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Comment on Demystifying the constancy of the Ermakov-Lewis invariant for a time-dependent oscillator

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We show that a simple modification of the Lagrangian proposed by Padmanabhan in the paper [Mod. Phys. Lett. A **33**, 1830005 (2018)] leads to the most general dynamical invariant in [Ray and Reid, Phys. Lett. A **71**, 317 (1979)].

Keywords: parametric oscillator; Ermakov-Lewis invariant; Ray-Reid invariant.

In an interesting paper, Padmanabhan[1] shows that the Lagrangian L_q of a parametric oscillator $q(t)$ with time-dependent mass can be transformed up to a total time derivative into the Lagrangian L_Q of a harmonic oscillator of unit mass and constant frequency Ω, where Ω^2 enters as the coupling constant of the inverse cubic nonlinearity in the associated Ermakov-Milne-Pinney equation of the original oscillator $q(t)$. Using this connection, Padmanabhan concludes that the energy of the conservative oscillator $Q(t)$ is precisely the Ermakov-Lewis invariant of the parametric oscillator $q(t)$. He goes a step further and presents a generalization of the procedure valid for any (anharmonic) potential $V(Q)$, not only for the harmonic one, $V(Q) = (1/2)\Omega^2 Q^2$. In this more general case, the nonlinear coupling constant is a Q-running coupling constant given by $F(Q) = V'(Q)/Q$, while in the harmonic oscillator case $F(Q)$ reduces to a constant. For this more general case, Padmanabhan shows again that the energy corresponds to a case of Ray-Reid invariants.[2]

However, Padmanabhan does not present the most general case of Ray-Reid invariants, as expressed by equation (15) in their paper, which refers to the system

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of nonlinear oscillator equations of the form

$$\ddot{x} + \omega^2(t)x = (1/\rho x^2)g(\rho/x) \ , \tag{1}$$

$$\ddot{\rho} + \omega^2(t)\rho = (1/\rho^2 x)h(x/\rho) \ , \tag{2}$$

where g and h are arbitrary functions.

Our aim here is to fill in this shortcoming. For this, we propose the more general Lagrangian L_Q given by

$$L_Q = \frac{1}{2}Q'^2 - V(Q) - W\left(\frac{1}{Q}\right), \tag{3}$$

where $'$ refers to the time derivative with respect to τ defined next. This Lagrangian has the additional potential term W with respect to that of Padmanabhan. Performing the transformations,[3,4,5,6]

$$q = fQ, \qquad dt = mf^2 d\tau \ , \tag{4}$$

the Lagrangian in the new variables, up to the total derivative $-\frac{1}{2}\frac{d}{dt}(\frac{q^2}{f}m\dot{f})$ which does not change the equation of motion,[7] is

$$\tilde{L}_q = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}\frac{q^2}{f}\frac{d}{dt}(m\dot{f}) - \frac{V\left(\frac{q}{f}\right)}{mf^2} - \frac{W\left(\frac{f}{q}\right)}{mf^2} \ . \tag{5}$$

If we apply the Euler-Lagrange procedure, we obtain the equation of motion

$$\frac{d}{dt}(m\dot{q}) = \frac{q}{f}\frac{d}{dt}(m\dot{f}) - \frac{V'\left(\frac{q}{f}\right)}{mf^3} + \frac{W'\left(\frac{f}{q}\right)}{mq^2 f} \ , \tag{6}$$

where now $'$ represents the derivative with respect to its argument and the point stands for the time derivative with respect to t.

If we now impose the equivalent of the first equation (1) from the Ray-Reid type system

$$\frac{d}{dt}(m\dot{q}) + m\tilde{\omega}^2(t)q = \frac{G\left(\frac{f}{q}\right)}{mq^3} \ , \tag{7}$$

and denote

$$F(Q) = \frac{V'(Q)}{Q}, \qquad G\left(\frac{1}{Q}\right) = W'\left(\frac{1}{Q}\right)Q \ , \tag{8}$$

we obtain the equivalent of the second equation (2) from the Ray-Reid system

$$\frac{d}{dt}(m\dot{f}) + m\tilde{\omega}^2(t)f = \frac{F\left(\frac{q}{f}\right)}{mf^3} \ . \tag{9}$$

This equivalence can be seen for example by using the change of variables $x = q\sqrt{m}$ and $\rho = f\sqrt{m}$ in the latter equation, which becomes

$$\ddot{\rho} + \omega^2(t)\rho = \frac{F\left(\frac{x}{\rho}\right)}{\rho^3},$$

where

$$\omega^2(t) = \frac{1}{4}m^{-2}\dot{m}^2 - \frac{1}{2}m^{-1}\ddot{m} + \tilde{\omega}^2(t) \ .$$

If one chooses $F(x/\rho) = h(x/\rho)/(x/\rho)$, then one obtains (2). The equivalence of (7) and (1) is proved in the same way.

The conserved energy associated to the Lagrangian (3) is then of the same form as the most general invariant proposed by Ray and Reid in the transformed system

$$E = \frac{1}{2}Q'^2 + V(Q) + W\left(\frac{1}{Q}\right) = \frac{1}{2}m^2(\dot{q}f - q\dot{f})^2 + \int \frac{q}{f}Fd\left(\frac{q}{f}\right) + \int \frac{f}{q}Gd\left(\frac{f}{q}\right) . \quad (10)$$

This also shows that energy type invariants can be associated to special pairs of Ermakov-Milne-Pinney type equations with nonlinear running coupling constants relating (an)harmonic and singular oscillators.

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