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Systems of 3-braid equations

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Dedicated to Professor Francisco Javier ("Fico") González Acuña

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Abstract The tangle model is a useful topological tool in the study of the mechanism of action of certain enzymes on DNA molecules. In particular, the model proves helpful to determine the topological structure of the DNA molecules resulting from those reactions. Roughly speaking, the tangle model consists in solving a system of three equations in which the *unknowns* on the left-hand-side of each equation are tangles, whereas the *known* data on the right-hand sides are 2-bridge knots in many cases. Initially [6], the model was successfully applied to study the Tn3 enzyme acting on rational 2-tangles, for which a complete classification exists [4]. By contrast, the Gin enzyme is known [8] to act on 3-tangles and, since no complete classification is known for general 3-tangles, the tangle model was used to study the mechanism of action of Gin under the assumption [2] that the 3-tangles involved were in fact 3-braids, a particular class of 3-tangles. Some questions derived from the application of the tangle model are of mathematical interest in themselves, e.g., given a system of equations that admits a solution, what kinds of 2-bridge knots may appear on the right-hand sides of the equations so that a (nonempty) solution is guaranteed to exist? In this paper we address and solve this question by showing that, while a system of two equations always admits a solution for any selection of 2-bridge knots, adding a third equation reduces the number of possible knots to only 6, 9 or 18, the exact value depending on the relationships satisfied by the knots in the first two equations. If a fourth equation is adjoined, however, exactly one 2-bridge knot may appear in its the right-hand side for the system to admit a solution. Furthermore, a new simple method that exploits an unexpected cyclic behavior of the solutions is presented and used to construct the proofs. The method relies on the continued fractions associated with 2-bridge knots and their behavior under the concatenation of 3-braids.

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1 Introduction

In nature there exist enzymes, such as DNA recombinases, which act on specific sites and manipulate the structure of a molecule after a recombination event. Starting in the decade of the 1990s, the method now known as the *tangle model* has been used to predict the structure of the substrate molecule after a certain number of iterated recombinations. This prediction is usually referred to as studying *the mechanism of action* of the given enzyme. In the application of the tangle model, an intermediate step consists in solving systems of equations with left-hand-sides that involve tangles as unknowns in the left-hand-side of each equation and 2-bridge knots on the opposite sides, also known as "products."

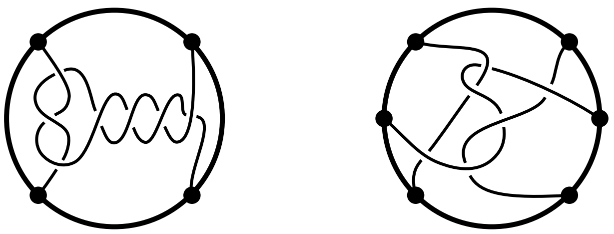
The model, introduced by Ernst and Sumners [6, 12], was applied to model the site-specific recombinase Tn3 resolvase which is an enzyme that acts on 2-tangles. In [1], the tangle model is considered for a generic member of the Integrase family, using the protein Flp which acts on 2-string tangles as an illustration. Using Dehn Surgery techniques a class of solutions is found. Some enzymes, however, such as Gin, Hin integrase recombinases and the bacteriophage Mu virus, act on 3-tangles instead of 2-tangles. In [13], Vazquez and Sumners gave a solution to the action of the Gin enzyme with inversely and directly repeated sites by assuming that one of the strings involved remains completely fixed and hence that the presence of this string can be neglected, thereby obtaining a 2-tangle. On the other hand, the case of the Gin enzyme acting on 3-braids was solved in [2], where the family of all of the 2-bridge knots than can be obtained as products is detailed as well. In [5], the Mu protein-DNA complex is analyzed and, after a simplification by using 2-string tangle analysis, a system of four 3-tangle equations is obtained which is then solved using wagon wheel graphs and tetrahedral graphs, ultimately reaching the conclusion that there is only one biologically reasonable solution for the shape of DNA bound by Mu transposase.

In each of the aforementioned analyses, the products involved come from experimental data involving the action of enzymes on molecules. Hence it is natural to expect the corresponding problems to admit solutions. An additional issue to deal with, however, is the complexity of the methods applied to obtain such solutions from a mathematical standpoint. Building on the theory and results developed in [3] and [7], in the present work we exploit the properties of standard 3-braid diagrams and apply the main ideas of the tangle model in order to solve arbitrary systems of equations. Among the contributions in this paper we show that, whereas a system of two equations with 3-braids as unknowns and arbitrarily chosen 2-bridge knots as products always admits a solution, when one considers a third equation only a few 2-bridge knots exist which preserve the solvability of the system. Moreover, if a fourth equation is considered, the choice of possible 2-bridge knots as products reduces to exactly one. These conclusions are reached by applying a very simple method that uses information derived from the 2-bridge knots in the first two equations and by taking advantage of the—rather surprising—cyclic nature of the braid solutions of a single equation. At this point, it is important to mention that all knots and tangles considered throughout are assumed to be unoriented.

This paper is organized as follows. In Section 2 we recall some basic facts about tangles and about 3-braids and their standard diagrams. Also recalled in the same section is the existing connection between 3-braids and continued fractions via an invariant F. In Section 3 we point out the relationship between 3-braids and 2-bridge knots based on the notion of 2-bridge knot type and the way it relates to the first component of the invariant F. In Section 4, systems of equations are analyzed by, firstly, finding all of the 3-braid solutions to a single equation and, secondly, extending the analysis to two or more equations. Finally, in Section 5 we conclude with some remarks regarding the relevance of our method and results.

2 Preliminary notions

An n-tangle is a pair (B^3, T), where B^3 denotes the 3-ball and T is a set of n disjoint, piecewise linear curves that intersect nontrivially with B^3 [10]. Examples of n-tangles are shown in Figure 1 for n =2 and $n = 3$, respectively.



(a) (b)

Fig. 1 Examples of 2- and 3-tangles, respectively.

An n-***braid*** is a set of n strings attached to vertical bars at their left and right endpoints (cf. Figure 3), with the property that each string heads rightwards at every point as it is traversed from left to right. It is easy to prove that n-braids are particular cases of n-tangles.

As in the study of general knot theory, tangles are studied via their two dimensional diagrams. Given two n-tangles A and B, their sum $A + B$ is defined as the n-tangle obtained by concatenation of A to the left of B, as shown in Figure 2. The restriction of the concatenation operation to the set of *n-braids* endows the latter with the structure of a *noncommutative* group with identity given by . The inverse of A under this operation is denoted by $-A$ and, therefore, the meaning of kA is clear for any $k \in \mathbb{Z}$.

A pair of diagrams D and D' are said to be related by Reidemeister moves if D may be transformed into D' using only a finite number of the Reidemeister moves. The Reidemeister moves define an equivalence relation, hence one writes, by mild abuse of notation, $D = D'$ when the two diagrams are related. One of the key problems in tangle theory is their classification up to this equivalence.

Since the knot-theoretical objects of main interest in this paper are 3-braids, let us now recall some basic notation taken from [2]. Given a 3-braid B, there exists a finite sequence of integers $a_1, \ldots, a_n$, such that B admits a diagram of the form $\mathcal{T}(a_1, \ldots, a_n)$, where $\mathcal{T}(a_1, \ldots, a_n)$ indicates $|a_1|$ crossings of the two uppermost strands, followed by $|a_2|$ crossings of the two lowermost strands, and then $|a_3|$

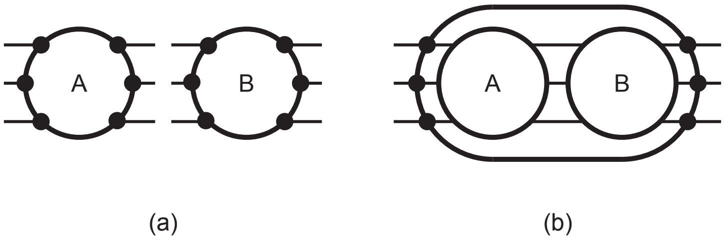


Fig. 2 The concatenation of two 3-braid diagrams. (a) Two 3-braids A and B; and (b) The diagram of concatenation $A+B$.

crossings of the two uppermost strands, and so on, with the following sign convention. For odd i, positive values of the a_i indicate that the uppermost strand passes over the middle strand, whereas for even i, a positive value of a_i indicates that the lowermost strand passes over the middle strand. This notation is illustrated in Figure 3, where examples of 3-braid diagrams are given.

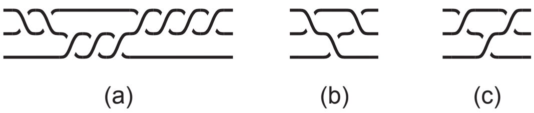


Fig. 3 Examples of 3-braid diagrams and the notation employed in this work: (a) $\mathcal{T}(2,3,-4)$; (b) $\mathcal{T}(1,-1,1)$; and (c) $\mathcal{T}(-1,1,-1)$.

In the sequel we define the diagrams $\mathcal{E}=\mathcal{T}(1,-1,1)$ and $-\mathcal{E}=\mathcal{T}(-1,1,-1)$, which will play a role in the ensuing developments. Note that

$$\mathcal{E}=\mathcal{T}(1,-1,1)=\mathcal{T}(0,-1,1,-1) \quad \text{and} \quad -\mathcal{E}=\mathcal{T}(-1,1,-1)=\mathcal{T}(0,1,-1,1),$$

which represent, respectively, the two braids on the right of Figure 3. It is clear that a diagram $\mathcal{T}(a_1,\dots,a_n)$ equals the sum of diagrams $\mathcal{T}(a_1)+\mathcal{T}(0,a_2)+\cdots+\mathcal{T}(0,a_n)$, if n is even, or $\mathcal{T}(a_1)+\mathcal{T}(0,a_2)+\cdots+\mathcal{T}(a_n)$, if n is odd. Consequently, the inverse of a diagram $\mathcal{T}(a_1,\dots,a_n)$ under concatenation is given by $\mathcal{T}(0,-a_n,-a_{n-1},\dots,-a_1)$, if n is even, and by $\mathcal{T}(-a_n,-a_{n-1},...,-a_1)$, if n is odd.

An n-braid is said to be alternating if, and only if, it admits an alternating diagram, that is, a diagram $\mathcal{T}(a_1,...,a_n)$ such that $a_i\geq 0$ for all $i=1,2,...,n$ or $a_i\leq 0$ for all $i=1,2,...,n$. As an example, the three braid diagrams in Figure 3 are not alternating. Nonalternating diagrams can be rearranged to yield equivalent diagrams having an alternating part and a nonalternating portion composed only of a finite number of summands $\pm\mathcal{E}$, as illustrated by the following example.

Example 1

$$\begin{aligned}\mathcal{T}(2,2,-3)&=\mathcal{T}(2,2)+\mathcal{E}+\mathcal{T}(0,3)-\mathcal{E}\\&=\mathcal{T}(2,2)+\mathcal{T}(0,-1,1,-1)+\mathcal{T}(0,3)-\mathcal{E}\\&=\mathcal{T}(2,1,1,2)-\mathcal{E}.\end{aligned}$$

This example conveys the main idea behind the proof of the following result.

Theorem 1 *Consider $\mathcal{T}(a_1, ..., a_k, ..., a_n)$ where $k, n \in \mathbb{N}$, $k < n$ and k is odd (the case for k even is dealt with similarly). If $a_1, ..., a_k > 0$ and $a_{k+1}, ..., a_n < 0$, then*

$$\mathcal{T}(a_1, ..., a_n) = \mathcal{T}(a_1, a_2, ..., a_{k-1}, a_k - 1, 1, -a_{k+1} - 1, -a_{k+2}, ..., -a_n) + \mathcal{E}.$$

If, however, $a_1, ..., a_k < 0$ and $a_{k+1}, ..., a_n > 0$, then

$$\mathcal{T}(a_1, ..., a_n) = \mathcal{T}(a_1, a_2, ..., a_{k-1}, a_k + 1, -1, -a_{k+1} + 1, -a_{k+2}, ..., -a_n) - \mathcal{E}.$$

Proof. Suppose that $a_1, ..., a_k > 0$ and $a_{k+1}, ..., a_n < 0$. Since k is assumed to be odd, one has (cf. Figure 4):

$$\begin{aligned}\mathcal{T}(a_1, ..., a_n) &= \mathcal{T}(a_1, ..., a_k) - \mathcal{E} + \mathcal{T}(-a_{k+1}, ..., -a_n) + \mathcal{E}\\ &= \mathcal{T}(a_1, ..., a_k) + \mathcal{T}(-1, 1, -1) + \mathcal{T}(-a_{k+1}, ..., -a_n) + \mathcal{E}\\ &= \mathcal{T}(a_1, a_2, ..., a_{k-1}, a_k - 1, 1, -a_{k+1} - 1, -a_{k+2}, ..., -a_n) + \mathcal{E}.\end{aligned}$$

The remaining cases are proven in an analogous way. □

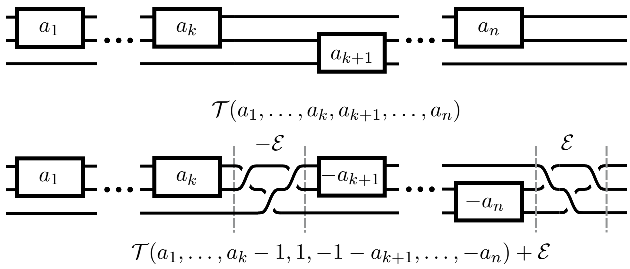


Fig. 4 The rightmost portion of the diagram (boxes a_{k+1} to a_n) is twisted by keeping the strand ends fixed, so these boxes swap their upper and lower positions and therefore change their signs. While the braid remains unaltered, an effect of the twist (also called a "flype") is that $-\mathcal{E}$ and $\mathcal{E}$ summands are inserted at the indicated positions. Although the figure illustrates the case n and k odd, the remaining three cases are analogous.

Using the above result along with an induction argument one readily proves the following.

Lemma 1 [2] *For every 3-braid T there exists a unique alternating diagram $\mathcal{A}$ and a unique integer k such that $T = \mathcal{A} + k\mathcal{E}$.*

If T is a 3-braid, the unique diagram $\mathcal{A} + k\mathcal{E}$ for T, whose existence is guaranteed by the previous result, is called the ***standard diagram*** of T.

Continued fractions are another key ingredient for the solution of equations with 3-braids. Given elements $a_1, ..., a_n \in \mathbb{Z}$, the associated continued fraction is the element $[a_1, \ldots, a_n] \in \mathbb{Q}$ with ***numerator*** $N([a_1, \ldots, a_n])$ and ***denominator*** $D([a_1, \ldots, a_n])$ defined inductively by:

$$N([a_1]) = a_1, \quad D([a_1]) = 1, \quad N([a_1, a_2]) = 1 + a_1 a_2, \quad D([a_1, a_2]) = a_2,$$

and, for $n > 2$:

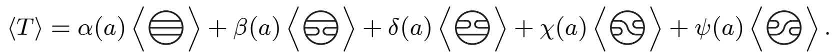
$$\begin{aligned}N([a_1, \ldots, a_n]) &= a_n N([a_1, \ldots, a_{n-1}]) + N([a_1, \ldots, a_{n-2}])\\ D([a_1, \ldots, a_n]) &= a_n D([a_1, \ldots, a_{n-1}]) + D([a_1, \ldots, a_{n-2}]).\end{aligned}$$

From the above definitions one easily proves that

$$[a_1, a_2, ..., a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}} = \frac{N([a_1, ..., a_n])}{D([a_1, ..., a_n])}.$$

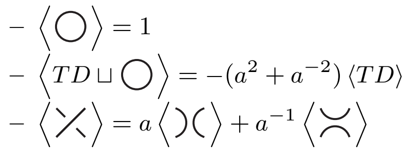
In order to simplify some expressions occurring in the manipulation of continued fractions, for any nonzero integer a we define $\frac{a}{0} = \infty$, $\infty + a = \infty = \infty \cdot a$, $\frac{a}{\infty} = 0$. Note that $[a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1] = [a_1, a_2, \ldots, a_n + 1, -1]$. In particular, if the entries of $[a_1, a_2, \ldots, a_n]$ do not alternate signs, that is, if $a_i \geq 0$ for $i = 1, \ldots, n$, or $a_i \leq 0$ for $i = 1, \ldots, n$, then exactly one from $[a_1, a_2, \ldots, a_n - 1, 1]$ or $[a_1, a_2, \ldots, a_n + 1, -1]$ satisfies the same property, namely, that all of its entries do not alternate signs.

It was shown in [3] that the Kauffman bracket polynomial $\langle T \rangle$ of a 3-tangle T may be expressed as



$$\langle T \rangle = \alpha(a) \langle \cdot \rangle + \beta(a) \langle \cdot \rangle + \delta(a) \langle \cdot \rangle + \chi(a) \langle \cdot \rangle + \psi(a) \langle \cdot \rangle .$$

Recall that the Kauffman bracket (cf. e.g. [9, Chap. 3]) is a function from unoriented link diagrams to Laurent polynomials with integer coefficients in an indeterminate a. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[a, a^{-1}]$, a polynomial characterized by the following three conditions:



- $\langle \bigcirc \rangle = 1$
- $\langle TD \sqcup \bigcirc \rangle = -(a^2 + a^{-2}) \langle TD \rangle$
- $\langle \cdot \rangle = a \langle \cdot \rangle + a^{-1} \langle \cdot \rangle$

Note that if $a = \sqrt{i}$, where i is the complex number such that $i^2 = -1$, then $a^2 + a^{-2} = 0$ and, therefore, $\langle TD \sqcup \bigcirc \rangle = 0$, which simplifies certain computations.

An invariant $\widehat{F}(T)$, defined in terms of the polynomial components of $\langle T \rangle$, was introduced in [3] by setting:

$$\widehat{F}(T)(a) = \left(\frac{\delta(a)}{\alpha(a) + \chi(a)}, \frac{\alpha(a) + \psi(a)}{\beta(a)} \right).$$

In the sequel, we let $F(T) = \widehat{F}(T)(\sqrt{i})$. In light of the above definitions we have $F(\mathcal{T}(n)) = \left(\frac{1}{i}n, \infty\right)$ and $F(\mathcal{T}(0, n)) = \left(0, \frac{1}{i}\frac{1}{n}\right)$. The following additional results in terms of the invariant F will prove useful in the sequel.

Theorem 2 [3] *Given $a_1, ..., a_n \in \mathbb{Z}$, one has*

$$F(\mathcal{T}(a_1, ..., a_n)) = \begin{cases} (\frac{1}{i}[a_1, \ldots, a_n], \frac{1}{i}[a_1, \ldots, a_{n-1}]), \textit{ for } n \textit{ odd;} \\ (\frac{1}{i}[a_1, \ldots, a_{n-1}], \frac{1}{i}[a_1, \ldots, a_n]), \textit{ for } n \textit{ even.} \end{cases}$$

The following result states that an effect of concatenating a 3-braid with $\mathcal{E}$ is that the components of F get swapped.

Theorem 3 [3] *If $F(\mathcal{T}(a_1, ..., a_n)) = (\frac{1}{i}\frac{\alpha}{\beta}, \frac{1}{i}\frac{\alpha'}{\beta'})$ then $F(\mathcal{T}(a_1, ..., a_n) + \mathcal{E}) = (\frac{1}{i}\frac{\alpha'}{\beta'}, \frac{1}{i}\frac{\alpha}{\beta})$.*

3 3-Braid closures and 2-bridge knots

The connection between 3-braids and knots becomes clear when one considers braid closures. If T is a 3-braid, its A ***closure***, denoted $A(T)$, is the knot or link that results after the addition of strands external to T, as illustrated in Figure 5.

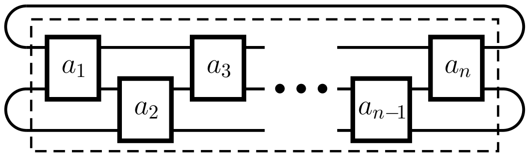


Fig. 5 The A closure of the 3-braid diagram $\mathcal{T}(a_1, a_2, ..., a_n)$ is the knot or link that results from the addition of external strands (outside the dotted box) as indicated.

According to [10, Chap. 9], knots thus obtained are called 2***-bridge knots***. The following theorem, derived from results in [3] and [10], shows that a close relationship exists between 2-bridge knots and A closures of 3-braids.

Theorem 4 *Every 2-bridge knot (or link) is the A closure of a 3-braid, and the A closure of any 3-braid is a 2-bridge knot (or link).*

Let T be a 2-bridge knot that admits a regular diagram $A(\mathcal{T}(a_1, a_2, \ldots, a_n))$, as shown in Figure 5. Then T is said to be a 2***-bridge knot of type*** $b(\alpha, \beta)$ if $\frac{\alpha}{-\beta} = [a_1, a_2, \ldots, a_n]$, where $\alpha, \beta \in \mathbb{Z}$ are relatively prime and $0 < |\beta| < \alpha$. In the remainder of this paper we write $\mathcal{K}_{b(\alpha,\beta)}$ to denote a 2-bridge knot of type $b(\alpha, \beta)$. As thoroughly described in [10, Chap. 9], given a knot type $b(\alpha, \beta)$, a specific diagram of the knot $\mathcal{K}_{b(\alpha,\beta)}$ may be constructed in a canonical way. Let us also remark that, by symmetry of the A closure, the diagrams $A(\mathcal{T}(1, 2, 3))$ and $A(\mathcal{T}(3, 2, 1))$ both represent the same 2-bridge knot, whereas their knot types are $b(10, -7)$ and $b(10, -3)$, respectively. As a consequence, $\mathcal{K}_{b(10,-7)} = \mathcal{K}_{b(10,-3)}$. This simple example illustrates the fact that a single 2-bridge knot may be represented by one or more knot types. What is more, the following strong result establishes criteria to assess the equivalence of 2-bridge knots in terms of their knot types. In the statement of the theorem, as well as in the ensuing exposition, one shall write $a \underset{\alpha}{\equiv} b$ as a shorthand notation for $a = b \mod \alpha$, with $a, b \in \mathbb{Z}$ and $\alpha \neq 0$.

Theorem 5 [11] *Given knot types $b(\alpha, \beta)$ and $b(\alpha', \beta')$, one has $\mathcal{K}_{b(\alpha,\beta)} = \mathcal{K}_{b(\alpha',\beta')}$ if, and only if, any of the following two conditions hold: (1) $\alpha = \alpha'$ and $\beta \underset{\alpha}{\equiv} \beta'$; or (2) $\alpha = \alpha'$ and $\beta\beta' \underset{\alpha}{\equiv} 1$.*

The following useful corollary speficies, in terms of α and β, the number of knot types associated with a 2-bridge knot of the form $\mathcal{K}_{b(\alpha,\beta)}$. Its proof, moreover, explicitly singles out the corresponding knot types.

Corollary 1 *Any 2-bridge knot admits exactly two or exactly four knot types. More specifically, if K is of type $b(\alpha, \beta)$, then K admits exactly two knot types if $\beta^2 \underset{\alpha}{\equiv} 1$ and exactly four knot types if $\beta^2 \underset{\alpha}{\not\equiv} 1$.*

Proof. Suppose that K is of knot type $b(\alpha,\beta)$, with α and β relatively prime and $0<|\beta|<\alpha$. Using condition (1) of Theorem 5, we seek β', with $\beta'\neq\beta$, such that $\beta \underset{\alpha}{\equiv} \beta'$. If $\beta>0$, then $\beta'=\beta-\alpha$ and, in that case, two types of equivalent knots are $b(\alpha,\beta)$ and $b(\alpha,\beta-\alpha)$. If $\beta<0$, then two types of equivalent knots are $b(\alpha,\beta)$ and $b(\alpha,\beta+\alpha)$. Thus, if Condition (1) of Theorem 5 holds, K admits at least two associated knot types. Moreover, since α and β are relatively prime, there exists a unique β' such that $\beta\beta' \underset{\alpha}{\equiv} 1$, hence either $\beta'=\beta$ or $\beta'\neq\beta$. If $\beta'=\beta$ we have $\beta^2 \underset{\alpha}{\equiv} 1$ and thus, in that case, condition (2) does not add any new knot type to the two already mentioned. If, on the other hand, $\beta'\neq\beta$, then $\beta^2 \underset{\alpha}{\not\equiv} 1$ and, again by condition (1), it follows that, besides the two types already singled out, $b(\alpha,\beta')$ is a knot type associated with K . Applying condition (1), the latter type yields $b(\alpha,\beta'-\alpha)$ if $\beta'>0$ or $b(\alpha,\beta'+\alpha)$ if $\beta'<0$, so in this case one ends up with exactly four knot types. □

The following result, which extends Theorem 4, allows one to determine the knot type corresponding to the closure of a 3-braid via the invariant F.

Theorem 6 [3] *Let T be a 3-braid. If $F(T)=\left(\frac{1}{i}\frac{\alpha_1}{\beta_1},\frac{1}{i}\frac{\alpha_2}{\beta_2}\right)$, with $|\frac{\alpha_1}{\beta_1}|>1$, then $A(T)$ $=\mathcal{K}_{b(\alpha_1,-\beta_1)}$.*

A direct consequence from this result and Theorem 2 is that, for n odd and $a_1\neq 0$,

$$A(\mathcal{T}(a_1,a_2,...,a_n))=\mathcal{K}_{b(N([a_1,a_2,...,a_n]),-D([a_1,a_2,...,a_n]))}.$$

Remark 1 One easily checks, by simple diagram inspection, the following equalities:

$$\begin{aligned} A(\mathcal{T}(0,a_2,\ldots,a_n)) &= A(\mathcal{T}(a_3,\ldots,a_n))\\ A(\mathcal{T}(a_1,\ldots,a_n)+2\mathcal{E}) &= A(\mathcal{T}(a_1,\ldots,a_n))\\ A(\mathcal{T}(a_1,\ldots,a_n)) &= A(\mathcal{T}(a_1,\ldots,a_{n-1})) \quad \text{ if } n \text{ is even,}\end{aligned}$$

and

$$A(\mathcal{T}(a_1,\ldots,a_n))=\begin{cases} A(\mathcal{T}(a_n,\ldots,a_1)), & \text{if } n \text{ is odd;}\\ A(\mathcal{T}(0,a_n,\ldots,a_1)), & \text{if } n \text{ is even.}\end{cases}$$

In order to simplify some expressions in the ensuing development, it is convenient to define two functions on the set of 3-braids. First, we set

$$h(\mathcal{T}(a_1,\ldots,a_n))=\begin{cases} \mathcal{T}(a_n,\ldots,a_1), & \text{if } n \text{ is odd;}\\ \mathcal{T}(0,a_n,\ldots,a_1), & \text{if } n \text{ is even,}\end{cases}$$

which is readily shown to be a mapping from the set of 3-braids onto itself. Intuitively, $h(T)$ is the result of flipping the braid T horizontally (i.e., around a vertical axis), and hence it is clear that, for any 3-braid T, $A(h(T))=A(T)$. To define the value of the second function on a 3-braid T, we consider its representation in standard form as $\mathcal{T}(a_1,\ldots,a_n)+k\mathcal{E}$ and set, for k even

$$c(\mathcal{T}(a_1,\ldots,a_n)+k\mathcal{E})=\begin{cases} \mathcal{T}(a_1,\ldots,a_n-1,1)+\mathcal{E}, & \text{if } a_i\geq 0 \text{ for all } i;\\ \mathcal{T}(a_1,\ldots,a_n+1,-1)+\mathcal{E}, & \text{if } a_i\leq 0 \text{ for all } i,\end{cases}$$

and, for k odd:

$$c(\mathcal{T}(a_1,\ldots,a_n)+k\mathcal{E}) = \begin{cases} \mathcal{T}(a_1,\ldots,a_n-1,1), & \text{if } a_i \geq 0 \text{ for all } i; \\ \mathcal{T}(a_1,\ldots,a_n+1,-1), & \text{if } a_i \leq 0 \text{ for all } i. \end{cases}$$

Similarly to h, c thus defined is a function from the set of 3-braids into itself, although c is not surjective since, for instance, no braid of the form $\mathcal{T}(1)+k\mathcal{E}$, with $k>1$, is in the image of c.

Lemma 2 *Let T be a 3-braid and let $\mathcal{T}(a_1,\ldots,a_n)+k\mathcal{E}$ be its representation in standard form. If n is odd and k is even, or n is even and k is odd, then $A(T)=A(c(T))$.*

Proof. Let us assume that n is odd and k is even. (The remaining case is proven in a similar manner.) In this case, Theorems 2 and 3 imply that

$$F(T)=\left(\frac{1}{i}[a_1,\ldots,a_n],\frac{1}{i}[a_1,\ldots,a_{n-1}]\right).$$

On the other hand, by virtue of Theorem 2 one has

$$F(\mathcal{T}(a_1,...,a_n-1,1))=\left(\frac{1}{i}[a_1,\ldots,a_n-1],\frac{1}{i}[a_1,\ldots,a_n-1,1]\right).$$

Hence, applying Theorem 3 yields

$$\begin{aligned} F(c(\mathcal{T})) &= \left(\frac{1}{i}[a_1,\ldots,a_n-1,1],\frac{1}{i}[a_1,\ldots,a_n-1]\right) \\ &= \left(\frac{1}{i}[a_1,\ldots,a_n],\frac{1}{i}[a_1,\ldots,a_n-1]\right). \end{aligned}$$

In view of Theorem 6, the knot type of $A(T)$ is $b(N([a_1,\ldots,a_n]),-D([a_1,\ldots,a_n]))$, which in turn equals the knot type of $A(c(T))$. Therefore, the latter two knots are equal, as was to be shown. □

4 Solving a system of equations

4.1 Solving a single equation

Given a 2-bridge knot $\mathcal{K}_{b(\alpha,\beta)}$, our first aim in this section will be to study the set of solutions to the equation

$$A(\mathcal{X})=\mathcal{K}_{b(\alpha,\beta)}, \tag{1}$$

that is, to explicitly determine the set of 3-braids $\mathcal{X}$ whose A closures equal the 2-bridge knot $\mathcal{K}_{b(\alpha,\beta)}$. Of course, Theorem 4 already states that the equation admits a solution irrespective of the choice of knot type $b(\alpha,\beta)$, so the next step would consist in finding those solutions. As a step in this direction, the following theorem outlines a procedure to determine solutions to Equation 1 from the given of a continued fraction expansion associated with the knot type $b(\alpha,\beta)$. According to Lemma 2 only the cases with or without $\mathcal{E}$ are listed.

Theorem 7 *Given a 2-bridge knot $\mathcal{K}_{b(\alpha,\beta)}$, the only 3-braids with $-\frac{\alpha}{\beta} = [a_1, ..., a_n]$ and n odd, which satisfy the equation $A(\mathcal{X}) = \mathcal{K}_{b(\alpha,\beta)}$ are:* ***(a)*** *If $a_i > 0$ for all i,*

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{T}(a_1, ..., a_n)\\
\mathcal{X}_2 &= \mathcal{T}(a_1, ..., a_n - 1, 1) + \mathcal{E}\\
\mathcal{X}_3 &= \mathcal{T}(-1, -(a_n - 1), ..., -a_2, -a_1) + \mathcal{E}\\
\mathcal{X}_4 &= \mathcal{T}(-1, -(a_n - 1), ..., -a_2, -(a_1 - 1), -1)\\
\mathcal{X}_5 &= \mathcal{T}(-1, -(a_1 - 1), -a_2, ..., -(a_n - 1), -1)\\
\mathcal{X}_6 &= \mathcal{T}(-1, -(a_1 - 1), -a_2, ..., -a_n) + \mathcal{E}\\
\mathcal{X}_7 &= \mathcal{T}(a_n, ..., a_1 - 1, 1) + \mathcal{E}\\
\mathcal{X}_8 &= \mathcal{T}(a_n, ..., a_1);
\end{aligned}$$

and ***(b)*** *If $a_i < 0$ for all i,*

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{T}(a_1, ..., a_n)\\
\mathcal{X}_2 &= \mathcal{T}(a_1, ..., a_n + 1, -1) + \mathcal{E}\\
\mathcal{X}_3 &= \mathcal{T}(1, -(a_n + 1), ..., -a_2, -a_1) + \mathcal{E}\\
\mathcal{X}_4 &= \mathcal{T}(1, -(a_n + 1), ..., -a_2, -(a_1 + 1), 1)\\
\mathcal{X}_5 &= \mathcal{T}(1, -(a_1 + 1), -a_2, ..., -(a_n + 1), 1)\\
\mathcal{X}_6 &= \mathcal{T}(1, -(a_1 + 1), -a_2, ..., -a_n) + \mathcal{E}\\
\mathcal{X}_7 &= \mathcal{T}(a_n, ..., a_1 + 1, -1) + \mathcal{E}\\
\mathcal{X}_8 &= \mathcal{T}(a_n, ..., a_1).
\end{aligned}$$

Proof. From Theorem 6 one has $\mathcal{K}_{b(\alpha,\beta)} = A(\mathcal{T}(a_1, ..., a_n)) = A(\mathcal{X}_1)$, so the first equality holds. Now, using Lemma 2 and Remark 1 repeatedly one deduces that

$$\begin{aligned}
A(\mathcal{X}_1) &= A(c(\mathcal{X}_1)) = A(\mathcal{X}_2) = A(h(\mathcal{X}_2)) = A(\mathcal{X}_3)\\
&= A(c(\mathcal{X}_3)) = A(\mathcal{X}_4) = A(h(\mathcal{X}_4)) = A(\mathcal{X}_5)
\end{aligned}$$

and

$$\begin{aligned}
A(\mathcal{X}_5) &= A(c(\mathcal{X}_5)) = A(\mathcal{X}_6) = A(h(\mathcal{X}_6)) = A(\mathcal{X}_7)\\
&= A(c(\mathcal{X}_7)) = A(\mathcal{X}_8) = A(h(\mathcal{X}_8)) = A(\mathcal{X}_1).
\end{aligned}$$

□

The following example shows an application of Theorem 7 to solve a single equation with one unknown.

Example 2 Consider the 2-bridge knot $\mathcal{K}_{b(7,-3)}$ and note that $\frac{7}{3} = [2, 2, 1]$. Then the only 3-braids $\mathcal{X}$ that satisfy the equation $A(\mathcal{X}) = \mathcal{K}_{b(7,-3)}$ are the following:

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{T}(2, 2, 1), & \mathcal{X}_5 &= \mathcal{T}(-1, -1, -3),\\
\mathcal{X}_2 &= \mathcal{T}(2, 3) + \mathcal{E}, & \mathcal{X}_6 &= \mathcal{T}(-1, -1, -2, -1) + \mathcal{E},\\
\mathcal{X}_3 &= \mathcal{T}(-3, -2) + \mathcal{E}, & \mathcal{X}_7 &= \mathcal{T}(1, 2, 1, 1) + \mathcal{E},\\
\mathcal{X}_4 &= \mathcal{T}(-3, -1, -1), & \mathcal{X}_8 &= \mathcal{T}(1, 2, 2).
\end{aligned}$$

Remark 2 Computing the invariant F of the 3-braids involved in the statement of Theorem 7 (in the case $a_i > 0$ for all i) yields
$F(\mathcal{X}_1) = (\frac{1}{i}[a_1, ..., a_n], \frac{1}{i}[a_1, ..., a_{n-1}])$
$F(\mathcal{X}_2) = (\frac{1}{i}[a_1, ..., a_n - 1, 1], \frac{1}{i}[a_1, ..., a_n - 1]) = (\frac{1}{i}[a_1, ..., a_n], \frac{1}{i}[a_1, ..., a_n - 1])$
$F(\mathcal{X}_3) = (\frac{1}{i}[-1, -(a_n - 1), ..., -a_2, -a_1], \frac{1}{i}[-1, -(a_n - 1), ..., -a_2])$
$F(\mathcal{X}_4) = (\frac{1}{i}[-1, -(a_n-1), ..., -a_2, -(a_1-1), -1], \frac{1}{i}[-1, -(a_n-1), ..., -a_2, -(a_1-1)])$
$F(\mathcal{X}_5) = (\frac{1}{i}[-1, -(a_1-1)-a_2, , ..., -(a_n-1), -1)], \frac{1}{i}[-1, -(a_1-1)-a_2, , ..., -(a_n-1)])$
$F(\mathcal{X}_6) = (\frac{1}{i}[-1, -(a_1 - 1), -a_2, ..., -(a_n)], \frac{1}{i}[-1, -(a_1 - 1), -a_2, ..., -(a_{n-1})])$
$F(\mathcal{X}_7) = (\frac{1}{i}[(a_n, ..., a_1 - 1, 1], \frac{1}{i}[(a_n, ..., a_1 - 1)])$
$F(\mathcal{X}_8) = (\frac{1}{i}[a_n, ..., a_1], \frac{1}{i}[a_n, ..., a_2])$.

Recall that the A closures of the $\mathcal{X}_1, \dots, \mathcal{X}_8$ are all equal to the same 2-bridge knot K, so let us now show that K admits exactly 4 or exactly 2 knot types. Using the properties of continued fractions mentioned above, one sees that for each $i \in \{1, \dots, 4\}$, the first component of $F(\mathcal{X}_{2i})$ coincides with the first component of $F(\mathcal{X}_{2i-1})$. By Theorem 6, it follows that the knot types of the closures $A(\mathcal{X}_{2i})$ and $A(\mathcal{X}_{2i-1})$ are equal as well, hence there are at most four different knot types involved. Moreover, note that if $(a_1, \dots, a_n) \neq (a_n, \dots, a_1)$ (i.e., if the finite sequence of the a_i is not palindromic), then the continued fraction defined by the first component of $F(\mathcal{X}_{2i})$ differs from the continued fraction defined by the first component of $F(\mathcal{X}_{2j})$ whenever $i \neq j$, so in this case one has exactly four different knot types. By contrast, if $(a_1, \dots, a_n) = (a_n, \dots, a_1)$ then $\mathcal{X}_1 = \mathcal{X}_8$, $\mathcal{X}_2 = \mathcal{X}_7$, $\mathcal{X}_3 = \mathcal{X}_6$ and $\mathcal{X}_4 = \mathcal{X}_5$. Therefore, since the first components of $F(\mathcal{X}_1)$, $F(\mathcal{X}_2)$, $F(\mathcal{X}_7)$ and $F(\mathcal{X}_8)$ are all equal, and the analogous situation holds for $F(\mathcal{X}_3)$, $F(\mathcal{X}_4)$, $F(\mathcal{X}_5)$ and $F(\mathcal{X}_6)$, in this case the number of different knot types reduces to exactly two.

The 3-braid solutions to the equation in the statement of Theorem 7 were found using a particular choice of α and β. Nevertheless, as mentioned in the previous paragraphs, several knot types are associated to a given 2-bridge knot, so one may naturally wonder whether the 3-braids obtained with a different choice α' and β' are the same. Let us now show that this is indeed the case. According to Theorem 5, if a 3-braid T satisfies $A(T) = \mathcal{K}_{b(\alpha,\beta)}$, then one has $A(T) = \mathcal{K}_{b(\alpha',\beta')}$ whenever (1) $\alpha = \alpha'$ and $\beta \underset{\alpha}{\equiv} \beta'$, or (2) $\alpha = \alpha'$ and $\beta\beta' \underset{\alpha}{\equiv} 1$. However, according to Corollary 1, all these other knot types for the same 2-bridge knot may amount to at most 4. Thus, the argument in remark 2 shows that there exists $i \in \{1, \dots, 8\}$ such that $A(\mathcal{X}_i) = \mathcal{K}_{b(\alpha',\beta')}$ or, in other words, if any 3-braid $\mathcal{X}$ is a solution of equation (1), then its A closure must be equal to one of the knot types associated with the A closures of $\mathcal{X}_1, \dots, \mathcal{X}_8$. In light of these observations, in order to study solutions to (1) one may focus exclusively on the eight 3-braids listed in Theorem 7.

As a simple byproduct of these results we obtain the following corollary, which is of a more number-theoretic flavor.

Corollary 2 *Let $\alpha, \beta \in \mathbb{Z}$ be relatively prime and such that $0 < |\beta| < \alpha$. If $\beta^2 \underset{\alpha}{\equiv} 1$, then there exist integers $b_1, \dots, b_n$ satisfying $\frac{\alpha}{\beta} = [b_1, b_2, \dots, b_n, \dots, b_2, b_1]$.*

Proof. In view of Lemma 1, there are exactly two knot types associated with the 2-bridge knot $\mathcal{K}_{b(\alpha,-\beta)}$. On the other hand, if $-\frac{\alpha}{\beta} = [a_1, ..., a_n] \neq [a_n, ..., a_1]$, by Remark 2, the 3-braids $\mathcal{X}_i$, with $i = 1, 2, \dots, 8$, are all different, satisfy $A(\mathcal{X}_i) = \mathcal{K}_{b(\alpha,-\beta)}$, and the first components of $F(\mathcal{X}_{2i})$ and $F(\mathcal{X}_{2j})$ are different whenever $i \neq j$. Hence, in this case there are exactly four different knot types, a contradiction. □

Note that, whenever $\beta^2 \underset{\alpha}{\equiv} 1$, Corollary 2 implies that the expansion of $\frac{\alpha}{\beta}$ is palindromic, i.e., $\frac{\alpha}{\beta} = [b_1, b_2, \ldots, b_n, \ldots, b_2, b_1]$. Substituting the latter in the expressions for the $\mathcal{X}_1, \ldots, \mathcal{X}_8$ of Theorem 7, one immediately concludes that, regardless of the sign of the $b_1, \ldots, b_n$, $\mathcal{X}_4 = \mathcal{X}_5$, $\mathcal{X}_3 = \mathcal{X}_6$, $\mathcal{X}_2 = \mathcal{X}_7$ and $\mathcal{X}_1 = \mathcal{X}_8$, hence in this case there are only 4 different 3-braid solutions. On the other hand, if $\beta^2 \underset{\alpha}{\not\equiv} 1$ then there are 8 different solutions.

4.2 Solving two or more equations

The main aim in this section is to explore the solutions of systems of equations involving 3-braids and 2-bridge knots. More precisely, suppose one is now given the system

$$A(\mathcal{X}) = \mathcal{K}_{b(\alpha,\beta)}, \qquad A(\mathcal{Y}) = \mathcal{K}_{b(\alpha',\beta')}. \tag{2}$$

As stated above, these systems arise in biological applications involving enzymes that act repeatedly on a substrate, e.g. DNA molecule. Owing to the model chosen to represent the enzymatic action, namely a model based on iterated recombinations, the unknown braids in (2) are required to satisfy additional conditions. Indeed, the original equations are stated in terms of unknown braids $\mathcal{S}$ and $\mathcal{T}$:

$$A(\mathcal{S} + \mathcal{T}) = \mathcal{K}_{b(\alpha,\beta)}, \qquad A(\mathcal{S} + 2\mathcal{T}) = \mathcal{K}_{b(\alpha',\beta')}. \tag{3}$$

Setting $\mathcal{X} = \mathcal{S} + \mathcal{T}$ and $\mathcal{Y} = \mathcal{S} + 2\mathcal{T}$ one retrieves (2), and by simple manipulations involving concatenation and inverses, one immediately checks that $\mathcal{S} = \mathcal{X} - \mathcal{Y} + \mathcal{X}$ and $\mathcal{T} = -\mathcal{X} + \mathcal{Y}$. Following the sequence of equations suggested by the recombination model, one may then consider, more generally, equations in which the unknowns are $\mathcal{S} + n\mathcal{T}$ for $n \geq 2$, in which case one may always rewrite them in terms of $\mathcal{X}$ and $\mathcal{Y}$ as long as one considers the additional equations $\mathcal{S} + n\mathcal{T} = \mathcal{Y} + (n-2)(-\mathcal{X} + \mathcal{Y})$ for $n \geq 2$.

Clearly, any knot types $b(\alpha, \beta)$ and $b(\alpha', \beta')$ can be used in the above system and the resulting equations are guaranteed to admit solutions in view of Theorem 4. What is more, Theorem 7 implies that if $\beta^2 \underset{\alpha}{\equiv} 1$ and $\beta'^2 \underset{\alpha'}{\equiv} 1$, then there exist four 3-braids $\mathcal{X}_i$ and four 3-braids $\mathcal{Y}_j$ satisfying Equation (2); in this case, by taking all possible combinations of values for i and j, one ends up with 16 systems of the form $\mathcal{X}_i = \mathcal{S} + \mathcal{T}$ and $\mathcal{Y}_j = \mathcal{S} + 2\mathcal{T}$, which implies in turn that the system of equations in (3) admits 16 pairs of solutions $(\mathcal{S}, \mathcal{T})$. On the other hand, if exactly one of $\beta^2 \underset{\alpha}{\not\equiv} 1$ or $\beta'^2 \underset{\alpha'}{\not\equiv} 1$ holds, System (3) admits 32 pairs of solutions $(\mathcal{S}, \mathcal{T})$, whereas if both conditions hold, the number of pairs of solutions reaches 64.

Assuming that one has solved the equations in (3) for $\mathcal{S}$ and $\mathcal{T}$, one may again augment the system to

$$A(\mathcal{S} + \mathcal{T}) = \mathcal{K}_{b(\alpha,\beta)}, \quad A(\mathcal{S} + 2\mathcal{T}) = \mathcal{K}_{b(\alpha',\beta')} \quad A(\mathcal{S} + 3\mathcal{T}) = \mathcal{K}_{b(\alpha'',\beta'')}. \tag{4}$$

Here, a natural question would be *what knot types $b(\alpha'', \beta'')$ may appear on the right hand side of the third equation so that the resulting system still admits solutions?* The following result answers this question by singling out knot types which still guarantee solutions of the augmented system *in terms of known solutions $\mathcal{X}$ and $\mathcal{Y}$*, i.e., without requiring the computation of $\mathcal{S}$ and $\mathcal{T}$.

Theorem 8 *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are solutions to (2). Then, depending on which of the conditions (C1), (C2) or (C3) hold, there exist exactly 6, 9 or 18 different 2-bridge knots, respectively, whose types $b(\alpha'', \beta'')$ guarantee that solutions exist for System (4):*

(C1) $\beta^2 \underset{\alpha}{\equiv} 1 \underset{\alpha'}{\equiv} \beta'^2$,

(C2) $(\beta^2 \underset{\alpha}{\not\equiv} 1$ *and* $\beta'^2 \underset{\alpha'}{\equiv} 1)$ *or* $(\beta^2 \underset{\alpha}{\equiv} 1$ *and* $\beta'^2 \underset{\alpha'}{\not\equiv} 1)$,

(C3) $\beta^2 \underset{\alpha}{\not\equiv} 1 \underset{\alpha'}{\not\equiv} \beta'^2$.

Proof. Only the proof of condition **(C1)** will be given; the remaining cases are treated in a similar way. From the knot types $b(\alpha, \beta)$ and $b(\alpha', \beta')$, and by Corollary 2, one obtains the following continued fraction expansions

$$\frac{\alpha}{-\beta} = [a_1, a_2, ..., a_n, ..., a_2, a_1] \quad \text{and} \quad \frac{\alpha'}{-\beta'} = [b_1, b_2, ..., b_m, ..., b_2, b_1].$$

By Theorem 7, a set of braids $\mathcal{X}_i$ and $\mathcal{Y}_j$ satisfying $A(\mathcal{X}_i) = \mathcal{K}_{b(\alpha,\beta)}$ and $A(\mathcal{Y}_j) = \mathcal{K}_{b(\alpha',\beta')}$, for all i, j, may be computed explicitly:

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{T}(a_1, a_2, ..., a_n, ..., a_2, a_1),\\
\mathcal{X}_2 &= \mathcal{T}(a_1, a_2, ..., a_n, ..., a_2, a_1 - 1, 1) + \mathcal{E},\\
\mathcal{X}_3 &= \mathcal{T}(-1, -(a_1 - 1), ..., -a_n, ..., -a_2, -a_1) + \mathcal{E},\\
\mathcal{X}_4 &= \mathcal{T}(-1, -(a_1 - 1), ..., -a_n, ..., -a_2, -(a_1 - 1), -1)\\
\mathcal{Y}_1 &= \mathcal{T}(b_1, b_2, ..., b_m, ..., b_2, b_1),\\
\mathcal{Y}_2 &= \mathcal{T}(b_1, b_2, ..., b_m, ..., b_2, b_1 - 1, 1) + \mathcal{E},\\
\mathcal{Y}_3 &= \mathcal{T}(-1, -(b_1 - 1), ..., -b_m, ..., -b_2, -b_1) + \mathcal{E},\\
\mathcal{Y}_4 &= \mathcal{T}(-1, -(b_1 - 1), ..., -b_m, ..., -b_2, -(b_1 - 1), -1).
\end{aligned}$$

In order to keep the notation compact, in the remainder of the proof we write $\mathcal{T}(b_1, B, b_1 - a_1, -A, a_1)$ as shorthand notation for

$$\mathcal{T}(b_1, b_2, ..., b_m, \ldots, b_2, b_1 - a_1, -a_2, ..., -a_n, \ldots, -a_2, -a_1).$$

From the above expression one gets

$$\mathcal{X}_1 = \mathcal{T}(a_1, A, a_1) \quad \text{and} \quad \mathcal{Y}_3 = \mathcal{T}(-1, 1 - b_1, -B, -b_1) + \mathcal{E}.$$

On the other hand, as suggested by the discussion in Remark 2, it suffices to consider only these 3-braids in order to study the full set of solutions to each of the two equations. The next step is the computation, using basic manipulations involving the operations in the group of 3-braid diagrams, of the sixteen corresponding combinations of 3-braids related by the equations $\mathcal{Z}_{ij} = \mathcal{Y}_j - \mathcal{X}_i + \mathcal{Y}_j = \mathcal{S}_{ij} + 3\mathcal{T}_{ij}$, with $i, j = 1, 2, 3, 4$. As a first case, we have the 3-braids

$$\begin{aligned}
\mathcal{Z}_{11} &= \mathcal{T}(b_1, B, b_1 - a_1, -A, -a_1 + b_1, B, b_1),\\
\mathcal{Z}_{22} &= \mathcal{T}(b_1, B, b_1 - a_1, -A, -a_1 + b_1, B, b_1 - 1, 1) + \mathcal{E},\\
\mathcal{Z}_{33} &= \mathcal{T}(-1, 1 - b_1, -B, -(b_1 + a_1), A, a_1 - b_1, -B, -b_1) + \mathcal{E},\\
\mathcal{Z}_{44} &= \mathcal{T}(-1, 1 - b_1, -B, -b_1 + a_1, A, a_1 - b_1, -B, 1 - b_1, -1).
\end{aligned}$$

Now, since by Theorems 2 and 3 the first component of $F(\mathcal{Z}_{11})$ equals the first component of $F(\mathcal{Z}_{22})$, by Theorem 6 it follows that $A(\mathcal{Z}_{11}) = A(\mathcal{Z}_{22})$. Moreover, from the definition of h it follows that $h(\mathcal{Z}_{22}) = \mathcal{Z}_{33}$ and hence, by Remark 1, one has $A(\mathcal{Z}_{22}) = A(\mathcal{Z}_{33})$. Finally, since the first component of $F(\mathcal{Z}_{33})$ is equal to the first component of $F(\mathcal{Z}_{44})$, by Theorem 6 it follows that $A(\mathcal{Z}_{33}) = A(\mathcal{Z}_{44})$. Therefore

$$A(\mathcal{Z}_{11}) = A(\mathcal{Z}_{22}) = A(\mathcal{Z}_{33}) = A(\mathcal{Z}_{44}).$$

In the case of the 3-braids

$$\begin{aligned}
\mathcal{Z}_{12} &= \mathcal{T}(b_1, B, b_1 - 1, 1 + a_1, A, a_1 - b_1, -B, 1 - b_1, -1) + 2\mathcal{E},\\
\mathcal{Z}_{13} &= \mathcal{T}(-1, 1 - b_1, -B, -b_1 + a_1, A, a_1 + 1, b_1 - 1, B, b_1) + 2\mathcal{E},\\
\mathcal{Z}_{24} &= \mathcal{T}(-1, 1 - b_1, -B, -b_1 + a_1, A, a_1 + 1, b_1 - 1, B, b_1 - 1, 1) - \mathcal{E}.\\
\mathcal{Z}_{34} &= \mathcal{T}(-1, 1 - b_1, -B, 1 - b_1, -1 - a_1, -A, -a_1 + b_1, B, b_1 - 1, 1) - \mathcal{E},
\end{aligned}$$

one has $h(\mathcal{Z}_{12}) = \mathcal{Z}_{13}$, so it follows, by Remark 1, that $A(\mathcal{Z}_{12}) = A(\mathcal{Z}_{13})$. Since, by Theorems 2 and 3, the first components of $F(\mathcal{Z}_{13})$ and $F(\mathcal{Z}_{24})$ are the same, by Theorem 6 it follows that $A(\mathcal{Z}_{13}) = A(\mathcal{Z}_{24})$. Finally, since $h(\mathcal{Z}_{24}) = \mathcal{Z}_{34}$ it follows that $A(\mathcal{Z}_{24}) = A(\mathcal{Z}_{34})$. Hence

$$A(\mathcal{Z}_{12}) = A(\mathcal{Z}_{13}) = A(\mathcal{Z}_{24}) = A(\mathcal{Z}_{34}).$$

Now, for

$$\begin{aligned}
\mathcal{Z}_{21} &= \mathcal{T}(b_1, B, b_1 + 1, a_1 - 1, A, a_1 - b_1, -B, -b_1) - \mathcal{E},\\
\mathcal{Z}_{31} &= \mathcal{T}(b_1, B, b_1 - a_1, -A, 1 - a_1, -1 - b_1, -B, -b_1) - \mathcal{E},\\
\mathcal{Z}_{42} &= \mathcal{T}(b_1, B, b_1 - a_1, -A, 1 - a_1, -1 - b_1, -B, 1 - b_1, -1) + 2\mathcal{E},\\
\mathcal{Z}_{43} &= \mathcal{T}(-1, 1 - b_1, -B, -b_1 - 1, 1 - a_1, -A, -a_1 + b_1, B, b_1) + 2\mathcal{E},
\end{aligned}$$

one has $h(\mathcal{Z}_{21}) = \mathcal{Z}_{31}$, hence it follows that $A(\mathcal{Z}_{21}) = A(\mathcal{Z}_{31})$. Since, by Theorems 2 and 3, the first components of $F(\mathcal{Z}_{31})$ and $F(\mathcal{Z}_{42})$ are equal, then it follows readily, from Theorem 6, that $A(\mathcal{Z}_{31}) = A(\mathcal{Z}_{42})$. As before, we have $h(\mathcal{Z}_{42}) = \mathcal{Z}_{43}$, therefore $A(\mathcal{Z}_{42}) = A(\mathcal{Z}_{43})$. Accordingly

$$A(\mathcal{Z}_{21}) = A(\mathcal{Z}_{31}) = A(\mathcal{Z}_{42}) = A(\mathcal{Z}_{43}).$$

On the other hand, since

$$\begin{aligned}
\mathcal{Z}_{23} &= \mathcal{T}(-1, 1 - b_1, -B, 1 - b_1, 1 - a_1, -A, -a_1 - 1, 1 - b_1, -B, -b_1) + \mathcal{E},\\
\mathcal{Z}_{32} &= \mathcal{T}(b_1, B, b_1 - 1, 1 + a_1, A, a_1 - 1, b_1 - 1, B, b_1 - 1, 1) + \mathcal{E},
\end{aligned}$$

and given that $h(\mathcal{Z}_{23}) = \mathcal{Z}_{32}$, by Remark 1, it follows that

$$A(\mathcal{Z}_{23}) = A(\mathcal{Z}_{32}).$$

Finally, from

$$\begin{aligned}
\mathcal{Z}_{14} &= \mathcal{T}(-1, 1 - b_1, -B, 1 - b_1, -1 - a_1, -A, -a_1 - 1, 1 - b_1, -B, 1 - b_1, -1).\\
\mathcal{Z}_{41} &= \mathcal{T}(b_1, B, b_1 + 1, a_1 - 1, A, a_1 - 1, 1 + b_1, B, b_1),
\end{aligned}$$

the knots given by $A(\mathcal{Z}_{14})$ and $A(\mathcal{Z}_{41})$ are obtained.

Note that the knots defined by $A(\mathcal{Z}_{11})$, $A(\mathcal{Z}_{12})$, $A(\mathcal{Z}_{21})$, $A(\mathcal{Z}_{23})$, $A(\mathcal{Z}_{14})$ and $A(\mathcal{Z}_{41})$ are all different, a fact that is easily deduced by analyzing their knot types together with the first components of $F(\mathcal{Z}_{11})$, $F(\mathcal{Z}_{12})$, $F(\mathcal{Z}_{21})$, $F(\mathcal{Z}_{23})$, $F(\mathcal{Z}_{14})$ and $F(\mathcal{Z}_{41})$. This analysis shows that the six knots above mentioned do not satisfy Theorem 5. Hence, it is concluded that there are six different knots determined by the closures of $A(\mathcal{Z}_{ij})$ with $i, j \in \{1, 2, 3, 4\}$.

The corresponding proofs under the assumptions that Condition **(C2)** or **(C3)** holds are similar and will be omitted for economy of space. In any case, however, we detail the sets of different knots in each case, as well as the way they arise as A closures of the corresponding 3-braids Z_{ij}. For the first case of **(C2)**, one has eight and four 3-braids that solve the equations in (2), respectively: $\mathcal{X}_i$ with $i = 1, 2, \ldots, 8$ and $\mathcal{Y}_j$ with $j = i, 2, 3, 4$. There are 32 combinations in this case, and the respective relations are:

$$A(\mathcal{Z}_{11}) = A(\mathcal{Z}_{22}) = A(\mathcal{Z}_{33}) = A(\mathcal{Z}_{44}) = A(\mathcal{Z}_{54}) = A(\mathcal{Z}_{63}) = A(\mathcal{Z}_{72}) = A(\mathcal{Z}_{81})$$
$$A(\mathcal{Z}_{12}) = A(\mathcal{Z}_{34}) = A(\mathcal{Z}_{74}) = A(\mathcal{Z}_{83}) \qquad A(\mathcal{Z}_{13}) = A(\mathcal{Z}_{24}) = A(\mathcal{Z}_{64}) = A(\mathcal{Z}_{82})$$
$$A(\mathcal{Z}_{21}) = A(\mathcal{Z}_{43}) = A(\mathcal{Z}_{52}) = A(\mathcal{Z}_{61}) \qquad A(\mathcal{Z}_{42}) = A(\mathcal{Z}_{31}) = A(\mathcal{Z}_{53}) = A(\mathcal{Z}_{71})$$
$$A(\mathcal{Z}_{14}) = A(\mathcal{Z}_{84}) \qquad A(\mathcal{Z}_{23}) = A(\mathcal{Z}_{62})$$
$$A(\mathcal{Z}_{41}) = A(\mathcal{Z}_{51}) \qquad A(\mathcal{Z}_{32}) = A(\mathcal{Z}_{73}).$$

Again, one shows that the knots $A(\mathcal{Z}_{ij})$, for

$$(i, j) \in \{(1, 1), (1, 2), (1, 3), (2, 1), (4, 2), (1, 4), (2, 3), (4, 1), (3, 2)\},$$

are all different, so there exist exactly 9 distinct knots. For the second case of **(C2)**, the relations that arise are:

$$A(\mathcal{Z}_{11}) = A(\mathcal{Z}_{18}) = A(\mathcal{Z}_{22}) = A(\mathcal{Z}_{27}) = A(\mathcal{Z}_{33}) = A(\mathcal{Z}_{36}) = A(\mathcal{Z}_{44}) = A(\mathcal{Z}_{45}),$$
$$A(\mathcal{Z}_{12}) = A(\mathcal{Z}_{16}) = A(\mathcal{Z}_{25}) = A(\mathcal{Z}_{34}), \qquad A(\mathcal{Z}_{13}) = A(\mathcal{Z}_{17}) = A(\mathcal{Z}_{24}) = A(\mathcal{Z}_{35}),$$
$$A(\mathcal{Z}_{21}) = A(\mathcal{Z}_{38}) = A(\mathcal{Z}_{43}) = A(\mathcal{Z}_{46}), \qquad A(\mathcal{Z}_{42}) = A(\mathcal{Z}_{46}) = A(\mathcal{Z}_{31}) = A(\mathcal{Z}_{28}),$$
$$A(\mathcal{Z}_{14}) = A(\mathcal{Z}_{15}), \qquad A(\mathcal{Z}_{23}) = A(\mathcal{Z}_{37}),$$
$$A(\mathcal{Z}_{26}) = A(\mathcal{Z}_{32}), \qquad A(\mathcal{Z}_{41}) = A(\mathcal{Z}_{48}).$$

For **(C3)**, one has eight solutions for each equation, from which one obtains, after the required manipulations:

$$A(\mathcal{Z}_{11}) = A(\mathcal{Z}_{22}) = A(\mathcal{Z}_{33}) = A(\mathcal{Z}_{44}) = A(\mathcal{Z}_{58}) = A(\mathcal{Z}_{67}) = A(\mathcal{Z}_{76}) = A(\mathcal{Z}_{85}),$$
$$A(\mathcal{Z}_{15}) = A(\mathcal{Z}_{26}) = A(\mathcal{Z}_{37}) = A(\mathcal{Z}_{48}) = A(\mathcal{Z}_{54}) = A(\mathcal{Z}_{63}) = A(\mathcal{Z}_{72}) = A(\mathcal{Z}_{81}),$$
$$A(\mathcal{Z}_{16}) = A(\mathcal{Z}_{38}) = A(\mathcal{Z}_{74}) = A(\mathcal{Z}_{83}), \qquad A(\mathcal{Z}_{17}) = A(\mathcal{Z}_{28}) = A(\mathcal{Z}_{64}) = A(\mathcal{Z}_{82}),$$
$$A(\mathcal{Z}_{12}) = A(\mathcal{Z}_{34}) = A(\mathcal{Z}_{78}) = A(\mathcal{Z}_{87}), \qquad A(\mathcal{Z}_{13}) = A(\mathcal{Z}_{24}) = A(\mathcal{Z}_{68}) = A(\mathcal{Z}_{86}),$$
$$A(\mathcal{Z}_{19}) = A(\mathcal{Z}_{43}) = A(\mathcal{Z}_{56}) = A(\mathcal{Z}_{65}), \qquad A(\mathcal{Z}_{21}) = A(\mathcal{Z}_{36}) = A(\mathcal{Z}_{61}) = A(\mathcal{Z}_{73}),$$
$$A(\mathcal{Z}_{25}) = A(\mathcal{Z}_{47}) = A(\mathcal{Z}_{52}) = A(\mathcal{Z}_{59}), \qquad A(\mathcal{Z}_{31}) = A(\mathcal{Z}_{42}) = A(\mathcal{Z}_{57}) = A(\mathcal{Z}_{75}),$$
$$A(\mathcal{Z}_{35}) = A(\mathcal{Z}_{46}) = A(\mathcal{Z}_{53}) = A(\mathcal{Z}_{71}),$$
$$A(\mathcal{Z}_{14}) = A(\mathcal{Z}_{88}), \quad A(\mathcal{Z}_{18}) = A(\mathcal{Z}_{84}), \quad A(\mathcal{Z}_{23}) = A(\mathcal{Z}_{66}), \quad A(\mathcal{Z}_{27}) = A(\mathcal{Z}_{62}),$$
$$A(\mathcal{Z}_{32}) = A(\mathcal{Z}_{77}), \qquad A(\mathcal{Z}_{41}) = A(\mathcal{Z}_{55}), \qquad A(\mathcal{Z}_{45}) = A(\mathcal{Z}_{51}).$$

Which determine 18 different knots. □

Let us now augment the system of equations (4) by adding a fourth equation

$$A(\mathcal{W}) = \mathcal{K}_{b(\alpha''', \beta''')},$$

and then prove that in this scenario there exists only one 2-bridge knot type $b(\alpha''', \beta''')$ for which the augmented system is solvable.

Theorem 9 *Assume that the system $A(\mathcal{X}) = \mathcal{K}_{b(\alpha,\beta)}$, $A(\mathcal{Y}) = \mathcal{K}_{b(\alpha',\beta')}$, $A(\mathcal{Z}) = \mathcal{K}_{b(\alpha'',\beta'')}$ admits a nonempty set of solutions. Then there exists exactly one 2-bridge knot which guarantees that the system augmented with the fourth equation $A(\mathcal{W}) = \mathcal{K}_{b(\alpha''',\beta''')}$ admits a nonempty set of solutions.*

Proof. As mentioned above, the 3-braids $\mathcal{W}$ are completely determined by the first and second equations, namely $\mathcal{W}_{ij} = \mathcal{Y}_j + 2(-\mathcal{X}_i + \mathcal{Y}_j)$. Then we have:

$\mathcal{W}_{11} = \mathcal{T}(b_1, B, b_1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1)$,
$\mathcal{W}_{12} = \mathcal{T}(b_1, B, b_1\text{-}1, 1 + a_1, A, a_1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1\text{-}1, 1) + 3\mathcal{E}$,
$\mathcal{W}_{13} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1 + 1, b_1\text{-}1, B, b_1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, \text{-}b_1) + 3\mathcal{E}$,
$\mathcal{W}_{14} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1)$.
$\mathcal{W}_{21} = \mathcal{T}(b_1, B, b_1 + 1, a_1\text{-}1, A, a_1\text{-}b_1, \text{-}B, \text{-}b_1\text{-}1, 1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1)$,
$\mathcal{W}_{22} = \mathcal{T}(b_1, B, b_1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1\text{-}1, 1) + 3\mathcal{E}$,
$\mathcal{W}_{23} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, 1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, \text{-}b_1\text{-}1, 1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, \text{-}b_1) + 3\mathcal{E}$,
$\mathcal{W}_{24} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1 + 1, b_1\text{-}1, B, b_1\text{-}a_1, \text{-}A, \text{-}a_1\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1)$.
$\mathcal{W}_{31} = \mathcal{T}(b_1, B, b_1\text{-}a_1, \text{-}A, 1\text{-}a_1, \text{-}1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1\text{-}1, 1 + b_1, B, b_1)$,
$\mathcal{W}_{32} = \mathcal{T}(b_1, B, b_1\text{-}1, 1 + a_1, A, a_1\text{-}1, 1 + b_1, B, b_1\text{-}1, 1 + a_1, A, a_1\text{-}1, 1 + b_1, B, b_1\text{-}1, 1) + 3\mathcal{E}$,
$\mathcal{W}_{33} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1\text{-}b_1, \text{-}B, \text{-}b_1) + 3\mathcal{E}$,
$\mathcal{W}_{34} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1\text{-}1, 1{+}a_1, A, a_1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1)$.
$\mathcal{W}_{41} = \mathcal{T}(b_1, B, b_1 + 1, a_1\text{-}1, A, a_1\text{-}1, 1 + b_1, B, b_1 + 1, a_1\text{-}1, A, a_1\text{-}1, 1 + b_1, B, b_1)$,
$\mathcal{W}_{42} = \mathcal{T}(b_1, B, b_1\text{-}a_1, \text{-}A, 1\text{-}a_1, \text{-}1\text{-}b_1, \text{-}B, \text{-}b_1 + a_1, A, a_1\text{-}1, 1 + b_1, B, b_1\text{-}1, 1) + 3\mathcal{E}$,
$\mathcal{W}_{43} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, \text{-}b_1\text{-}1, 1\text{-}a_1, \text{-}A, b_1\text{-}a_1, B, b_1 + 1, a_1\text{-}1, A, a_1\text{-}b_1, \text{-}B, \text{-}b_1) + 3\mathcal{E}$,
$\mathcal{W}_{44} = \mathcal{T}(\text{-}1, 1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1\text{-}b_1, \text{-}B, a_1\text{-}b_1, A, a_1\text{-}b_1, \text{-}B, 1\text{-}b_1, \text{-}1)$.

Surprisingly, in all these cases the exact same relations as those obtained in the case with three equations hold. In other words, if $\beta^2 \underset{\alpha}{\equiv} 1 \underset{\alpha'}{\equiv} \beta'^2$ then with the same argument, just replacing $\mathcal{W}_{ij}$ instead of $\mathcal{Z}_{ij}$ for $i, j \in \{1, 2, 3, 4\}$, it is proven that

$A(\mathcal{W}_{11}) = A(\mathcal{W}_{22}) = A(\mathcal{W}_{33}) = A(\mathcal{W}_{44})$, $A(\mathcal{W}_{12}) = A(\mathcal{W}_{13}) = A(\mathcal{W}_{24}) = A(\mathcal{W}_{34})$,
$A(\mathcal{W}_{21}) = A(\mathcal{W}_{42}) = A(\mathcal{W}_{43}) = A(\mathcal{W}_{31})$, $A(\mathcal{W}_{23}) = A(\mathcal{W}_{32})$, $A(\mathcal{W}_{14})$, $A(\mathcal{W}_{41})$,

describe six different 2-bridge knots. Therefore, if there is a third equation, it means that for the third equation one of the six possible different 2-bridge knots has been chosen, say $A(\mathcal{Z}_{11})$. Hence there are four pairs of solutions $(\mathcal{S}_{ii}, \mathcal{T}_{ii})$ for $i = 1, 2, 3, 4$. Since for all i the knots $A(\mathcal{W}_{ii})$ are equivalent, the conclusion follows. The remaining cases are treated similarly. □

5 Conclusions

The tangle model emerged as a useful tool to explain the mechanism of action of some enzymes on DNA molecules. From the biological assumptions it turns out that, in order to explicitly determine such mechanism, some systems of equations with tangles as unknowns and 2-bridge knots as products need to be solved. In this case, the equations of the system are provided by experimental data. A natural, related question is: May arbitrary systems of equations, of the class under consideration but not necessarily arising from experimental data, be solved as well? In this work, it is assumed that the tangles involved are 3-braids and, under this assumption, it is shown that a system with two equations always admits a solution. It is also shown that, when one attempts to solve a system of three equations, only a few 2-bridge knots may appear on the right-hand side of the third equation for the system to admit a nonempty set of solutions. Finally, if a system of three equations admits a nonempty set of solutions, and of a fourth equation is added, then only one 2-bridge knot may appear on the right-hand side of the fourth equation for the system to be solvable. Another important contribution in this work is that the method presented here to compute complete sets of solutions is both novel and quite simple. In order to apply the method, it suffices to know the knot types of the first two equations and then to calculate the invariant F of the braids involved. It is expected that the method may easily be implemented as an algorithm in a software implementation. Perhaps another fact worth mentioning, which is rather surprising and certainly deserves further study, is the cyclic nature of the 3-braid solutions to the one equation problem, a nature which suggests that the set of solutions is the orbit of one solution under the action of some group on the set of 3-braids.

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References

1. D. Buck and C. Verjovsky Marcotte. *Tangle solutions for a family of DNA-rearranging proteins.* Math. Proc. Cambridge Philos. Soc., 139, (2005), 59-80. doi:10.1017/S0305004105008431.
2. H. Cabrera and D. Lizarraga, *Braid solutions to the action of the Gin enzyme*, J. Knot Theory Ramifications, 19, No. 8 (2010) 1051-1074.
3. H. Cabrera-Ibarra, *On the classification of rational 3-tangles*, J. Knot Theory Ramifications, 12, No. 7 (2003) 921-946.
4. J.H. Conway, *An Enumeration of Knots and Links, and Some of Their Algebraic Properties.* In J. Leech (editor), Computational Problems in Abstract Algebra. Oxford, England. Pergamon Press, 329-358, 1970.
5. I.K. Darcy, J. Luecke and M. Vazquez, *Tangle analysis of difference topology experiments: Applications to a Mu protein-DNA complex*, Algebr. Geom. Topol., 9 (2009) 2247 - 2309. MR 2558311 (2010j:57006)
6. C. Ernst and D.W. Sumners, *A calculus for rational tangles: applications to DNA*, Math. Proc. Cambridge Philos. Soc., 108 (1990), 489-515.
7. M.A. Guevara, *Solución de sistemas de ecuaciones con 3-ovillos del tipo 3-trenzas*, B. Sc. Thesis, Universidad Autónoma de San Luis Potosí, San Luis Potosí, México (2010).
8. R. Kanaar, A. Klippel, E. Shekhtman, J.M. Dungan, R. Kahmann and N.R. Cozzarelli, *Processive recombination by the phage Mu Gin system: Implications for the mechanisms of DNA strand exchange, DNA site alignment, and enhancer action*, Cell 62 (1990) 353 - 366.
9. W.B.R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag Inc., New-York, 1997.
10. K. Murasugi, Knot theory and its applications, Birkähuser, 1996.
11. H. Schubert, *Knoten mit zwei Brücken*, Math. Zeit. 65 (1956) 133-170.
12. D.W. Sumners, C. Ernst, S. Spengler and N. Cozzarelli, *Analysis of the mechanisms of DNA recombinations using tangles*, Q. Rev. Biophys. **28**(1995), 253-313.
13. M. Vazquez and D.W.L. Sumners, *Tangle analysis of Gin site-specific recombination*, Math. Proc. Cambridge Philos. Soc. 136 (3) (2004) 565 - 582.