Traveling wave solutions for wave equations with two exponential nonlinearities

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We use a simple method which leads to the integrals involved in obtaining the traveling wave solutions of wave equations with one and two exponential nonlinearities. When the constant term in the integrand is zero, implicit solutions in terms of hypergeometric functions are obtained while when that term is nonzero all the basic traveling wave solutions of Liouville, Tzitzéica and variants, and sine/sinh-Gordon equations with important applications in the phenomenology of nonlinear physics and dynamical systems are found through a detailed study of the corresponding elliptic equations.

Keywords: Liouville equation, Tzitzéica, Dodd-Bullough, Dodd-Bullough-Mikhailov, sine-Gordon, sinh-Gordon, Weierstrass function

PACS numbers: 02.30.Hq, 04.20.Jb, 02.30.Ik

I. INTRODUCTION

Some of the best known and well-studied hyperbolic nonlinear second-order differential equations are the sine-Gordon equation [1], its variant the sinh-Gordon equation, the Tzitzéica equation [2–4] and its variants, such as the Dodd-Bullough equation [5] and the Dodd-Bullough-Mikhailov equation [6, 7], and last but not least, the Liouville equation [8], a simpler case in this class. Discovered in the realm of differential geometry of surfaces with particular properties of the curvature, like in the sine-Gordon (1862) and Tzitzéica (1907) cases, or during the study of such surfaces as stated by Liouville (1853) in his short note, all of them have been revived much later when it became clear that they have important applications in solid state physics, nonlinear optics, biological physics, and quantum field theory through their soliton type solutions which can describe a variety of dynamical entities. This is especially true for the sine-Gordon equation whose soliton solutions have been identified with dislocations in crystals, fluxons in long Josephson junctions, waves in magnetic materials and superfluids, nonlinear DNA and microtubule excitations, neural impulses, and muscular contractions, among others [9, 10].

In this paper, we will approach all these equations as particular cases of second-order differential equations with two exponential nonlinear terms of the form

$$\psi_{uv} = \alpha e^{a\psi} + \beta e^{b\psi} \ , \tag{1}$$

where a and b are nonzero real constants, while α and β are real constants not simultaneously zero. The main advantage of this approach is to have a unifying treatment of these famous equations which in general are considered separately by the majority of authors, as illustrative examples of their solution methods. In the 1970s, during the remarkable advance in the solution method for nonlinear evolution equations brought by the inverse scattering method, Dodd and Bullough [11] posed and solved the problem of which equations of the form $y_{xt} = f(y)$ admit infinitely many integrals of motion, a property of soliton evolution equations discovered by Zakharov and Shabat in their breakthrough paper of 1972 [12]. Dodd and Bullough showed that beyond the linear case, the only allowed hyperbolic nonlinear equations with this property are precisely of the Liouville, sine/sinh-Gordon, and Tzitzéica form and the variants of the latter. This was a confirmation of the fact that these type of equations have soliton solutions, some of which were already known at that time.

On the other hand, this kind of equations can be turned into polynomial nonlinear equations

$$\frac{\partial^2}{\partial u \partial v} \log h = \alpha h^a + \beta h^b, \tag{2}$$

by using the change of variables $\psi = \log h$.

Along the two characteristics $z = u - \lambda v$, $t = u + \lambda v$, $\lambda \neq 0$, Eq. (1) becomes the nonlinear wave equation

$$\psi_{tt} - \psi_{zz} = \frac{1}{\lambda}\left(\alpha e^{a\psi} + \beta e^{b\psi}\right) \tag{3}$$

or in the logarithmic variable

$$h(h_{tt} - h_{zz}) - ({h_t}^2 - {h_z}^2) = \frac{h^2}{\lambda}\left(\alpha h^a + \beta h^b\right). \tag{4}$$

Furthermore, the usage of the traveling wave ansatz $h(z,t) = h(\xi)$ with $\xi = kz - \omega t$, and $k \neq \pm\omega$ in Eq. (4)

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yields the following ordinary differential equation (ODE)

$$h h_{\xi\xi} - {h_\xi}^2 = \frac{h^2}{\lambda\gamma}\left(\alpha h^a + \beta h^b\right) \equiv f(h), \tag{5}$$

with $\gamma = \omega^2 - k^2 \neq 0$. Once the traveling variable reduction is performed, one cannot avoid to recall that there is a multitude of papers on a variety of effective methods to solve the resulting ODE's based on polynomial ansatze of the solution, in general stemming from the breakthrough tanh-method of Malfliet and Hereman [13–15]. We mention the equivalent G'/G-expansion method [16], the sinh-cosh [17] and tanh-coth methods [18], the Q-function method [19, 20], and the more powerful Jacobi elliptic function method and their extended versions which can be used for more complicated equations such as the double sine-Gordon [21] or to derive doubly periodic wave solutions of a variety of Boussinesq-like equations [18]. In fact, the G'/G-expansion method and the tanh-method have been already applied to Tzitzéica's equation and its variants in [7, 22]. However, for the ODE in (5), we will apply here a simple trick that we used in a previous paper [23] to reduce it to a Bernoulli equation thus allowing us to obtain easily all the basic solutions both in its full generality and simplified to the important cases mentioned above. This reduction is apparently not well known in this context which motivated us to write the present paper.

II. THE IMPLICIT SOLUTION

A simple method to solve ODEs of type (5) is to let $h_\xi = u(h(\xi))$, and use $h_{\xi\xi} = u\frac{du}{dh}$ [23], to obtain

$$hu\frac{du}{dh} - u^2 = f(h), \tag{6}$$

which can be turned into a Bernoulli equation using the substitution $u^2 = z$

$$\frac{dz}{dh} - \frac{2}{h}z = \frac{2}{h}f(h). \tag{7}$$

The solution for this equation is

$$z = h^2\left(c_0 + 2\int \frac{f(h)}{h^3}dh\right) , \tag{8}$$

and using back the transformations $u = \pm\sqrt{z} = \frac{dh}{d\xi}$, then h is obtained by the quadrature

$$\int \frac{dh}{h\sqrt{c_1 + \frac{\alpha}{a}h^a + \frac{\beta}{b}h^b}} = \pm\sqrt{\frac{2}{\lambda\gamma}}\int d\xi . \tag{9}$$

In general, this quadrature can be performed only if the function h satisfies the elliptic equation or if $c_1 = 0$. In the latter case, one obtains the following implicit solution involving the hypergeometric function

$$h^{-b}\sqrt{\frac{\alpha h^a}{a} + \frac{\beta h^b}{b}}\, {}_2F_1\left(1, 1 - \frac{a}{2(a-b)}; 1 - \frac{b}{2(a-b)}; -\frac{b\alpha}{a\beta}h^{a-b}\right) = \mp\frac{\beta}{\sqrt{2\lambda\gamma}}(\xi - \xi_0) , \tag{10}$$

which also implies $\beta \neq 0$. We will show in the next section how the quadrature in (9) is solved in the important particular cases mentioned in the introduction for $c_1 \neq 0$.

III. PARTICULAR CASES

A. Liouville equation

We start with the simplest case, which is the Liouville equation, 'the widely known example of an exactly integrable non-linear partial differential equation' [24] in mathematical physics. For example, in cosmology, the inflationary expansion epoch of the early universe is usually generated by means of one or several scalar fields with an exponential potential and are dubbed Liouville cosmologies [25].

The Liouville equation corresponds to $\alpha = 1,\ \beta = 0,\ a = 1,\ b = 0$, which is

$$\frac{\partial^2}{\partial u \partial v}\log h = h . \tag{11}$$

The quadrature in Eq. (9) takes the form

$$\int \frac{dh}{h\sqrt{c_1 + h}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{12}$$

and the function h is obtained by solving the elliptic equation

$${h_\xi}^2 = a_3 h^3 + a_2 h^2 + a_1 h + a_0 \tag{13}$$

with the coefficients given by the system

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= \tfrac{2c_1}{\lambda\gamma} \\ a_3 &= \tfrac{2}{\lambda\gamma} . \end{aligned} \tag{14}$$

For convenience, denote $r = \frac{1}{\lambda\gamma}$, and $p = \frac{2c_1}{\lambda\gamma}$; then Eq. (13) becomes the reduced elliptic equation

$$h_\xi^2 = 2rh^3 + ph^2 , \tag{15}$$

with soliton solution if $p > 0$, or periodic solution if $p < 0$

$$\begin{aligned} h(\xi) &= -\tfrac{p}{2r}\operatorname{sech}^2\left[\tfrac{1}{2}\sqrt{p}(\xi - \xi_0)\right] , \quad p > 0 \\ h(\xi) &= -\tfrac{p}{2r}\sec^2\left[\tfrac{1}{2}\sqrt{-p}(\xi - \xi_0)\right] , \quad p < 0 , \end{aligned} \tag{16}$$

which by using system (14) give the solutions

$$\begin{aligned} h(\xi) &= -c_1\operatorname{sech}^2\left(\sqrt{\tfrac{c_1}{2\lambda\gamma}}(\xi - \xi_0)\right) , \quad \tfrac{c_1}{2\lambda\gamma} > 0 \\ h(\xi) &= -c_1\sec^2\left(\sqrt{\tfrac{-c_1}{2\lambda\gamma}}(\xi - \xi_0)\right) , \quad \tfrac{c_1}{2\lambda\gamma} < 0. \end{aligned} \tag{17}$$

Since we are in the case $\beta = 0$, we cannot apply (10) when $c_1 = 0$. However, the integration of (12) is easily performed and corresponds to the most degenerate case of the Weierstrass elliptic equation (26) when both germs g_2 and g_3 are zero, which leads to the rational solution

$$h(\xi) = \frac{2\lambda\gamma}{(\xi-\xi_0)^2} \ . \tag{18}$$

Plots of all these three types of Liouville solutions are given in Fig. (1). The only nonsingular solution is the soliton one which is the usual solution employed in the literature.

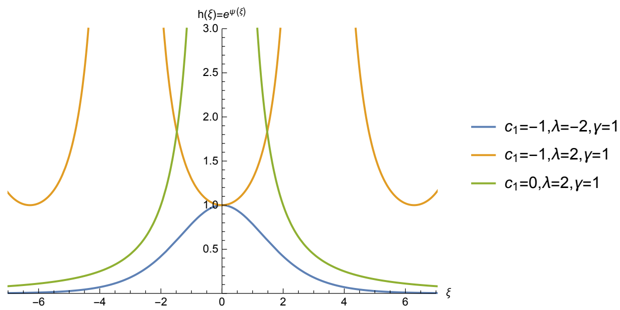

Figure 1: The soliton and periodic solutions (17) and the rational solution (18) of the Liouville equation.

B. The Tzitzéica equation

Tzitzéica's equation,

$$\frac{\partial^2}{\partial u \partial v}\log h = h - \frac{1}{h^2} \ , \tag{19}$$

emerged in 1907-1910 in the area of geometry, but only after eighty years it has been found to have applications in physics. For example, Euler's equations for an ideal gas with a special equation of motion can be reduced to the Tzitzéica equation, and a 2+1-dimensional system in magneto-hydrodynamics has been shown to be in one-to-one correspondence with it [26, 27]. Very recently, dark optical solitons and traveling waves of Tzitzéica type have been also discussed in the literature [28, 29].

For Tzitzéica's equation, we identify the constants in (1) as $\alpha = 1,\ \beta = -1,\ a = 1,\ b = -2$, which gives the quadrature

$$\int \frac{dh}{\sqrt{h^3 + c_1 h^2 + \frac{1}{2}}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{20}$$

For $c_1 = 0$, the quadrature reads:

$$\int \frac{dh}{\sqrt{h^3 + \frac{1}{2}}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{21}$$

with solution given by the equianharmonic function of case (2) i) below, while implicitly, the equianharmonic h satisfies Eq. (10) and simplifies to

$$h\, {}_2F_1\left(\frac{1}{2},\frac{1}{3};\frac{4}{3};-2h^3\right) = \pm\frac{\xi-\xi_0}{\sqrt{\lambda\gamma}} \ . \tag{22}$$

The solution h is obtained explicitly by solving the elliptic equation (13) with coefficients given by the system

$$\begin{aligned} a_0 &= r \\ a_1 &= 0 \\ a_2 &= 2c_1 r \\ a_3 &= 2r \ . \end{aligned} \tag{23}$$

which becomes

$$h_\xi{}^2 = 2rh^3 + ph^2 + r \ . \tag{24}$$

Using the scale shift transformation

$$h(\xi) = \frac{1}{r}\left(2\wp(\xi; g_2, g_3) - \frac{p}{6}\right), \tag{25}$$

Eq. (24) becomes the Weierstrass equation

$$\wp_\xi{}^2 = 4\wp^3 - g_2\wp - g_3 \ . \tag{26}$$

The germs of the Weierstrass function are given by

$$\begin{aligned} g_2 &= \tfrac{a_2^2 - 3a_1a_3}{12} = \tfrac{p^2}{12} = 2(e_1^2+e_2^2+e_3^2) \\ g_3 &= \tfrac{9a_1a_2a_3 - 27a_0a_3^2 - 2a_2^3}{432} = -\tfrac{1}{4}\left(r^3 + \tfrac{p^3}{54}\right) = 4e_1e_2e_3 \end{aligned} \tag{27}$$

and together with the modular discriminant

$$\begin{aligned} \Delta &= g_2{}^3 - 27g_3{}^2 = -\frac{r^3}{16}(p^3 + 27r^3) \\ &= 16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2 \end{aligned} \tag{28}$$

are used to classify the solutions of Eq. (24), where the constants e_i are the roots of the cubic polynomial

$$s_3(t) = 4t^3 - g_2t - g_3 = 4(t-e_1)(t-e_2)(t-e_3) = 0. \tag{29}$$

Case (1). If $\Delta \equiv 0 \Rightarrow p = -3r \Rightarrow c_1 = -\frac{3}{2}$. This degenerate case implies that $s_3(t)$ has repeated root of multiplicity two. Then the Weierstrass solutions can be simplified since $\wp$ degenerates into *hyperbolic* or *trigonometric* functions. Because of the degeneracy, Eq. (20) can be factored as

$$\int \frac{dh}{\sqrt{(h-1)^2\left(h+\frac{1}{2}\right)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{30}$$

Depending on the sign of g_3 we have the sub-cases:

Case (1a). $r > 0$ with $g_2 > 0 \Rightarrow g_3 = -\frac{r^3}{8} < 0 \Rightarrow \lambda\gamma > 0$, so Eq. (30) has the dark soliton solution

$$h(\xi) = 1 - \frac{3}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right) \ . \tag{31}$$

By letting $e_1 = e_2 = \hat{e} > 0$ then $e_3 = -2\hat{e} < 0$, hence

$$\begin{aligned} g_2 &= 12\hat{e}^2 > 0 \\ g_3 &= -8\hat{e}^3 < 0 \end{aligned} \tag{32}$$

the Weierstrass $\wp$ solution to Eq. (26) reduces to

$$\wp(\xi; 12\hat{e}^2, -8\hat{e}^3) = \hat{e} + 3\hat{e}\ \mathrm{csch}^2(\sqrt{3\hat{e}}\xi)\ . \tag{33}$$

For $\hat{e} = \frac{1}{4\lambda\gamma} > 0$, the Weierstrass solution replaced by (33) gives the singular (blow-up) soliton

$$h(\xi) = 1 + \frac{3}{2}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right). \tag{34}$$

Case (1b). $r < 0$ with $g_2 > 0 \Rightarrow g_3 = -\frac{r^3}{8} > 0 \Rightarrow \lambda\gamma < 0$, so Eq. (30) has the following solution with periodic negative singularities

$$h(\xi) = 1 - \frac{3}{2}\sec^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right) \tag{35}$$

By letting $e_2 = e_3 = -\tilde{e} < 0$ with $\tilde{e} > 0$, then $e_1 = 2\tilde{e} > 0$, hence

$$\begin{aligned} g_2 &= 12\tilde{e}^2 > 0 \\ g_3 &= 8\tilde{e}^3 > 0 \end{aligned} \tag{36}$$

the Weierstrass $\wp$ solution reduces to

$$\wp(\xi; 12\tilde{e}^2, 8\tilde{e}^3) = -\tilde{e} + 3\tilde{e}\ \csc^2(\sqrt{3\tilde{e}}\xi). \tag{37}$$

For $\tilde{e} = -\frac{1}{4\lambda\gamma} > 0$, the Weierstrass solution replaced by (37) gives

$$h(\xi) = 1 - \frac{3}{2}\csc^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right)\ , \tag{38}$$

which is similar to (35). All the Tzitzéica solutions corresponding to these cases are displayed in Fig. (2). The dark soliton solution may have physical applications, especially in optics and hydrodynamics, while the other singular solutions look unphysical for the time being. However, we notice that there are already detailed mathematical studies of the blow up problem in the Tzitzéica case [30].

Case (2). If $\Delta \neq 0 \Rightarrow p \neq -3r \Rightarrow c_1 \neq -\frac{3}{2}$ we include two particular solutions which will fix the integration constant c_1 as follows: the equianharmonic ($g_2 = 0$) and lemniscatic case ($g_3 = 0$), respectively.

i) For the equianharmonic case $g_2 = 0 \Rightarrow p = 0 \Rightarrow g_3 = -\frac{r^3}{4}$. Because $\Delta = -\frac{27}{16}r^6 < 0$, then $s_3(t)$ has

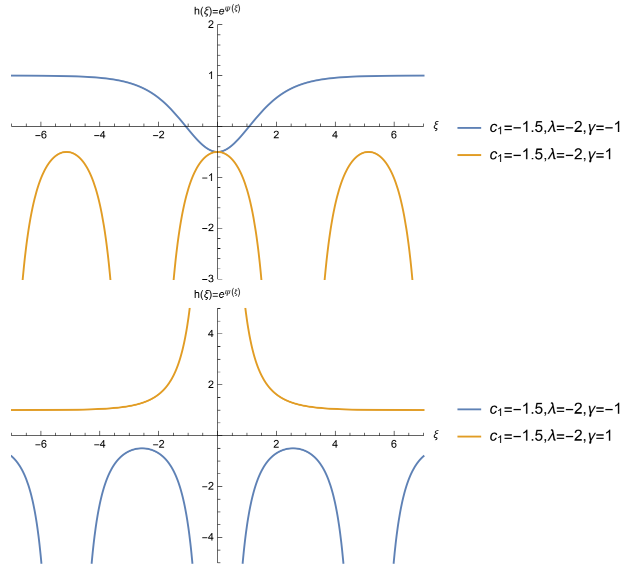

Figure 2: The Tzitzéica dark soliton (31) and the periodic singular solution (35) (top). The Tzitzéica solution (34), which can be also called Tzitzéica's singular soliton, and the periodic singular solution (38) (bottom).

a pair of conjugate complex roots, and since $c_1 = 0$ the solution to Eq. (20) reduces to

$$h(\xi) = 2\lambda\gamma\wp\left(\xi - \xi_0; 0, -\frac{1}{4\lambda^3\gamma^3}\right). \tag{39}$$

ii) For the lemniscatic case $g_3 = 0 \Rightarrow p = -3\sqrt[3]{2}r \Rightarrow g_2 = \frac{3\sqrt[3]{4}}{4}r^2$. Because $\Delta = \frac{27}{16}r^6 > 0$, then $s_3(t)$ has three distinct real roots given by $e_3 = -\frac{\sqrt{g_2}}{2}$, $e_2 = 0$, and $e_1 = \frac{\sqrt{g_2}}{2}$. Although the Weierstrass unbounded function has poles aligned on the real axis of the $\xi - \xi_0$ complex plane, we can choose ξ_0 in such a way to shift these poles a half of period above the real axis, so that the elliptic function simplifies using the formula [31]

$$\wp(\xi; g_2, 0) = e_3 + (e_2 - e_3)\mathrm{sn}^2\left(\sqrt{e_1 - e_3}(\xi - \xi_0'); m\right) \tag{40}$$

with elliptic modulus $m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$. Using the values of the roots together with $\xi_0' = 0$ we obtain

$$\wp(\xi; g_2, 0) = -\frac{\sqrt{g_2}}{2}\mathrm{cn}^2\left(\sqrt[4]{g_2}\xi; \frac{\sqrt{2}}{2}\right). \tag{41}$$

Because $c_1 = -\frac{3}{\sqrt[3]{4}}$ the solutions for the lemniscatic case are reduced using the transformation (25) to periodic cnoidal waves, and they become

$$h(\xi) = \frac{1}{\sqrt[3]{4}}\left[1 - \sqrt{3}\ \mathrm{cn}^2\left(\frac{\sqrt[4]{3}}{\sqrt[3]{2}\sqrt{\lambda\gamma}}\xi; \frac{\sqrt{2}}{2}\right)\right]\ . \tag{42}$$

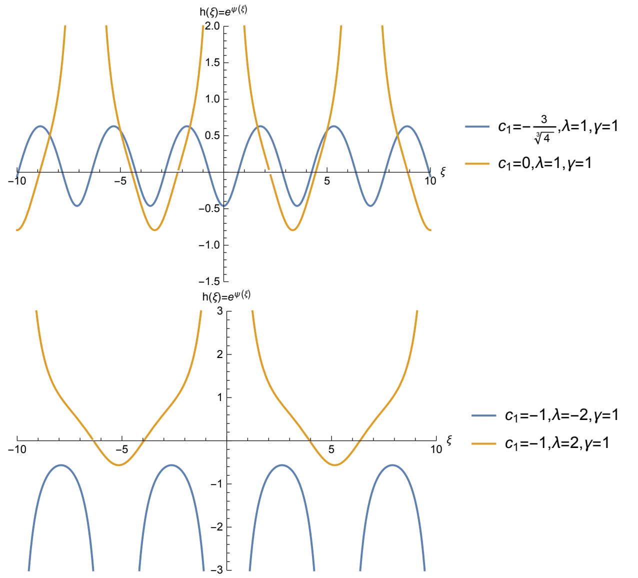

Figure 3: The lemniscatic (cnoidal) and equianharmonic solutions, (42) and (39), respectively, of the Tzitzéica equation (top). The Weierstrass solution (43) of the Tzitzéica equation (bottom).

iii) For the most general case, $g_2 \neq 0$, $g_3 \neq 0$ the general solution to Eq. (20) is

$$h(\xi) = \lambda\gamma \left[2\wp\left(\xi - \xi_0; \frac{{c_1}^2}{3\lambda^2\gamma^2}, -\frac{4{c_1}^3 + 27}{108\lambda^3\gamma^3} \right) - \frac{c_1}{3\lambda\gamma} \right]. \tag{43}$$

The equianharmonic, lemniscatic, and Weierstrass solutions of Tzitzéica's equation are displayed in Fig. (3). We notice that the only regular solutions are the periodic cnoidal ones corresponding to the lemniscatic case. All the other solutions have periodic positive or negative blow ups and are interesting rather from the mathematical standpoint [30] than for applications to the physical phenomenology.

C. The Dodd-Bullough equation

This variant of Tzitzéica's equation has been introduced in the first dedicated study of the polynomial conserved quantities of the sine-Gordon equation [5]. Its form in the h variable is

$$\frac{\partial^2}{\partial u \partial v} \log h = -h + \frac{1}{h^2} \ , \tag{44}$$

thus, we identify the constants to be $\alpha = -1, \ \beta = 1, \ a = 1, \ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{-h^3 + c_1 h^2 - \frac{1}{2}}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{45}$$

In implicit form, the h function satisfies Eq. (10) which simplifies to

$$h\, {}_2F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -2h^3\right) = \mp\frac{\xi - \xi_0}{\sqrt{-\lambda\gamma}} \tag{46}$$

and up to a sign, the implicit solution is the same as the Tzitzeica solution.

Using Tzitzéica solutions, provided that $r \to -r \Rightarrow \lambda\gamma \to -\lambda\gamma$ and $c_1 \to -c_1$, we have:

Case (1). If $\Delta \equiv 0 \Rightarrow c_1 = \frac{3}{2}$, then

Case (1a). $\lambda\gamma < 0$ so Eq. (45) has soliton solution

$$h(\xi) = 1 - \frac{3}{2}\text{sech}^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right) \ . \tag{47}$$

while the solution corresponding to (34) is

$$h(\xi) = 1 + \frac{3}{2}\text{csch}^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right). \tag{48}$$

Case (1b). $\lambda\gamma > 0$ so Eq. (45) has periodic solution

$$h(\xi) = 1 - \frac{3}{2}\sec^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right) \tag{49}$$

and the periodic solution corresponding to (38) is

$$h(\xi) = 1 - \frac{3}{2}\csc^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right). \tag{50}$$

Case (2). If $\Delta \neq 0 \Rightarrow c_1 \neq \frac{3}{2}$ then the Weierstrass solution of Eq. (45) is

$$h(\xi) = -\lambda\gamma \left[2\wp\left(\xi - \xi_0; \frac{{c_1}^2}{3\lambda^2\gamma^2}, -\frac{4{c_1}^3 - 27}{108\lambda^3\gamma^3} \right) - \frac{c_1}{3\lambda\gamma} \right]. \tag{51}$$

i) For the equianharmonic case, $c_1 = 0$, we have:

$$h(\xi) = -2\lambda\gamma\wp\left(\xi - \xi_0; 0, \frac{1}{4\lambda^3\gamma^3}\right). \tag{52}$$

ii) For the lemniscatic case, $c_1 = \frac{3}{\sqrt[3]{4}}$, we have:

$$h(\xi) = \frac{1}{\sqrt[3]{4}}\left[1 - \sqrt{3}\ \text{cn}^2 \quad \frac{\sqrt[4]{3}}{\sqrt[3]{2}\sqrt{-\lambda\gamma}}\xi; \frac{\sqrt{2}}{2}\right)\right]. \tag{53}$$

Respecting the rules of changing the signs of the parameters, the Dodd-Bullough solutions are identical to the Tzitzéica solutions and consequently we will not plot them here.

D. The Tzitzéica-Dodd-Bullough equation

This variant equation reads

$$\frac{\partial^2}{\partial u \partial v} \log h = h + \frac{1}{h^2}, \tag{54}$$

thus, we identify the constants to be $\alpha = 1,\ \beta = 1,\ a = 1,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{h^3 + c_1 h^2 - \frac{1}{2}}} = \mp\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0)\ . \tag{55}$$

We can use the Dodd-Bullough solutions, Eqs. (47)-(51,) with $h \to -h$ and $\xi \to -\xi$.

E. The Dodd-Bullough-Mikhailov equation

The last Tzitzéica variant equation reads

$$\frac{\partial^2}{\partial u \partial v} \log h = -h - \frac{1}{h^2}, \tag{56}$$

thus the constants are identified as $\alpha = -1,\ \beta = -1,\ a = 1,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{-h^3 + c_1 h^2 + \frac{1}{2}}} = \mp\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0)\ . \tag{57}$$

From the polynomial in the integrand, one can see that the solutions of this variant equation are obtained from Tzitzéica solutions by $h \to -h$ and $\xi \to -\xi$.

F. The sine-Gordon equation

According to [32], Tzitzéica's equation is the "nearest relative" of the well-known sine-Gordon equation which can be written as

$$\frac{\partial^2}{\partial u \partial v} \log h = \frac{1}{2i}\left(h^i - h^{-i}\right) = \sin(\log h)\ . \tag{58}$$

Thus, we identify the constants to be $\alpha = \frac{1}{2i},\ \beta = -\frac{1}{2i},\ a = i,\ b = -i$ which gives the quadrature

$$\int \frac{dh}{h\sqrt{c_1 - \cos(\log h)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{59}$$

that is equivalent to

$$\int \frac{d\psi}{\sqrt{c_1 - \cos(\psi)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{60}$$

In implicit form, h satisfies Eq. (10) which simplifies to

$$h^i\sqrt{\cos(\log h)}\ {}_2F_1\left(1, \frac{3}{4}; \frac{5}{4}; -h^{2i}\right) = \mp\frac{\xi - \xi_0}{2\sqrt{2\lambda\gamma}}. \tag{61}$$

The explicit solution satisfies Eq. (60), with $c_1 = 0$, which gives

$$\int \frac{d\psi}{\sqrt{\cos(\psi)}} = \mp\sqrt{\frac{2}{-\lambda\gamma}}(\xi - \xi_0) \tag{62}$$

Using the definition of the elliptic integral of the first kind $\xi = F(\phi; m) = \int_0^\phi \frac{d\psi}{\sqrt{1 - m\sin^2\psi}}$ [33, 34], then (62) becomes

$$F\left(\frac{\psi}{2}; 2\right) = \mp\frac{1}{2}\sqrt{\frac{2}{-\lambda\gamma}}(\xi - \xi_0). \tag{63}$$

By inverting, the explicit solution is

$$\psi(\xi) = 2\ \mathrm{am}\left(\mp\frac{1}{2}\sqrt{\frac{2}{-\lambda\gamma}}(\xi - \xi_0); 2\right)\ , \tag{64}$$

where the function $\mathrm{am}(\xi; m)$ is the Jacobi Amplitude function, which can also be obtained from (70) for $c_1 = 0$.

For the special case of $c_1 = 1$, Eq. (60) simplifies to

$$\int \frac{d\psi}{\sin\frac{\psi}{2}} = \pm\frac{2}{\sqrt{\lambda\gamma}}(\xi - \xi_0). \tag{65}$$

for $\lambda\gamma > 0$, and using the identity $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$ we obtain the kink-antikink solutions

$$\psi(\xi) = 4\arctan\left(e^{\pm\frac{\xi - \xi_0}{\sqrt{\lambda\gamma}}}\right). \tag{66}$$

When $c_1 = -1$, Eq. (60) simplifies to

$$\int \frac{d\psi}{\cos\frac{\psi}{2}} = \pm\frac{2}{\sqrt{-\lambda\gamma}}(\xi - \xi_0) \tag{67}$$

for $\lambda\gamma < 0$, and using the identity $\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{1 - \cos\theta}{\sin\theta}$ the shifted kink-antikink solution is obtained

$$\psi(\xi) = -\pi + 4\arctan\left(e^{\pm\frac{\xi - \xi_0}{\sqrt{-\lambda\gamma}}}\right)\ . \tag{68}$$

Plots of the $c_1 = \pm 1$ arctan solutions (66) and (68) are displayed in Fig. (4). As well known, these kink solutions are overwhelmingly encountered in applications [9].

When $c_1 \neq \pm 1$, then ψ satisfies the elliptic equation

$${\psi_\xi}^2 = \frac{2}{\lambda\gamma}(c_1 - \cos\psi), \tag{69}$$

with solution given by

$$\psi(\xi) = 2\ \mathrm{am}\left(\mp\sqrt{\frac{c_1 - 1}{2\lambda\gamma}}(\xi - \xi_0); \frac{2}{1 - c_1}\right)\ . \tag{70}$$

For plots of the sine-Gordon Jacobi amplitude solutions, see Fig. (5). Only the Jacobi amplitude solutions of superunitary modulus $|m|$ are periodic and bounded, and are physically acceptable.

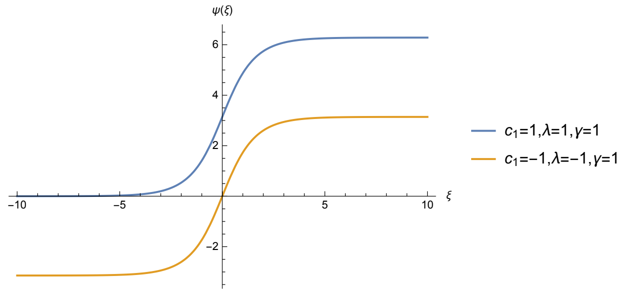

Figure 4: The arctan kink solutions (66) and (68) of the sine-Gordon equation.

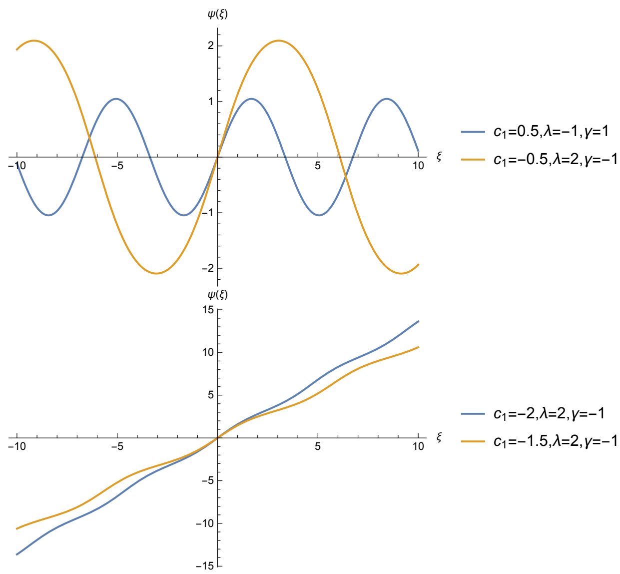

Figure 5: The Jacobi amplitude solution (70) of the sine-Gordon equation for $|m| > 1$ (top) and $|m| < 1$ (bottom).

G. The sinh-Gordon equation

The hyperbolic version of the sine-Gordon equation, the sinh-Gordon equation, is also extensively used in integrable quantum field theory [35, 36], kink dynamics [37], and hydrodynamics [38]. Here, we will use the parametrization in [35] and write the corresponding equation in the variable h in the form

$$\frac{\partial^2}{\partial u \partial v} \log h = \frac{1}{2}\left(h^2 - h^{-2}\right) = \sinh(2\log h)\ . \tag{71}$$

Thus, we identify the constants to be $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $a = 2$, $b = -2$ which gives the quadrature

$$\int \frac{dh}{h\sqrt{c_1 + \frac{1}{2}\cosh(2\log h)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{72}$$

which is equivalent to

$$\int \frac{d\psi}{\sqrt{c_1 + \frac{1}{2}\cosh(2\psi)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{73}$$

In implicit form, h satisfies Eq. (10) which simplifies to

$$h\, {}_2F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}; -h^4\right) = \pm\frac{\xi - \xi_0}{\sqrt{2\lambda\gamma}}. \tag{74}$$

By a simple transformation, the explicit solution for $c_1 = 0$ is obtained by solving the integral

$$\int \frac{d\psi}{\sqrt{\cosh(2\psi)}} = \pm\sqrt{\frac{1}{\lambda\gamma}}(\xi - \xi_0), \tag{75}$$

and using (62) and (76) we obtain

$$\psi(\xi) = i\ \mathrm{am}\left(\pm\sqrt{\frac{1}{-\lambda\gamma}}(\xi - \xi_0); 2\right), \tag{76}$$

which can also be obtained from (82) for $c_1 = 0$.

For the special case of $c_1 = -\frac{1}{2}$, Eq. (75) simplifies to

$$\int \frac{d\psi}{\sinh\psi} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{77}$$

with the kink-antikink solutions

$$\psi(\xi) = 2\ \mathrm{arctanh}\left(e^{\pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0)}\right). \tag{78}$$

When $c_1 = \frac{1}{2}$, Eq. (75) simplifies to

$$\int \frac{d\psi}{\cosh\psi} = \mathrm{gd}(\psi) = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{79}$$

where the Gudermannian function $\mathrm{gd}(\psi) = 2\ \arctan\left(\tanh\frac{\psi}{2}\right)$ gives also the kink-antikink solutions

$$\psi(\xi) = 2\ \mathrm{arctanh}\left[\tan\left(\pm\frac{1}{\sqrt{2\lambda\gamma}}(\xi - \xi_0)\right)\right]. \tag{80}$$

The arctanh solutions are plotted in Fig. (6). Being singular, they have only theoretical interest from the point of view of blow-up analysis.

When $c_1 \neq \pm\frac{1}{2}$, then ψ satisfies the elliptic equation

$${\psi_\xi}^2 = \frac{2}{\lambda\gamma}\left(c_1 + \frac{1}{2}\cosh(2\psi)\right), \tag{81}$$

with solutions given by the Jacobi amplitude function

$$\psi(\xi) = i\ \mathrm{am}\left(\pm\sqrt{\frac{2c_1 + 1}{-\lambda\gamma}}(\xi - \xi_0); \frac{2}{2c_1 + 1}\right). \tag{82}$$

Plots of (82) are given in Fig. (7). Similarly to the sine-Gordon case, only the Jacobi amplitude solutions of superunitary modulus $|m|$ are bounded periodic functions.

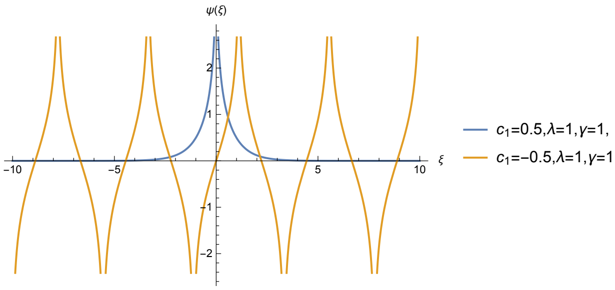

Figure 6: The arctanh solutions (78) and (80) of the sinh-Gordon equation.

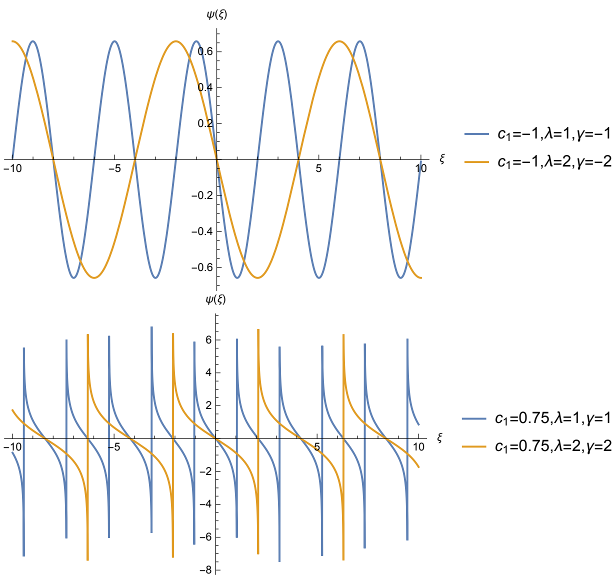

Figure 7: The Jacobi amplitude solution (82) of the sinh-Gordon equation for $|m| > 1$ (top) and $|m| < 1$ (bottom).

IV. CONCLUSION

In summary, we have used a very simple method to obtain all the basic soliton, periodic and Weierstrass solutions of wave equations with two exponential nonlinearities whose particular cases correspond to celebrated equations in mathematical physics, such as Liouville, Tzitzéica and its variants, sine-Gordon, and sinh-Gordon equations. All these solutions are obtained consistently in the traveling variable by a thorough analysis of the elliptic equation. Novel implicit solutions in terms of a generic hypergeometric function are also obtained through a direct integration. Although there are other methods to obtain these translation-invariant solutions, e.g., the integral bifurcation method [39], some of these solutions, in particular the Weierstrass solutions of the Tzitzéica class of equations and the amplitude Jacobi solutions of the sine/sinh-Gordon equations cannot be obtained by the tanh method usually employed in the literature. Consequently, with a few exceptions in the case of the amplitude Jacobi solutions [33, 34], their potential for realistic physical applications has been ignored in the past. As for the the Weierstrass solutions of the Tzitzéica class of equations, their potential in the area of optical solitons is still to be assessed [28, 29].

We also plan a future publication with the aim to extend the unifying approach presented in this paper to the hyperelliptic cases [40].

Finally, for more complicated multiple-soliton (multiphase soliton) solutions, one should use Darboux and Bäcklund transformations [32, 41, 42].

Acknowledgements

SCM would like to acknowledge partial support from the Dean of Research & Graduate Studies at Embry-Riddle Aeronautical University while on a short visit to IPICyT, San Luis Potosí, Mexico. MPM thanks CONACyT for a doctoral fellowship.

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