$$E_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}. \tag{37}$$

If the vector $c = (c_0, c_1, \ldots, c_n)^T \geq 0$ is a solution of inequalities system $E_{(n,n)} c \gneq 0$ and the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=0}^{n} c_i t^{n-i}$, then $p_0(t) + kp_1(t)$ is a Hurwitz polynomial $\forall k > k_0$, where $k_0 = \max_{i=1,\ldots,n,n+1}(-Z^i a / E^i c)$; that is, $k_{\min} \leq k_0$.

Open Problem 8. With this approach find conditions for calculating k_0 as $\deg(p_0) > \deg(p_1)$ or as $\deg(p_0) = \deg(p_1)$.

If we consider the set of the Hadamardized Hurwitz polynomials, then we can plant some open problems also. It is the motivation to include the next section.

6. Hadamardized Hurwitz Polynomials

In the present section, some results and open problems related to Hadamardized Hurwitz polynomials are postulated.

Another property of Hurwitz polynomials is related to Hadamard product. Consider two real polynomials $p(t)$ and $q(t)$:

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$
$$q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0. \tag{38}$$

The Hadamard product of $p(t)$ and $q(t)$, denoted by $(p * q)(t)$, is defined by

$$(p * q)(t) = a_n b_n t^n + a_{n-1} b_{n-1} t^{n-1} + \cdots + a_1 b_1 t + a_0 b_0 t. \tag{39}$$

In [17], it was proved that the set of Hurwitz polynomials is closed under Hadamard product. However in [18] it was shown that there are Hurwitz polynomials of degree 4 that do not have a Hadamard factorization in two Hurwitz polynomials. In [19, 20], some conditions to Hadamard factorization existence were presented. The set of Hurwitz polynomials that admits a Hadamard factorization, we called Hadamardized Hurwitz polynomials, is denoted by $\mathscr{H}(\text{Had})_n$. Recently, in [21] some topological properties about this set were shown. But regarding the convex combination, we can propose the following problems.

Open Problem 9. Given $f, g \in \mathscr{H}(\text{Had})_n$, what conditions are required to guarantee that $\lambda f + (1 - \lambda)g \in \mathscr{H}(\text{Had})_n$?

Open Problem 10. Is there any property in H_n set that does not share with $\mathscr{H}(\text{Had})_n$ set?

7. Conclusions

In this paper, a brief overview of some results concerning the stability of the convex combination of Hurwitz polynomials is provided. Also some criteria for deciding if a segment of polynomials is a Hurwitz polynomial were presented. Furthermore some open problems about these polynomials in order to motivate the study of them were given. As mentioned above, the study and solution of these problems also have their field of opportunity in the framework of robust systems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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