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A class of Piecewise Linear Systems without equilibria with 3-D grid multiscroll chaotic attractors

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***Abstract*—In this paper a new class of piecewise linear (PWL) dynamical system without equilibria which exhibits a three dimensional (3D) grid multiscroll chaotic attractor is presented. The number of scrolls of the attractor generated can be easily changed by the number of linear parts of three piecewise constant functions. A particular system with a 3D grid multiscroll attractor whose scrolls appear in an arrangement of $3 \times 3 \times 3$ is taken as a case study. Moreover, an electronic circuit realization is proposed for the particular system and simulation data as well as experimental data is provided.**

***Index Terms*—dynamical systems; systems without equilibrium; linear systems.**

I. INTRODUCTION

THE study of dynamical systems along the history has been constantly focused on the behavior of the system around the equilibria. There is a special interest in the chaotic behavior exhibited by some systems. Since several of the mathematical tools and theory used for analyzing and classifying dynamical systems consider the existence of at least an equilibrium point, they cannot be applied to the study of systems without equilibria which presents an interesting research area. For instance, the Hartman-Grobman theorem is a very important result in the local qualitative theory of ordinary differential equations and assumes an equilibrium point. The theorem states that the behavior of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point, where hyperbolicity means that no eigenvalue of the linearization has real part equal to zero.

One of the first dynamical systems with an oscillating behavior but no equilibria was described by Arnold Sommerfeld in 1902 [1] while the system sprott case A (1994) is the first reported chaotic system without equilibria [2]. This last one is a particular case of the Nose-Hoover system [3]. According to [4] systems without equilibria can be considered as hidden attractors since the basin of attraction does not intersect with small neighborhoods of equilibria. More recently, three-dimensional systems with chaotic attractor and no equilibria have been reported in [5]–[8]. Also, four-dimensional systems with chaotic or hyperchaotic attractors have been reported in [9]–[12].

PWL systems with chaotic attractors have been reported. One of the most studied PWL systems is the Chua's circuit whose attractor exhibit a double scroll. The idea of increasing the number of scrolls has been studied in [13]–[18].

A PWL system is defined by using a partition $\{D_1, \ldots, D_m\}$ of the phase space $\mathbb{R}^n$ that has an associated vector field of the form:

$$\dot{x} = A_i x + B_i, \text{ if } x \in D_i, \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$ is the state vector of the system, B_i are constant vectors that one of them could be zero, the domains D_i, with $i = 1, 2, \ldots, m$, of the partition fulfill $\bigcup_{i=1}^m D_i = \mathbb{R}^n$ and $\bigcap_{i=1}^m D_i = \emptyset$.

A PWL system has no equilibria in all $\mathbb{R}^n$ when each subsystem presents one of the next two cases. The former is when $x \in D_i$ and for each equilibrium point x^* of the linear affine vector field $\dot{x} = A_i x + B_i$, $x^* \in D_j$ with $i \neq j$. The latter is when the linear affine vector field $\dot{x} = A_i x + B_i$ has no equilibria in all $\mathbb{R}^n$ due to all A_i are singular.

Even tough there are reported multiscroll hidden attractors as those in [19] and [20], or multiscroll chaotic sea [21], we did not find any reported method to design a three dimensional grid attractor with a three-dimensional system without equilibria. Thus, a class of PWL systems that exhibits any desired number of scrolls in each direction of the 3D-grid is proposed.

The approach uses a singular matrix A, however, the way how the functions are defined allows the use of non singular matrix whose eigenvalues have positive real parts.

In section II a new class of dynamical system without equilibria whose attractor presents 1D, 2D, 3D-grid multiscroll attractor is introduced. A particular system with a 3D grid multiscroll attractor of twenty seven scrolls in an arrangement of $3 \times 3 \times 3$ is taken as a case study in section III-A. In section IV a possible circuit realization for the case study is proposed and electronic simulation data along with experimental results are shown.

II. NEW PWL SYSTEM CLASS

Consider a dynamical system given by (1) in $\mathbb{R}^3$ with linear operators $A_i = A$ given as follows:

$$A = \begin{bmatrix} \frac{a+c}{2} & -b & \frac{c-a}{2} \\ \frac{b}{2} & a & \frac{-b}{2} \\ \frac{c-a}{2} & b & \frac{a+c}{2} \end{bmatrix}, \tag{2}$$

where $a, b \in \mathbb{R} - \{0\}$ and $c \in \mathbb{R}$. The eigenvalues are $\lambda_1 = c, \lambda_{2,3} = a \pm ib$. According to the Jordan canonical

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form theorem, that states that a real matrix A can be reduced to its Jordan canonical form J, *i.e.*, $J = P^{-1}AP$, P is given by a basis of generalized eigenvectors $\{p_1, p_2, p_3\}$. Thereupon, matrices J and P are given as follows:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}, \; J = \begin{bmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}, \tag{3}$$

So, the dynamical system has an associated vector field of the form:

$$\dot{x} = PJP^{-1}x + B(x). \tag{4}$$

We define $PJP^{-1} = A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ and $B(x)$ is defined as follows:

$$\begin{aligned} B(x) = \; & -f_1(x)a_1 - f_2(x, f_1)a_2 - f_3(x)a_3 + \\ & f_4(x, f_1, f_3)p_1 - f_5(x, f_1, f_3)(a_1 - a_3), \end{aligned} \tag{5}$$

where $f_1, \ldots, f_5$ are piecewise constant functions given by

$$f_1(x) = \begin{cases} 0, & \text{if} \quad x \leq S_{11}; \\ \Delta_{x_1}, & \text{if} \quad S_{11} < x \leq S_{12}; \\ \vdots & \\ p\Delta_{x_1}, & \text{if} \quad S_{1p} < x; \end{cases} \tag{6}$$

where $\Delta_{x_1} \in \mathbb{R}_{>0}$, S_{1i} for $i = 1, \ldots, p$ are the switching surfaces given by $S_{1i} = \left\{ x \in \mathbb{R}^3 | x_1 = \frac{(2i-1)\Delta_{x_1}}{2} \right\}$. We use the notation $x > S_{1i}$ if x is in the set pointed by the vector $[1, 0, 0]^T$, $x \leq S_{ix}$ if x is in the opposite set or on the plane.

$$f_2(x, f_1) = \begin{cases} 0, & \text{if} \quad x \leq S_{21}; \\ \Delta_{x_2}, & \text{if} \quad S_{22} < x \leq S_{22}; \\ \vdots, & \\ q\Delta_{x_2}, & \text{if} \quad x > S_{2q}. \end{cases} \tag{7}$$

where $\Delta_{x_2} \in \mathbb{R}_{>0}$, S_{2i} for $i = 1, \ldots, r$ are the switching surfaces given by $S_{2i} = \left\{ x \in \mathbb{R}^3 | x_2 = \frac{(2i-1)\Delta_{x_3}}{2} - kf_1(x) \right\}$ where $k \in \mathbb{R}$. We use the notation $x > S_{2i}$ if x is in the set pointed by the vector $[0, 1, 0]^T$, $x \leq S_{ix}$ if x is in the opposite set or on the plane.

$$f_3(x) = \begin{cases} 0, & \text{if} \quad x \leq S_{31}; \\ \Delta_{x_3}, & \text{if} \quad S_{31} < x \leq S_{32}; \\ \vdots & \\ r\Delta_{x_3}, & \text{if} \quad x > S_{3r}; \end{cases} \tag{8}$$

where $\Delta_{x_3} \in \mathbb{R}_{>0}$, S_{3i} for $i = 1, \ldots, r$ are the switching surfaces given by $S_{3i} = \left\{ x \in \mathbb{R}^3 | x_3 = \frac{(2i-1)\Delta_{x_3}}{2} \right\}$. We use the notation $x > S_{3i}$ if x is in the set pointed by the vector $[0, 0, 1]^T$, $x \leq S_{ix}$ if x is in the opposite set or on the plane. These three piecewise constant functions f_1, f_2, f_3 generate a partition of the phase space $\mathbb{R}^3$ where the PWL system (4) under an appropriate selection of parameters can display 1D, 2D or 3D-grid scroll attractor of $p + 1 \times q + 1 \times r + 1$. Now we need to define a function that assures the location of the scrolls no mater the value of λ_1, as follows:

$$f_4(x, f_1, f_3) = \begin{cases} v, & \text{if} \quad x < S_4; \\ 0, & \text{if} \quad x = S_4; \\ -v, & \text{if} \quad x > S_4; \end{cases} \tag{9}$$

where $v \in \mathbb{R}_{>0}$ and the switching plane $S_4 = \{x \in \mathbb{R}^3 | x_1 + x_3 = f_1(x) + f_3(x)\}$, we use the notation $x > S_4$ if x is in the set pointed by the vector $[1, 0, 1]^T$, $x < S_4$ if x is in the opposite set and $x = S_4$ if x is on the plane. Notice that a trajectory could be trapped on a point in S_4 located at the center of the scroll even for the case $\lambda_1 = 0$ ($c = 0$) which could be called a virtual point, in order to avoid this situation a new function is defined as follows:

$$f_5(x, f_1, f_3) = \begin{cases} -w, & \text{si} \quad x \leq S_5; \\ w, & \text{si} \quad x > S_5; \end{cases} \tag{10}$$

where $w \in \mathbb{R}_{>0}$ and the switching plane $S_5 = \{x \in \mathbb{R}^3 | -x_1 + x_3 = f_3(x) - f_1(x)\}$, we use the notation $x > S_5$ if x is in the set pointed by the vector $[-1, 0, 1]^T$, $x \leq S_{ix}$ if x is in the opposite set or on the plane.

III. Analysis of the solution

The electronic implementation of the system (4) for $c = 0$ results in a system without equilibria. But it is impossible to guarantee experimentally the value of $c = 0$ due to tolerance and noise on all electronic circuits. So the interest is to work with a system without equilibria $c = 0$, but considering perturbations δ, *i. e.*, $c = \delta \in \mathbb{R}$ such that $|\delta| << 1$.

The system (4) can be rewritten considering (5) as:

$$\begin{aligned} \dot{x} = PJP^{-1} & \begin{bmatrix} x_1 - f_1(x) - f_5(x, f_1, f_3) \\ x_2 - f_2(x, f_1) \\ x_3 - f_3(x) + f_5(x, f_1, f_3) \end{bmatrix} + \\ & \begin{bmatrix} f_4(x, f_1(x), f_3(x)) \\ 0 \\ f_4(x, f_1(x), f_3(x)) \end{bmatrix}. \end{aligned} \tag{11}$$

Considering a change of variables $y_1 = x_1 - f_1(x) - f_5(x, f_1, f_3)$, $y_2 = x_2 - f_2(x, f_1)$ and $y_3 = x_3 - f_3(x) + f_5(x, f_1, f_3)$ with the appropiate transformation of the function f_4 we get:

$$\dot{y} = PJP^{-1}y + \begin{bmatrix} f_4(y), & 0, & f_4(y) \end{bmatrix}^T, \tag{12}$$

with the switching plane $S_4 = \{y \in \mathbb{R}^3 | y_1 + y_3 = 0\}$. It can be seen that:

$$f_4(y)P^{-1}p_1 = \begin{bmatrix} f_4(y) & 0 & 0 \end{bmatrix}^T. \tag{13}$$

thus considering a change of variable $z = P^{-1}y$:

$$\dot{z} = Jz + \begin{bmatrix} f_4(z), & 0, & 0 \end{bmatrix}^T, \tag{14}$$

with the switching plane $S_4 = \{z \in \mathbb{R}^3 | z_1 = 0\}$. The flows of the systems (11), (12) and (14) are topological equivalent. When $c = 0$ the solution of the systems (14), (12) and (11) are given as follows:

$$z = \exp(Jt)z_0 + \begin{bmatrix} f_4(z)t, & 0, & 0 \end{bmatrix}^T, \tag{15}$$

$$y = \exp(At)y_0 + \begin{bmatrix} f_4(y)t, & 0, & f_4(y)t \end{bmatrix}^T, \tag{16}$$

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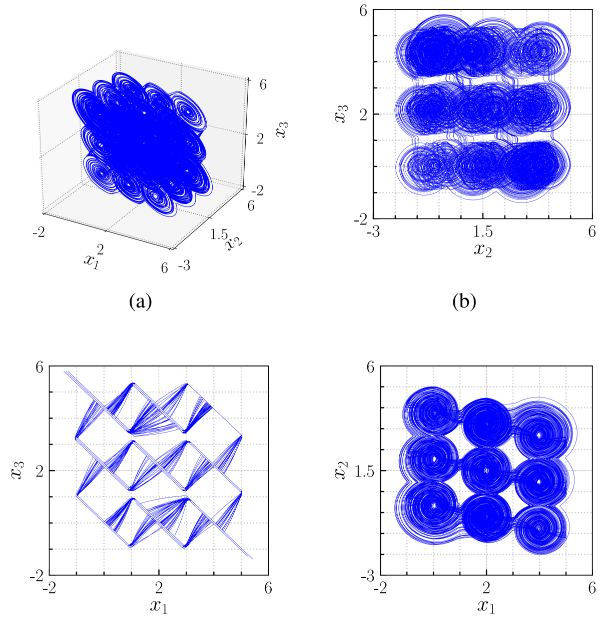


(a) (b)

(c) (d)

Fig. 1. Attractor of the system (4) with the affine part given in (5) and the the parameters $p = q = r = 2$, $a = 0.7$, $b = 10$, $c = 0$, $v = 11$, $w = 0.1$, $k = 0.25$, $\Delta x_1 = \Delta_{x_2} = 2$ and $\Delta_{x_3} = 2.2$ for the initial condition $x_0 = (0,\ 0,\ 0)^T$ in the space (a) $\mathbb{R}^3$ and its projections onto the planes: (b) (x_1, x_2), (c) (x_1, x_3), and (d) (x_2, x_3).

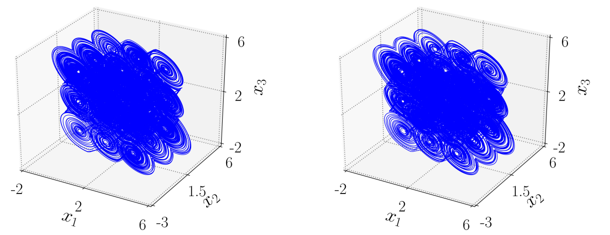
$$x = \exp(At)\begin{bmatrix} x_1(0) - f_1(x) - f_5(x, f_1, f_3) \\ x_2(0) - f_2(x, f_1) \\ x_3(0) - f_3(x) + f_5(x, f_1, f_3) \end{bmatrix} + \begin{bmatrix} f_4(x, f_1, f_3)t \\ 0 \\ f_4(x, f_1, f_3)t \end{bmatrix} - \begin{bmatrix} -f_1(x) - f_5(x, f_1, f_3) \\ -f_2(x) \\ -f_3(x) + f_5(x, f_1, f_3) \end{bmatrix}. \quad (17)$$

When $c \neq 0$ the solution of the systems (11), (12) and (14) are given as:

$$z = \exp(Jt)z_0 + \begin{bmatrix} \frac{f_4(z)}{c}(\exp(ct) - 1) \\ 0 \\ 0 \end{bmatrix}, \quad (18)$$

$$y = \exp(At)y_0 + \begin{bmatrix} \frac{f_4(y)}{c}(\exp(ct) - 1) \\ 0 \\ \frac{f_4(y)}{c}(\exp(ct) - 1) \end{bmatrix}, \quad (19)$$

$$x = \exp(At)\begin{bmatrix} x_1(0) - f_1(x) - f_5(x, f_1, f_3) \\ x_2(0) - f_2(x, f_1) \\ x_3(0) - f_3(x) + f_5(x, f_1, f_3) \end{bmatrix} + \begin{bmatrix} \frac{f_4(x, f_1, f_3)}{c}(\exp(ct) - 1) \\ 0 \\ \frac{f_4(x, f_1, f_3)}{c}(\exp(ct) - 1) \end{bmatrix} - \begin{bmatrix} -f_1(x) - f_5(x, f_1, f_3) \\ -f_2(x, f_1) \\ -f_3(x) + f_5(x, f_1, f_3) \end{bmatrix}. \quad (20)$$



(a) (b)

Fig. 2. Attractor of the system (4) with the affine part given in (5) and the parameters $p = q = r = 2$, $a = 0.7$, $b = 10$, $c = 0$, $v = 11$, $w = 0.1$, $k = 0.25$, $\Delta x_1 = \Delta_{x_2} = 2$ and $\Delta_{x_3} = 2.2$ plus a perturbation in the parameter c of (a) -0.1 and (b) 0.1.

The function $f_4(z)$ can be written in z coordinates as:

$$f_4(z) = \begin{cases} v, & \text{if} \quad z_1 < 0; \\ 0, & \text{if} \quad z_1 = 0; \\ -v, & \text{if} \quad z_1 > 0; \end{cases} \quad (21)$$

Looking at the solution in (15) when $z_1(0) \neq 0$ and $c = 0$, $z_1(t) = z_1(0) + f_4(z)t$, thus the trajectories goes towards the plane S_4.

When $c \neq 0$, *i.e.* when there is a perturbation in the eigenvalue $\lambda_1 = 0$, there are two cases, when $c > 0$ and $c < 0$. The solution for z_1 is $z_1(t) = \exp(ct)z_1(0) + \frac{f_4(z)}{c}(\exp(ct) - 1)$ and from the equation (14) the equilibrium point is given by:

$$z^* = \left[-\frac{f_4(z)}{c}, 0, 0\right]^T. \quad (22)$$

For the case $z_1(0) \neq 0$ and $c < 0$, $\text{sgn}(z_1^*) \neq \text{sgn}(z_1(0))$, thus the trajectories go towards to the plane $\{z \in \mathbb{R}^3 | z_1 = z_1^*\}$ which guarantees these reach the plane $S_4 = \{z \in \mathbb{R}^3 | z_1 = 0\}$.

For the case when $z_1(0) \neq 0$ and $c > 0$ $\text{sgn}(z_1^*) = \text{sgn}(z_1(0))$, however it is assumed that the perturbation on λ_1 it is small and $|z_1^*| >> 1$. Then the trajectories move away from the plane $\{z \in \mathbb{R}^3 | z_1 = z_1^*\}$ and reach S_4.

The absence of equilibrium points in all $\mathbb{R}^3$ even when λ_1 is perturbed is stated with the following theorem.

Theorem. *Let the system* (4) *with* (5), (6), (7), (8), (9), *and* (10) *be a PWL system, then the system* (4) *has no equilibria for* $c = 0$ *or* $0 < |c| << 1$.

Proof: Let us rewrite the system as

$$\dot{x} = PJP^{-1}\begin{bmatrix} x_1 - f_1(x) \\ x_2 - f_2(x, f_1(x)) \\ x_3 - f_3(x) \end{bmatrix} + PJP^{-1}\begin{bmatrix} -f_5(x) \\ 0 \\ f_5(x) \end{bmatrix} + \begin{bmatrix} f_4(x, f_1, f_3) \\ 0 \\ f_4(x, f_1, f_3) \end{bmatrix}. \quad (23)$$

Considering a change of variables $y_1 = x_1 - f_1(x)$, $y_2 = x_2 - f_2(x, f_1(x))$ and $y_3 = x_3 - f_3(x)$:

$$\dot{y} = PJP^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} -f_5(y)a \\ -f_5(y)b \\ f_5(y)a \end{bmatrix} + \begin{bmatrix} f_4(y) \\ 0 \\ f_4(y) \end{bmatrix}, \quad (24)$$

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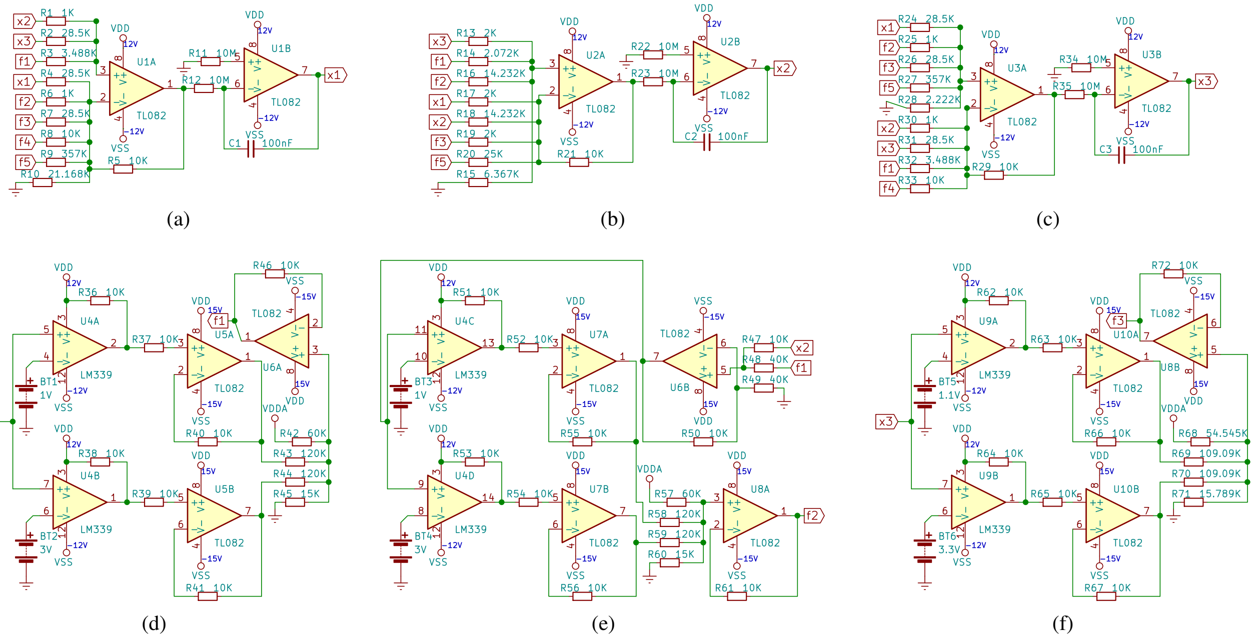

Fig. 3. Sub-circuits for the signals: (a)x_1, (b)x_2, (c)x_3, (d)f_1, (e)f_2, and (f)f_3.

considering the change of variable $z = P^{-1}y$

$$\dot{z} = J\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -f_5(z)b \\ -f_5(z)a \end{bmatrix} + \begin{bmatrix} f_4(z) \\ 0 \\ 0 \end{bmatrix}, \tag{25}$$

$$\dot{z} = J\begin{bmatrix} z_1 \\ z_2 \\ z_3 - f_5(z) \end{bmatrix} + \begin{bmatrix} f_4(z) \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

Then $f_5(z)$ can be written as:

$$f_5(z) = \begin{cases} -w, & \text{if} \quad z_1 > 0; \\ w, & \text{if} \quad z_1 \leq 0; \end{cases} \tag{27}$$

When $c = 0$ and $f_4(z) \neq 0$ there is no equilibria because $[f_4(z), 0, 0]^T$ belongs to the eigenspace associated to λ_1 [8]. When $f_4(z) = 0$, *i.e.*, the flow on the surface S_4 the equilibrium point is virtually located at $[0, 0, f_5(z)]^T$, then there is no equilibrium points in all $\mathbb{R}^3$. When $c \neq 0$ the equilibrium point is located at $[-f_4(z)/c, 0, f_5(z)]^T$ then there are not equilibria in all $\mathbb{R}^3$. This complete the proof.

□

A. *Particular system*

Consider the system (4) with the affine part given by (5) $p = q = r = 2$, $a = 0.7$, $b = 10$, $c = 0$, $v = 11$, $w = 0.1$, $k = 0.25$, $\Delta x_1 = \Delta_{x_2} = 2$ and $\Delta_{x_3} = 2.2$. The system presents a $3 \times 3 \times 3$ grid scroll chaotic attractor.

The resulting attractor for the ideal case when $c = 0$ is shown in Figure 1. In Figure 2 the $3 \times 3 \times 3$ grid scroll chaotic attractor generated for $\lambda_1 = -0.1$ and $\lambda_1 = 0.1$ are shown. As it can be seen the attractor is preserved even when there is a perturbation in the eigenvalue λ_1.

The Maximum Lyapunov exponent (MLE) was calculated using a Fourth order Runge-Kutta method with a fixed step of 0.0001s as 1.02 by approximating the average separation of ten trajectories. Wolf's algorithm was also performed with a time step of 0.001, the calculated exponents are $\{1.16, 0, -20.42\}$ which gives a Kaplan-York dimension of 2.056. Approximated functions $f_1, \ldots, f_5$ via $tanh(\cdot)$ were used for the Wolf's algorithm.

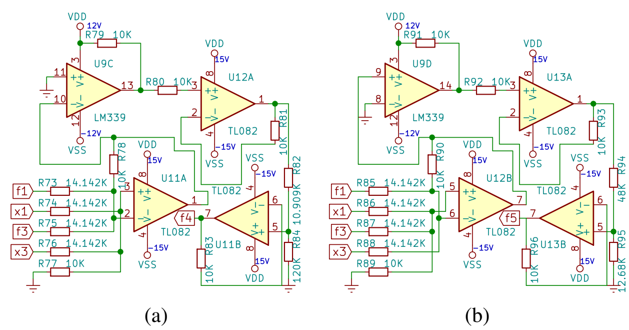

Fig. 4. Sub-circuits for the signals: (a)f_4 and (b)f_5.

IV. Circuit Realization

In this section an electronic realization for the previous system is proposed. The electronic diagram has been divided in eight sub-circuits. The responsible sub-circuits for the output signals x_1, x_2 and x_3 are shown in Figures 3a, 3b and 3c, and are composed basically of an adder-subtractor and an integrator.

The sub-circuits in Figures 3d and 3f are composed by two comparators followed by buffers that go to an adder and are responsible for the signals f_1 and f_3, respectively. The sub-circuit in the diagram of Figure 3e has the same structure but with an additional adder before the comparators and is responsible for the signal f_2. The sub-circuits in Figures 4a

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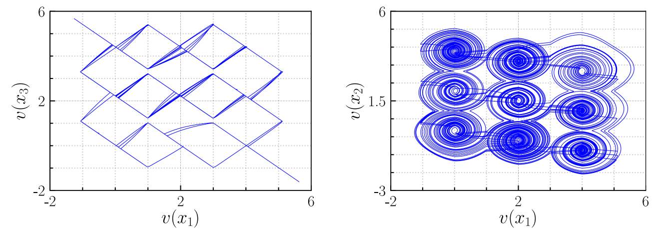


(a) (b)

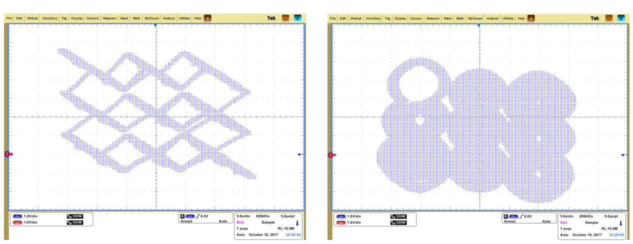
Fig. 5. Attractor generated by electronic simulation in ngspice projected on the planes: (a) (x_1, x_2) and (c) (x_1, x_3).

and 4b are composed of an adder-subtractor followed by a comparator, a buffer and finally an attenuator, and are responsible for the signals f_4 and f_5, respectively.

The proposed electronic realization makes use of the general purpose JFET-input dual Operational amplifier TL082CP and the quad differential comparator LM339AN. The calculated resistor values were approximated to achievable values either by one or two resistors from the E12 series configured either in parallel or series.

An electronic simulation of the circuit was run and the resulting projections on the planes (x_1, x_2) and (x_1, x_3) are shown in Figure 5.

The circuit was also implemented physically and the result is presented in the Figure 6. The measurement was done with the Tektronix DPO 5054B Digital Phosphor Oscilloscope with a sample rate of $20.0kS/s$ and a resolution of $50\mu s$.



(a) (b)

Fig. 6. Attractor of the physically implemented circuit in Figure 3 projected on: (a) (x_1, x_2) and (c) (x_1, x_3).

V. Conclusion

In this paper a new class of PWL dynamical system without equilibria whose chaotic attractor can display a 3D-grid scroll structure has been introduced. A detailed mathematical analysis has been performed to the solution of the system in order to show the absence of equilibria and the persistence of the behavior under perturbation of the eigenvalue $\lambda_1 = 0$. The number of scrolls as well as their distribution in the grid attractor can been easily modified using the functions f_1, f_2 and f_3. A particular case with an scroll arrangement of $3\times3\times3$ has been studied and its electronic realization has been tested by numerical simulation as experimentally.

The extension of the approach for a four-dimensional systems in order to obtain hyperchaotic grid attractors has not been addressed, however, since there are reported hyperchaotic multiscroll attractors in systems with equilibria, it is reasonable to think in a hyperchaotic grid attractor and is considered as future work.

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