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Finite-time continuous control for mechanical systems with bounded inputs

Tesis que presenta

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Director de la Tesis:

Dr. Arturo Zavala Río

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To my family and friends
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Resumen

En esta tesis se presenta el diseño de esquemas continuos de control en tiempo finito o exponencial para sistemas mecánicos con entradas acotadas. El trabajo involucra el reciente marco teórico de la homogeneidad local, lo cual amplía la flexibilidad de diseño y resulta fundamental para resolver los problemas formulados ante restricciones de entrada. Los esquemas propuestos involucran acciones de corrección en errores de posición y velocidad con estructuras generalizadas. Más aún, dan al usuario la posibilidad de elegir entre convergencia en tiempo finito y exponencial a través de un simple parámetro. En el contexto analítico considerado, se aborda primero el problema de regulación tanto por retroalimentación de estado como de salida. Después, se aborda el problema de control de movimiento por retroalimentación de estado, soportando el análisis en lazo cerrado a través de una función estricta de Lyapunov. Posteriormente, para el control de seguimiento de trayectorias, se realiza un estudio de robustez considerando un término de perturbación acotado que se añade a la entrada. Este estudio permite concluir que para un término de perturbación suficientemente pequeño, las trayectorias de las variables de error convergen al interior de una bola con centro en el origen cuyo radio se vuelve más pequeño en el caso de la convergencia en tiempo finito, implicando variaciones post-transitorias más pequeñas que en el caso exponencial. Más aún, esto se cumple para cualquier condición inicial y evitando restringir cualquier parámetro involucrado en el diseño de control. Todos los controladores propuestos son implementados a través de simulaciones y experimentos.

Palabras clave: Control continuo en tiempo finito, homogeneidad local, robustez, sistemas mecánicos, entradas acotadas.

Abstract

In this dissertation the design of finite-time/exponential continuous control schemes for mechanical systems with bounded inputs is presented. The work involves the recent theoretical framework of local homogeneity, which expands the design flexibility and it results to be fundamental to solve problems with constrained inputs. The proposed schemes include corrective actions on the position and velocity errors with generalized structures. Moreover, all the proposed schemes give the freedom to choose among finite-time and exponential convergence through a simple parameter. Under the considered analytical framework, the regulation problem is first studied for both cases: state-feedback and output-feedback. Further, the state-feedback motion control problem is studied, supporting the closed-loop system through a strict Lyapunov function. Subsequently, a robustness study is developed for the trajectory-tracking control, under the consideration of an input-matching bounded perturbation term. Such a study has permitted to conclude that for a perturbation term with sufficiently small bound, the error variable trajectories converge into an origin-centered ball whose radius becomes smaller in the finite-time convergence case, entailing smaller post-transient variations than in the exponential case. Moreover, this is shown to be achieved for any initial condition and avoiding to restrain any of the parameters involved in the control design. All the proposed controllers are further tested through simulation and experimental implementations.

Key words: Finite-time continuous control, local homogeneity, robustness, mechanical systems, constrained inputs.

Contents

CHAPTER 1

Introduction

This dissertation focuses on the finite-time continuous control of mechanical systems. Particularly, control schemes for bounded-input mechanical systems are designed so as to achieve both the regulation and the trajectory-tracking objectives avoiding input saturation. This work is developed under the consideration of the analytical framework of local homogeneity. In a coordinate-dependent context, such an analytical tool allows —contrarily to the conventional homogeneity theory— to involve input constraints, which permits to solve the formulated problems throughout the dissertation not only guaranteeing input saturation avoidance but also giving a wide range of design flexibility. The developed work succeeds on innovating not only on the control structures but also on the analytical procedures in order to achieve the design objectives. Divers contexts related to the data involved in the designed controllers are considered, and a robustness study that compares the resulting performance of the finite-time controllers with analog algorithms that achieve the objective by inducing exponential convergence on the closed-loop system trajectories is included.

The finite-time control study has been an interesting topic for the research control community since this does not only imply a challenging analytical problem but also for the so-claimed advantages over the asymptotic controllers, such as faster convergence, precise performance, robustness to uncertainties, and disturbance-rejection, [1, 2, 3, 4]. Finite-time controllers had been proposed through either discontinuous or continuous schemes. Particularly, finite-time stabilization of an equilibrium induced through continuous control becomes a topic of significant importance since, in addition to the previous mentioned advantages, its application avoids some unwanted effects like chattering due to measurement noise, contrarily to the discontinuous controllers [5]. Moreover, finite-time continuous stabilizers turn out to be suitable for certain tasks such as consensus [6] and formation of multi-agent systems [7], process supervision (monitoring) [8], and secure communication [9].

A first approach to finite-time continuous control appeared in the seminal work of Haimo [10], which introduced the concept of *finite time differential equations*. These were defined therein as differential equations whose trivial solution is asymptotically stable and all their solutions that converge to zero do it in finite time. Such a work stated the necessary and sufficient conditions to define a first order finite time differential equation, namely (reproducing verbatim [10, Fact 1]):

Fact 1.1. $\dot{x} = r(x),\ r(0) = 0,\ x \in \mathbb{R}$, *is finite time iff*

a) $xr(x) \leq 0$ *and equals* 0 *only at* $x = 0$, *for* x *in a neighborhood of* 0, *and*

b) $\int_p^0 (dx/r(x)) < \infty$, *for all* p *in* $\mathbb{R}$

Further, finite-time stability —in the finite-time-differential-equation sense stated in [10]— on second order systems of the form

$$\ddot{x} = u(x, \dot{x})$$

with u continuous, was studied in [10], particularly proving the referred stability property for

$$u(x, \dot{x}) = -|x|^a \operatorname{sign}(x) - |\dot{x}|^b \operatorname{sign}(\dot{x})$$

with $b \in (0, 1)$ and $a > b/(2 - b)$ —or equivalently $a \in (0, 1)$ and $b < 2a/(1 + a)$— [10, Corollary 1], and even stating finite-time stability preservation under (some type of) additional vanishing perturbation terms [10, Corollary 2].

Later on, several researchers focused on settling down a suitable underlying analytical framework for the subject. Remarkable contributions in this direction are those developed by Bhat and Bernstein [11, 12, 13, 14, 15], who stated —for continuous autonomous systems— a formal and precise definition of a *finite-time stable* equilibrium point gathering Lyapunov stability and finite-time convergence, as well as a Lyapunov-function-based criterion for its determination, and a suitable characterization of its relationship with homogeneous vector fields. This last is advantageous in view of its simplicity: for a homogeneous vector field with asymptotically stable equilibrium at the origin, verifying negativity of the degree of homogeneity suffices to conclude finite-time stability (of the origin). Consequently, homogeneity becomes an attractive analytical tool in control design to easily achieve finite-time stabilization. Nevertheless, for finite-time control design purposes, such a criterion might be restrictive in view of the requirements naturally imposed by homogeneity, which is conventionally a *global* property. For instance, in a constrained-input context, the closed-loop system would include bounded components which would preclude the corresponding vector field to be homogeneous [15]. However, such a restriction has been proven to be relaxed through alternative notions of homogeneity [16].

Finite-time stability and stabilization for time-varying vector fields has evolved more slowly and is still in progress. Important extensions and generalizations of the previously cited works from Bhat and Bernstein have been developed for instance in [17] by stating precise definitions and Lyapunov-type characterizations for non-autonomous systems. Uniform stability has been very recently studied within the framework of homogeneity in [18] where, in particular, the characterization of global uniform finite-time stability has been extended to time-varying vector fields. These contributions show the complexity entailed in the non-autonomous case in relation to the previously cited time-invariant case. For instance, the existence of a homogeneous Lyapunov function characterized for autonomous vector fields in [19] does not apply for time-varying ones, and a similar extension for the latter case does not exist. Consequently, results based on such a fundamental work of [19], like the finite-time-stability-preservation *approximation* approach of [20], do not apply in the non-autonomous case. Stability/stabilization studies in the time-varying context shall take into account such important analytical limitations and consequently entail more complex analysis.

1.1 Previous works

An initial work on finite-time continuous control was presented in [21] for mechanical manipulators, whose dynamical model is defined as

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where position q, velocity $\dot{q}$ and acceleration $\ddot{q}$ vectors, the inertia $H(q)$ and Centrifugal and Coriolis effect $C(q,\dot{q})$ matrices, and the conservative-force $g(q)$ and external force τ vectors, are involved. The controller design adopted proportional (P) and derivative (D) type actions, assuming unconstrained inputs, with two options on the structure:

- the first one compensating for the whole system dynamics, with the following form:

  $$\tau = g(q) + C(q,\dot{q})\dot{q} - H(q)\big[l_1\bar{s}_1(q-q_d) + l_2\bar{s}_2(\dot{q})\big]$$

  where $l_1,\ l_2 > 0$, $\bar{s}_i(\varsigma) = \big(\sigma_{i1}(\varsigma_1),\ldots,\sigma_{in}(\varsigma_n)\big)^T$, $i=1,2$, with $\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\alpha_i}$, $i=1,2$, $j=\overline{1,n}$, $\varsigma\in\mathbb{R}$, $0<\alpha_1<1$, $\alpha_2 = 2\alpha_1/(1+\alpha_1)$, and q_d is the desired position;

- the other one compensating only for the conservative-force terms, that takes the form:

  $$\tau = g(q) - l_1\bar{s}_1(q-q_d) - l_2\bar{s}_2(\dot{q})$$

  with l_i and $\bar{s}_i$, $i=1,2$, as previously defined.

The closed-loop analysis was developed based on the conventional analytical framework of homogeneity. Although a variation of the latter structure with the P and D type actions included (each of them separately) within conventional saturation functions was further contemplated, no formal closed-loop analysis was presented for this case, which does not fit within the analytical framework where the unconstrained versions were developed.

Another work oriented to the finite-time control of robotic manipulators, disregarding input constraints, appeared later in [22]. The scheme proposed therein is structured aiming at the compensation for the whole system nominal dynamics. The rest of the synthesis is developed applying the backstepping design technique, by viewing the velocity vector variable as a virtual input to achieve finite-time control of the positions, and using the force input vector to impose a closed-loop continuous dynamics that guarantees finite-time stabilization of the consequent error variables. The design is then complemented through a Lyapunov-redesign type procedure that results in the addition of a control term in charge to reject perturbations due to system uncertainties. Alternative approximations of certain control terms are suggested in order to avoid discontinuities and singularities implied by the developed approach, expecting close enough (to the desired position) stabilization through their replacement. Although a bounded version of the developed controller is also contemplated by involving conventional saturation functions, no further analysis is included for this case, which is claimed to be left for future research.

A further finite-time continuous stabilization scheme for mechanical systems was proposed in [23] similarly assuming unconstrained inputs. The approach is based on the definition of a manifold where the system is proven to converge to the zero (desired) state in a finite time T_1. A suitable closed-loop form ensuring convergence of the system variables to such manifold in a finite time T_2 is then found. The control law is then designed through exact dynamic compensation so as to impose the closed-loop form found in the precedent step, taking the following form:

$$\tau = H(q)\left(k(\alpha-1)\frac{qq^T}{\|q\|^{3-\alpha}}\dot{q} - \gamma\frac{l(q,\dot{q})}{(l^T(q,\dot{q})l(q,\dot{q}))^{1-\alpha}} - k\frac{\dot{q}}{\|q\|^{1-\alpha}}\right) + C(q,\dot{q})\dot{q} + g(q)$$

where $1/2 < \alpha < 1$, $k, \gamma > 0$, and $l(q,\dot{q}) = \dot{q} + \frac{kq}{\|q\|^{1-\alpha}}$. However, the extension of such an approach to the constrained input case was not developed.

Additionally, the works presented in [24] and [25] appeared during the development of this dissertation (actually, after the first publications derived from this research work). Such works focus on the extension of the energy-shaping methodology to the finite-time regulation problem; while [24] presents such extension for the state-feedback control study, the work in [25] does so for the output-feedback case. Further, both works present divers examples of control laws obtained from the developed methodology, among them bounded input versions are included. Such bounded controllers are characterized by the use of specific saturation functions and the application of the control gains to the shaped error correction actions (and not directly to the error variables prior to the shaping). In particular, such external weighting leads the control gains to act on the PD-action bounds, generating the need (at every setting or change on the control gain values) for an additional verification and eventual adjustment on the considered saturation function bounds to guarantee the input saturation avoidance requirements.

All the previous mentioned works mainly give rise to finite-time regulators and are consequently developed within the framework of autonomous systems. On the other hand, for the tracking control problem, which naturally implies a time-varying closed-loop dynamics, the stability analysis suffers from the impossibility to involve analytical tools exclusively addressed to time-invariant vector fields, and shall consequently be developed within the framework of non-autonomous systems, for instance through the use of a suitable strict Lyapunov function. Strict Lyapunov functions have been very recently constructed in [26] to support finite-time control of robot manipulators disregarding input constraints and focused on the regulation problem, thus leaving the tracking-under-bounded-input case unsolved.

1.2 Motivation and objectives

Based on the above mentioned, it is clear that divers advances have been made in the study of finite-time continuous control for mechanical systems, which highlighted the advantages of applying finite-time controllers to such kind of systems. However the bounded input case has barely been discussed. Several works have included some proposals, lacking from analytical support or missing important details in the design and/or analysis. The bounded-input case becomes an interesting issue since conventional analytical tools, as the (standard) homogeneity framework, result to be inappropriate in order to deal with such a problem. This generates an opportunity to approach the problem with more recent and suitable analytical tools, which constitutes one of the main motivations of the present work. The former section cited several works related to finite-time continuous control schemes addressed to the regulation problem of mechanical systems; however, most of them disregard both the bounded-input case (a natural characteristic of real systems) and the bounded-input-tracking problem. Moreover, all the mentioned works have been motivated arguing benefits of the finite-time algorithms over the asymptotic (infinite-time) ones, such as faster convergence and improved robustness under uncertainties. However, this has not yet been exhaustively explored or brought to the fore through formal analysis or implementation tests, which creates the opportunity to address it through this dissertation.

In this direction, the present work has the following general objective:

To design finite-time continuous control schemes for mechanical systems with constrained inputs, through generalized structures that increase the design flexibility to achieve either finite-time or asymptotic (with local exponential) stability, and avoiding input saturation.

The particular objectives are focused on giving a solution to:

- the regulation-under-bounded-input problem taking into account two cases: the on-line conservative force compensation and the desired conservative force compensation. The goal, for the both mentioned cases, is to design state-feedback control schemes and output feedback control schemes,

- the tracking-under-bounded input problem and study the robustness problem under perturbations.

1.3 Notation

Let X, $Y \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. Throughout this dissertation, X_{ij} denotes the element of X at its i^{th} row and j^{th} column, X_i represents the i^{th} row of X and x_i stands for the i^{th} element of x. With $m = n$, $X > 0$, resp. $X \geq 0$, (conventionally) denotes that X is positive definite, resp. semidefinite, and $X > Y$, resp. $X \geq Y$, that $X - Y$ is positive definite, resp. semidefinite, while, for a symmetric matrix X, $\lambda_m(X)$ and $\lambda_M(X)$ stand for its minimum and maximum eigenvalues, respectively. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. We denote $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ for scalars, and $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$ and $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$ for vectors. For a subset $A \subset \mathbb{R}^n$, ∂A stands for its boundary. $\|\cdot\|$ will conventionally denote the standard Euclidean norm, *i.e.*, the 2-norm for vectors and induced 2-norm for matrices. Other p-norms will be denoted $\|\cdot\|_p$. An n-dimensional ball and an $(n-1)$-dimensional sphere, both of radius $c > 0$, on $\mathbb{R}^n$, are denoted $\mathcal{B}^n_c$ and $\mathcal{S}^{n-1}_c$, respectively, *i.e.*, $\mathcal{B}^n_c = \{x \in \mathbb{R}^n : \|x\| \leq c\}$ and $S^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$. Let $\mathcal{A}$ and $\mathcal{E}$ be subsets (with non-empty interior) of some vector spaces $\mathbb{A}$ and $\mathbb{E}$, respectively. For any integer $m \geq 0$, we denote $\mathcal{C}^m(\mathcal{A};\mathcal{E})$ the set of continuous functions from $\mathcal{A}$ to $\mathcal{E}$, being m times continuously differentiable when m is strictly positive (with differentiability at any point on the boundary of $\mathcal{A}$ meant as the limit from the interior of $\mathcal{A}$). Consider a function $h \in \mathcal{C}^2(\mathbb{R}_{\geq 0};\mathcal{E})$. The first- and the second- order rates of change of h are respectively denoted $\dot{h}$ and $\ddot{h}$. For a continuously differentiable scalar function $f \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ and a vector function $g : \mathbb{R}^n \to \mathbb{R}^n$, we denote $D_g f$ the directional derivative of f along g, *i.e.*, $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. This work involves class $\mathcal{K}$ functions, *i.e.*, continuous strictly increasing functions $\alpha : \mathcal{A} \to \mathbb{R}_{\geq 0}$ with $\mathcal{A} \subset [0, a), a \in (0, \infty]$, and $\alpha(0) = 0$, and refers to $\mathcal{K}_\infty$ functions, *i.e.*, class $\mathcal{K}$ functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\alpha(\varsigma) \to \infty$ as $\varsigma \to \infty$. We will consider the sign function to be zero at zero, *i.e.*,

$$\operatorname{sign}(\varsigma) = \begin{cases} \frac{\varsigma}{|\varsigma|} & \text{if } \varsigma \neq 0 \\ 0 & \text{if } \varsigma = 0 \end{cases}$$

and denote $\operatorname{sat}(\cdot)$ the standard (unitary) saturation function, *i.e.*, $\operatorname{sat}(\varsigma) = \operatorname{sign}(\varsigma)\min\{|\varsigma|, 1\}$. Fundamental facts that will be involved in this study are *Young's inequality*, *i.e.*, for any $\phi,\ \psi \in (1, \infty)$ such that $\frac{1}{\phi} + \frac{1}{\psi} = 1$ and any $a,\ b \in \mathbb{R}_{\geq 0} : ab \leq \frac{a^\phi}{\phi} + \frac{b^\psi}{\psi}$; *Hölder inequality*, *i.e.*, for any $\phi,\ \psi \in [1, \infty]$ such that $\frac{1}{\phi} + \frac{1}{\psi} = 1$ and any $x,\ y \in \mathbb{R}^n : |x^T y| \leq \|x\|_\phi \|y\|_\psi$; and the following properties of p-norms.

Lemma 1.1. *For any $x \in \mathbb{R}^n$, $\|x\|_p$ is non increasing in p.*

Proof. Since $\|0_n\|_p = 0$ for any p-norm ($p \geq 1$), it is clear that $\left[\frac{\partial}{\partial p}\|x\|_p\right]_{x=0_n} = 0$. For $x \neq 0_n$:

$$\begin{aligned}
\frac{\partial}{\partial p}\|x\|_p = \frac{\partial}{\partial p}\left[\sum_{i=1}^n |x_i|^p\right]^{1/p} &= \frac{\|x\|_p^{1-p}}{p}\left[\sum_{i=1}^n |x_i|^p \ln |x_i|\right] - \frac{\|x\|_p}{p^2}\ln \|x\|_p^p \\
&= \frac{\|x\|_p^{1-p}}{p^2}\left[\sum_{i=1}^n |x_i|^p \ln |x_i|^p - \sum_{i=1}^n |x_i|^p \ln \|x\|_p^p\right] \\
&= \frac{\|x\|_p^{1-p}}{p^2}\left[\sum_{i=1}^n |x_i|^p \ln \frac{|x_i|^p}{\|x\|_p^p}\right] \leq 0
\end{aligned}$$

since $0 < \frac{|x_i|^p}{\|x\|_p^p} \leq 1 \iff \ln \frac{|x_i|^p}{\|x\|_p^p} \leq 0,\ i = 1, \dots, n.$ ∎

Remark 1.1. By equivalence of p-norms, for any $\|\cdot\|_\phi$ and $\|\cdot\|_\psi$, with $\phi \neq \psi$, there exist constants $\bar{c}_{\phi,\psi} > c_{\phi,\psi} > 0$ such that $c_{\phi,\psi}\|x\|_\psi \leq \|x\|_\phi \leq \bar{c}_{\phi,\psi}\|x\|_\psi$, $\forall x \in \mathbb{R}^n$. In particular, by Lemma 1.1 and the fact that

$$\|x\|_\phi = \left[\sum_{i=1}^{n} |x_i|^\phi\right]^{1/\phi} \leq \left[\sum_{i=1}^{n} \|x\|_\psi^\phi\right]^{1/\phi} = n^{1/\phi}\|x\|_\psi$$

we have: $c_{\phi,\psi} = n^{(\text{sign}(\psi-\phi)-1)/(2\psi)}$ and $\bar{c}_{\phi,\psi} = n^{(\text{sign}(\psi-\phi)+1)/(2\phi)}$.

1.4 Mechanical systems

In order to provide a characterization of the kind of systems involved in this work, the following facts are presented. Consider the n-degree-of freedom (DOF) fully actuated mechanical system dynamics with linear damping effects

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau \tag{1.1}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors, $H(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind, $F \in \mathbb{R}^{n\times n}$ is the (*a priori* symmetric positive semidefinite) damping effect matrix, $g(q) = \nabla\mathcal{U}_{ol}(q)$, with $\mathcal{U}_{ol}(q) : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the open-loop system, or equivalently

$$\mathcal{U}_{ol}(q) = \mathcal{U}_{ol}(q_0) + \int_{q_0}^{q} g^T(z)dz \tag{1.2}$$

for any $q, q_0 \in \mathbb{R}^n$ (the integration in (1.2) takes into account the conservative nature of g, as pointed for instance in [27, Note 1, p. 2009]); and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector. Some well-known properties characterizing the terms of such a dynamical model are recalled here [28], [29], [30]. Subsequently, we denote $\dot{H}$ the rate of change of H, *i.e.*, $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i,j = 1,\ldots,n$.

Property 1.1. *$H(q)$ is a continuously differentiable positive definite symmetric matrix function, and actually $H(q) \geq \mu_m I_n$ —whence $\|H(q)\| \geq \mu_m$— $\forall q \in \mathbb{R}^n$ for some $\mu_m > 0$.*

Property 1.2. *The Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind satisfies:*

1.2.1 $\dot{H}(q,\dot{q}) = C(q,\dot{q}) + C^T(q,\dot{q})$, $\forall q,\dot{q} \in \mathbb{R}^n$, *and consequently*

$$z^T\left[\frac{1}{2}\dot{H}(x,y) - C(x,y)\right]z = 0, \quad \forall x,y,z \in \mathbb{R}^n$$

1.2.2 $C(w,x+y)z = C(w,x)z + C(w,y)z$, $\forall w,x,y,z \in \mathbb{R}^n$;

1.2.3 $C(x,y)z = C(x,z)y$, $\forall x,y,z \in \mathbb{R}^n$*;*

1.2.4 $\|C(x,y)\| \leq \vartheta(x)\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $\vartheta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *or equivalently* $\|C_i(x,y)\| \leq \vartheta_i(x)\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $\vartheta_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i = 1,\ldots,n$.

Throughout this work, we consider the (realistic) bounded input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound $T_i > 0$, *i.e.*, $|\tau_i| \leq T_i$, $i = 1,\ldots,n$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \operatorname{sat}\left(\frac{u_i}{T_i}\right)$$

Further assumptions are stated next.

Assumption 1.1. *The inertia matrix is bounded, i.e.,* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, *for some* $\mu_M \geq \mu_m > 0$, *or equivalently* $\|H_i(q)\| \leq \mu_{Mi}$, $\forall q \in \mathbb{R}^n$, *for some* $\mu_{Mi} > 0$, $i = 1,\ldots,n$.

Assumption 1.2. $\vartheta(\cdot)$ *—hence, each* $\vartheta_i(\cdot)$, $i = 1,\ldots,n$*— in Property 1.2.4 is bounded, and consequently* $\|C(x,y)\| \leq k_C\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $k_C \geq 0$, *or equivalently* $\|C_i(x,y)\| \leq k_{Ci}\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $k_{Ci} \geq 0$, $i = 1,\ldots,n$.

Assumption 1.3. *The conservative (generalized) force vector* $g(q)$ *is a continuously differentiable bounded vector function with bounded Jacobian matrix* $\frac{\partial g}{\partial q}$*, which can be equivalently stated as follows.*

1.3.1 Every element of the conservative force vector, $g_i(q)$, $i = 1,\ldots,n$, *satisfies* $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, *for some* $B_{gi} > 0$.

1.3.2 $\frac{\partial g}{\partial q}$ *exists and is continuous and such that* $\left\|\frac{\partial g}{\partial q}(q)\right\| \leq k_g$, $\forall q \in \mathbb{R}^n$, *for some* $k_g > 0$, *and consequently* $\|g(x) - g(y)\| \leq k_g\|x - y\|$, $\forall x,y \in \mathbb{R}^n$.

Assumption 1.4. $T_i > B_{gi}$, $\forall i \in \{1,\ldots,n\}$.

Assumptions 1.1–1.3 apply *e.g.* for robot manipulators having only revolute joints [30]. Assumption 1.4 renders it possible to hold the system at any desired equilibrium configuration $q_d \in \mathbb{R}^n$.

Remark 1.2. By Property 1.1, the inverse matrix of $H(q)$, denoted $H^{-1}(q)$, exists and keeps analog analytical properties. More precisely, $H^{-1}(q)$ is a continuously differentiable positive definite matrix function, and actually, under the additional consideration of Assumption 1.1:

$$\left(\frac{1}{\mu_M}\right) I_n \leq H^{-1}(q) \leq \left(\frac{1}{\mu_m}\right) I_n \quad \Longrightarrow \quad \frac{1}{\mu_M} \leq \|H^{-1}(q)\| \leq \frac{1}{\mu_m}$$

$\forall q \in \mathbb{R}^n$, with $\mu_M \geq \mu_m$ being the positive constants characterized through Property 1.1 and Assumption 1.1.

Figure 1.1: 2-DOF robot manipulator at *Instituto Tecnológico de la Laguna*.

1.5 Robot manipulators: particular structures

The following model descriptions correspond to divers robot manipulators that will be subsequently involved throughout this dissertation.

1.5.1 2-DOF robot manipulator

The experimental robotic arm shown in Figure 1.1, is located at *Instituto Tecnológico de la Laguna* (Torreón, México). The arm movement relies on a vertical plane. The actuators are direct-drive brushless motors located at the shoulder (base) and at the elbow, which are operated in torque mode, so they act as torque source and accept an analog voltage as a reference of torque signal. The control algorithm is executed at a 2.5-ms sampling period in a control board (based on a DSP 32-bit floating-point microprocessor from Texas Instrument) mounted on a PC-host computer. The robot software is in open architecture, whose platform is based on C language to run the control algorithm in real time. The system dynamics of this robot is characterized, in accordance to Eq. (1.1), by

$$H(q) = \begin{pmatrix} 2.351 + 0.168 \cos q_2 & 0.102 + 0.084 \cos q_2 \\ 0.102 + 0.084 \cos q_2 & 0.102 \end{pmatrix}$$

$$C(q, \dot{q}) = \begin{pmatrix} -0.084 \dot{q}_2 \sin q_2 & -0.084(\dot{q}_1 + \dot{q}_2) \sin q_2 \\ 0.084 \dot{q}_1 \sin q_2 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} 38.465 \sin q_1 + 1.825 \sin(q_1 + q_2) \\ 1.825 \sin(q_1 + q_2) \end{pmatrix}$$

$$F = \begin{pmatrix} 2.288 & 0 \\ 0 & 0.175 \end{pmatrix}$$

Property 1.1 and Assumptions 1.1–1.3 are thus satisfied with $\mu_m = 0.088$ kg m^2, $\mu_M = 2.533$ kg m^2, $k_C = 0.1422$ kg m^2/s, $B_{g1} = 40.29$ N m, and $B_{g2} = 1.825$ N m. More precisely, $\mu_{M1} = 2.5259$ kg m^2, $\mu_{M2} = 0.2121$ kg m^2, $k_{C1} = 0.1359$ kg m^2/s and $k_{C2} = 0.084$ kg m^2/s (in Appendix A, a detailed analysis to obtain the listed bounds is shown). Furthermore, the input saturation bounds are $T_1 = 150$ N m, and $T_2 = 15$ N m, for the first and second links, respectively.

1.5.2 3-DOF robot manipulators

Anthropomorphic-type arm

The experimental robotic arm, shown in Figure 1.2, is a 3-revolute-joint anthropomorphic-type arm located at the *Benemérita Universidad Autónoma de Puebla*, Mexico. The arm links —made of 6061 aluminum— are actuated by direct-drive brushless servomotors from *Parker Compumotor* —models: DM1015B (base), DM1050A (shoulder) and DM1004C (elbow)— operated in torque mode, *i.e.*, they act as ideal torque sources —without gear reduction— upon reception of an analogue voltage as torque reference signal. Joint positions are measured using incremental encoders on the motors and the standard backwards difference algorithm was used to calculate the velocity signals. The control algorithm is written in C language and executed at a 2.5 ms sampling period on a PC host computer. The dynamical characterization of this robot, in accordance to Eq. (1.1), is described by

$$H(q) = \begin{pmatrix} h_{11}(q) & h_{12}(q) & h_{13}(q) \\ h_{12}(q) & h_{22}(q) & h_{23}(q) \\ h_{13}(q) & h_{23}(q) & \zeta_{13} \end{pmatrix}$$

with

Figure 1.2: 3-DOF robot manipulator at *Benemérita Universidad Autónoma de Puebla*.

$$
\begin{aligned}
h_{11}(q) &= \zeta_1 + \zeta_2 \cos^2 q_2 + \zeta_3 \sin 2q_2 + \zeta_4 \sin(2q_2 + 2q_3) + \zeta_5 \cos^2(q_2 + q_3) \\
&\quad + 2l_2\zeta_6 \sin q_2 \sin(q_2 + q_3) + 2l_2\zeta_7 \sin q_2 \cos(q_2 + q_3) \\
h_{12}(q) &= h_{13}(q) + \zeta_{10} \cos q_2 + \zeta_{11} \sin q_2 \\
h_{13}(q) &= \zeta_8 \cos(q_2 + q_3) + \zeta_9 \sin(q_2 + q_3) \\
h_{22}(q) &= \zeta_{12} + 2l_2\zeta_6 \cos q_3 - 2l_2\zeta_7 \sin q_3 \\
h_{23}(q) &= \zeta_{13} + l_2\zeta_6 \cos q_3 - l_2\zeta_7 \sin q_3
\end{aligned}
$$

$$
C(q,\dot{q}) = \begin{pmatrix} a_1(q)\dot{q}_2 + a_2(q)\dot{q}_3 & a_1(q)\dot{q}_1 + a_3(q)\dot{q}_2 + a_4(q)\dot{q}_3 & a_2(q)\dot{q}_1 + a_4(q)\big(\dot{q}_2 + \dot{q}_3\big) \\ -a_1(q)\dot{q}_1 & -a_5(q)\dot{q}_3 & -a_5(q)\big(\dot{q}_2 + \dot{q}_3\big) \\ -a_2(q)\dot{q}_1 & a_5(q)\dot{q}_2 & 0 \end{pmatrix}:
$$

$$
\begin{aligned}
a_1(q) &= -\frac{\zeta_2}{2} \sin 2q_2 + \zeta_3 \cos 2q_2 + \zeta_4 \cos(2q_2 + 2q_3) - \frac{\zeta_5}{2} \sin(2q_2 + 2q_3) \\
&\quad + l_2\zeta_6 \sin(2q_2 + q_3) + l_2\zeta_7 \cos(2q_2 + q_3) \\
a_2(q) &= \zeta_4 \cos(2q_2 + 2q_3) - \frac{\zeta_5}{2} \sin(2q_2 + 2q_3) + l_2\zeta_6 \sin(q_2) \cos(q_2 + q_3) \\
&\quad - l_2\zeta_7 \sin(q_2) \sin(q_2 + q_3) \\
a_3(q) &= a_4(q) - \zeta_{10}\dot{q}_2 \sin q_2 + \zeta_{11}\dot{q}_2 \cos q_2 \\
a_4(q) &= -\zeta_8 \sin(q_2 + q_3) + \zeta_9 \cos(q_2 + q_3) \\
a_5(q) &= l_2\big(\zeta_6 \sin q_3 + \zeta_7 \cos q_3\big)
\end{aligned}
$$

$$
g(q) = \begin{pmatrix} 0 \\ g_0\big(\zeta_6 \sin(q_2 + q_3) + \zeta_7 \cos(q_2 + q_3)\big) + \zeta_{14} \sin q_2 + \zeta_{14} \cos q_2 \\ g_0\big(\zeta_6 \sin(q_2 + q_3) + \zeta_7 \cos(q_2 + q_3)\big) \end{pmatrix}
$$

$$
F = \begin{pmatrix} \zeta_{16} & 0 & 0 \\ 0 & \zeta_{17} & 0 \\ 0 & 0 & \zeta_{18} \end{pmatrix}
$$

In the expressions above: $l_2 = 0.35$ m and $g_0 = 9.81 \mathrm{m/s^2}$. The system parameters ζ_i, $i = \overline{1,18}$, were identified using the recursive-least-squares-based filtered dynamic model method [31], giving the following set of estimations:

$\zeta_1 = 2.8968$	$\zeta_7 = 6.828 \times 10^{-3}$	$\zeta_{13} = 8.9485 \times 10^{-2}$
$\zeta_2 = -0.35456$	$\zeta_8 = 7.9808 \times 10^{-2}$	$\zeta_{14} = 16.4906$
$\zeta_3 = 7.6885 \times 10^{-4}$	$\zeta_9 = -4.5168 \times 10^{-3}$	$\zeta_{15} = 0.202478$
$\zeta_4 = -6.5428 \times 10^{-5}$	$\zeta_{10} = 0.89111$	$\zeta_{16} = 0.4$
$\zeta_5 = -5.5442 \times 10^{-3}$	$\zeta_{11} = 2.5675 \times 10^{-3}$	$\zeta_{17} = 1.2806$
$\zeta_6 = 0.11152$	$\zeta_{12} = 1.2607$	$\zeta_{18} = 0.64$

For such a robot, Property 1.1 and Assumptions 1.1–1.3 are satisfied with $\mu_m = 0.0761$ kg m^2, $\mu_M = 3.0846$ kg m^2, $k_C = 1.1116$ kg m^2/s, $B_{g1} = 0$, $B_{g2} = 17.5879$ N m, and $B_{g3} = 1.09606$ N m, and more precisely: $\mu_{M1} = 2.975$ kg m^2, $\mu_{M2} = 1.6589$ kg m^2, $\mu_{M3} = 0.1757$ kg m^2, $k_{C1} = 0.989$ kg m^2/s, $k_{C2} = 0.4681$ kg m^2/s and $k_{C3} = 0.1997$ kg m^2/s (the analysis to obtain the listed bounds can be found in Appendix A). Furthermore, the input saturation bounds are $T_1 = 15$ N m, $T_2 = 50$ N m, and $T_2 = 4$ N m, for the first, second and third links, respectively.

Geometric Touch Haptic Device

Figure 1.3: 3-DOF Geometric Touch Haptic Device at *Universidad de Guadalajara*.

The experimental setup, shown in Figure 1.3, is a haptic device (called Geometric Touch Haptic Device), located at *Universidad de Guadalajara*, usually involved in 3D modelling [32]. This haptic device is involved in a consensus work presented in [33], where technical characteristics of such device are specified. We reproduced here, from [33], the gravity (conservative) force vector since it is the only force vector (from the system model) required for regulation experiments. Its model takes the form:

$$g(q) = \begin{pmatrix} 0 \\ 105\sin(q_2 + q_3) + 137\cos q_2 \\ 105\sin(q_2 + q_3) \end{pmatrix} \times 10^{-4}$$

whence the values introduced through Assumption 1.3 are obtained as $B_{g1} = 0$, $B_{g2} = 242 \times 10^{-4}$ N m, and $B_{g3} = 105 \times 10^{-4}$ N m, while $k_g = 299 \times 10^{-4}$ Nm/rad. Furthermore, the input saturation bounds are $T_j = 1$ N m, $j = 1, 2, 3$.

Phantom haptic interface robot

Figure 1.4 shows the Phantom$^{\text{TM}}$ (model 1.5) haptic interface robot, whose dynamic model will be used for simulation tests in this dissertation. Thorough technical description and model derivation of such a 3-DOF robotic device are presented in [34]. The elements H, C and g involved in (1.1) are reproduced here,

$$H(q) = \begin{pmatrix} h_{11}(q) & 0 & 0 \\ 0 & 24.26 & -4.56\sin(q_2 - q_3) \\ 0 & -4.56\sin(q_2 - q_3) & 9.32 \end{pmatrix} \times 10^{-4}$$

Figure 1.4: 3-DOF Phantom haptic interface robot.

with

$$h_{11}(q) = 28.33 + 11.32\cos(2q_2) - 3.91\cos(2q_3) + 9.12\cos(q_2)\sin(q_3)$$

$$C(q,\dot{q}) = \begin{pmatrix} c_{11}(q,\dot{q}) & c_{12}(q,\dot{q}) & c_{13}(q,\dot{q}) \\ c_{21}(q,\dot{q}) & 0 & c_{23}(q,\dot{q}) \\ c_{31}(q,\dot{q}) & c_{32}(q,\dot{q}) & 0 \end{pmatrix} \times 10^{-4}$$

where

$$\begin{aligned} c_{11}(q,\dot{q}) &= -[11.32\sin(2q_2) + 4.56\sin(q_2)\sin(q_3)]\dot{q}_2 \\ &\quad + [3.91\sin(2q_3) + 4.56\cos(q_2)\cos(q_3)]\dot{q}_3 \\ c_{12}(q,\dot{q}) &= -c_{21}(q,\dot{q}) = -[11.32\sin(2q_2) + 4.56\sin(q_2)\sin(q_3)]\dot{q}_1 \\ c_{13}(q,\dot{q}) &= -c_{31}(q,\dot{q}) = [3.91\sin(2q_3) + 4.56\cos(q_2)\cos(q_3)]\dot{q}_1 \\ c_{23}(q,\dot{q}) &= 4.56\cos(q_2 - q_3)\dot{q}_3 \\ c_{32}(q,\dot{q}) &= -4.56\cos(q_2 - q_3)\dot{q}_2 \end{aligned}$$

$$g(q) = \begin{pmatrix} 0 \\ -162.98\cos(q_2) \\ -737.55\sin(q_3) \end{pmatrix} \times 10^{-4}$$

For such a robot, Property 1.1 and Assumptions 1.1–1.3 are satisfied with $\mu_m = 8.04 \times 10^{-4}$ kg m^2, $\mu_M = 53 \times 10^{-4}$ kg m^2, $k_C = 24 \times 10^{-4}$ kg m^2/s, $B_{g1} = 0$, $B_{g2} = 162.98 \times 10^{-4}$ N m, and $B_{g3} = 737.55 \times 10^{-4}$ N m, and more precisely: $\mu_{M1} = 53 \times 10^{-4}$ kg m^2, $\mu_{M2} = 25 \times 10^{-4}$ kg m^2, $\mu_{M3} = 10 \times 10^{-4}$ kg m^2, $k_{C1} = k_{C2} = 15 \times 10^{-4}$ kg m^2/s, and $k_{C3} = 7.41 \times 10^{-4}$ kg m^2/s (the analysis to obtain the listed bounds can be found in Appendix A). Furthermore, the input saturation bounds are $T_j = 1.8$ N m, $j = 1, 2, 3$.

1.6 Structure of the dissertation

The work is organized as follows: Chapter 2 presents definitions and results that were used in the analyzes developed in this dissertation. In addition, the models of divers multi-degree-of-freedom robot manipulators are presented.

Chapter 3 introduces the proposed finite-time/exponential control schemes and develops the corresponding closed-loop stability proofs. First, the results corresponding to position control with on-line compensation are shown in the case of availability of all system states. Then a control scheme that does not involve system velocities is presented. Further, the desired-conservative force compensation case of the position control problem is developed including both the state-feedback and the output-feedback approaches. Moreover, a trajectory tracking controller is presented assuming availability of states. Finally, the control law proposed in the tracking problem is involved in a robustness study.

Throughout Chapter 4 the simulation results of all developed schemes are presented including the nominal dynamic models of the robot manipulators described in Chapter 2. For each proposed scheme, a comparison among the two types of convergence, finite-time *vs* exponential, is presented.

Chapter 5 shows experimental results obtained using the different robot manipulators described in Chapter 2. Through the experiments, the aim is to corroborate the capability of the proposed control laws and the conclusions drawn from the analytical results presented in Chapter 3, as well as to compare the different types of convergence (between finite-time and exponential).

Conclusions, future work and perspectives are also given in Chapter 6.

CHAPTER 2

Mathematical background

This part settles down the framework within which the analysis of this work is developed.

2.1 Lyapunov stability

Consider a system of the form

$$\dot{x} = f(t, x) \tag{2.1}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and continuous in x, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$; for any given initial condition (t_0, x_0), $x(t)$ denotes a solution of (2.1) such that $x(t_0) = x_0$ and the set of all solutions is denoted by $\mathcal{S}_{t_0,x_0}$. Then, we recall the following definition [35].

Definition 2.1. *The equilibrium point $x = 0$ of (2.1) is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that for all $\|x(t_0)\| < \delta$ and all solutions $x(t) \in \mathcal{S}_{t_0,x_0}$ one has that*

$$\|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0 \tag{2.2}$$

- *uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that for each $\|x(t_0)\| < \delta$ and all solutions $x(t) \in \mathcal{S}_{t_0,x_0}$, (2.2) is satisfied.*

- *unstable if is it not stable.*

- *asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$ and all solutions $x(t) \in \mathcal{S}_{t_0,x_0}$.*

- *uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < c$ and all solutions $x(t) \in \mathcal{S}_{t_0,x_0}$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall\, t_0 + T(\eta)$$

for all $\|x(t_0)\| < c$ *and all solutions* $x(t) \in \mathcal{S}_{t_0,x_0}$.

- *globally uniformly asymptotically stable if it is uniformly stable,* $\delta(\varepsilon)$ *can be chosen to satisfy* $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$*, and, for each pair of positive numbers* η *and* c*, there is* $T = T(\eta) > 0$ *such that*

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta, c)$$

for all $\|x(t_0)\| < c$ *and all solutions* $x(t) \in \mathcal{S}_{t_0,x_0}$.

Remark 2.1. When the system is autonomous, *i.e.*, for any $x \in D$: $f(t,x) \equiv f(x)$ in (2.1), the stability properties are uniform in t_0, which is understood without the need to explicitly specify the term *uniformly* in the given terminology.

2.2 Homogeneity

The concept of *homogeneity* whose definition is presented next as a conventional notion, more precisely corresponds to that of *weighted homogeneity* [36], which is related to the family of dilations δ_ε^r, defined as $\delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1, \ldots, r_n)^T$, where the dilation coefficients $r_1, \ldots, r_n$ are positive scalars.

Definition 2.2. *[35]*

- *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *is* δ^r*–homogeneous of degree* m *(*$m \in \mathbb{R}$*) if*

$$V(\delta_\varepsilon^r(x)) = \varepsilon^m V(x) \quad \forall\, x \in \mathbb{R}^n,\, \forall\, \varepsilon > 0$$

- *A vector field* $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ *is said to be* δ^r*–homogeneous of degree* k *if the component* f_i *is* δ^r*–homogeneous of degree* $k + r_i$*, for each* i*; that is,*

$$f_i(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^{k+r_i} f_i(x), \quad \forall\, x \in \mathbb{R}^n,\, \forall\, \varepsilon > 0,\, \forall\, i \in \{1, \ldots, n\}$$

The next definition states a local notion of homogeneity.

Definition 2.3. *[16]: A function* $V : \mathbb{R}^n \to \mathbb{R}$*, resp. a vector field* $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$*, is locally homogeneous of degree* α *with respect to the family of dilations* δ_ε^r *—or equivalently, it is said to be locally* r*-homogeneous of degree* α*— if there exists an open neighborhood of the origin* $D \subset \mathbb{R}^n$ *—referred to as the domain of homogeneity— such that, for every* $x \in D$ *and all* $\varepsilon \in (0,1] : \delta_\varepsilon^r(x) \in D$ *and*

$$V(\delta_\varepsilon^r(x)) = \varepsilon^\alpha V(x)$$

resp.

$$f_i(\delta_\varepsilon^r(x)) = \varepsilon^{\alpha + r_i} f_i(x)$$

$i = 1, \ldots, n.$

Definition 2.4. *[37] Given $r \in \mathbb{R}^n_{>0}$, a continuous function mapping $x \in \mathbb{R}^n$ to $\mathbb{R}$, denoted $\|x\|_r$, is called a homogeneous norm with respect to the family of dilations δ^r_ε —or equivalently, it is said to be an r-homogeneous norm— if $\|x\|_r \geq 0$ with $\|x\|_r = 0 \Leftrightarrow x = 0_n$, and $\|\delta^r_\varepsilon(x)\|_r = \varepsilon\|x\|_r$ for any $\varepsilon > 0$.*

Notice that an r-homogeneous norm is a positive definite continuous function being r-homogeneous of degree 1. A special subset of r-homogeneous norms is defined as follows.

Definition 2.5. *Given $r \in \mathbb{R}^n_{>0}$, an* r-homogeneous *p-norm ($p \geq 1$) is defined as*

$$\|x\|_{r,p} = \left[\sum_{i=1}^{n} |x_i|^{p/r_i}\right]^{1/p}$$

Subsequently an r-homogeneous norm $\|\cdot\|_r$ will be considered to refer to an r-homogeneous p-norm with $p > \max_i\{r_i\}$.

Lemma 2.1. *[16] Suppose that, for every $i = 1, 2$, V_i is a continuous scalar function being locally r-homogeneous of degree $\alpha_i > 0$, with domain of homogeneity $\mathcal{D}_i$. Suppose further that V_1 is positive definite on $\mathcal{D}_1$. Let $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ and consider an $(n-1)$-dimensional sphere $\mathcal{S}^{n-1}_c$ of some radius $c > 0$ such that $\mathcal{S}^{n-1}_c \subset \mathcal{D}$. Then, for every $x \in \mathcal{D}$: $c_1[V_1(x)]^{\alpha_2/\alpha_1} \leq V_2(x) \leq c_2[V_1(x)]^{\alpha_2/\alpha_1}$, with $c_1 = [\min_{z\in\mathcal{S}^{n-1}_c} V_2(z)] \cdot [\max_{z\in\mathcal{S}^{n-1}_c} V_1(z)]^{-\alpha_2/\alpha_1}$ and $c_2 = [\max_{z\in\mathcal{S}^{n-1}_c} V_2(z)] \cdot [\min_{z\in\mathcal{S}^{n-1}_c} V_1(z)]^{-\alpha_2/\alpha_1}$.*

Remark 2.2. Observe that if V_2 happens to be positive —resp. negative— definite, then c_1 and c_2 in Lemma 2.1 are both positive —resp. negative— constants. △

2.3 Finite-time and δ-exponential stability

Consider an nth order autonomous system

$$\dot{x} = f(x) \tag{2.3}$$

where $f : D \to \mathbb{R}^n$ is continuous on an open neighborhood of the origin $D \subset \mathbb{R}^n$ and $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$.

Definition 2.6. *[15]: The origin is said to be a finite-time stable equilibrium of system* (2.3) *if it is Lyapunov stable and there exist an open neighborhood of the origin, $\mathcal{N} \subset \mathcal{D}$, being positively invariant with respect to* (2.3)*, and a positive definite function $T : \mathcal{N} \to \mathbb{R}_{\geq 0}$, called the settling-time function, such that $x(t; x_0) \neq 0_n$, $\forall t \in [0, T(x_0))$, $\forall x_0 \in \mathcal{N} \setminus \{0_n\}$, and $x(T(x_0); x_0) = 0_n$, $\forall x_0 \in \mathcal{N}$. The origin is said to be a globally finite-time stable equilibrium if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.*

Remark 2.3. Note, from Definition 2.6, that the origin is a globally finite-time stable equilibrium of system (2.3) if and only if it is globally asymptotically stable and finite-time stable.

Theorem 2.1. *[16]: Consider system* (2.3) *with* $D = \mathbb{R}^n$*. Suppose that* f *is a locally* r*-homogeneous vector field of degree* α *with domain of homogeneity* $D \subset \mathbb{R}^n$*. Then, the origin is a globally finite-time stable equilibrium of system* (2.3) *if and only if it is globally asymptotically stable and* $\alpha < 0$*.*

The next definition is stated under the additional consideration that, for some $r \in \mathbb{R}^n_{>0}$, f in (2.3) is locally r-homogeneous with domain of homogeneity $D \subset \mathcal{D}$.

Definition 2.7. *[38, 37]: The equilibrium point* $x = 0_n$ *of* (2.3) *is* δ*-exponentially stable with respect to the homogeneous norm* $\|\cdot\|_r$ *if there exist a neighborhood of the origin,* $\mathcal{V} \subset D$*, and constants* $a \geq 1$ *and* $b > 0$ *such that*

$$\|x(t;x_0)\|_r \leq a\|x_0\|_r e^{-bt}, \qquad \forall x_0 \in \mathcal{V}$$

Remark 2.4. Observe that Definition 2.7 becomes equivalent to the usual definition of exponential stability when the standard dilation is concerned, *i.e.*, when $r_i = 1$, $i = 1, \ldots, n$.

The next lemma is a trivial extension to the local homogeneity context of [38, Lemma 2.4]. Analogously to [38, Lemma 2.4], it is stated under the additional consideration that solutions of (2.3) with $x_0 \in D$ remain unique (while belonging to D).

Lemma 2.2. *Suppose that* f *in* (2.3) *is a locally* r*-homogeneous vector field of degree* $\alpha = 0$ *with domain of homogeneity* $D \subset \mathcal{D}$*. Then, the origin is a* δ*-exponentially stable equilibrium if and only if it is asymptotically stable.*

Remark 2.5. Observe that the assumptions of Lemma 2.2 imply the existence of a neighborhood of the origin $\mathcal{V} \subset D$ such that $x_0 \in \mathcal{V} \Rightarrow x(t;x_0) \in D$, $\forall t \geq 0$. The proof of Lemma 2.2 is thus analogous to the one developed in [39] for the special case of $r = (r_1, \ldots, r_n)^T$ with $r_i = 1$, $i = 1, \ldots, n$. One further concludes from [39] that asymptotic stability for $\alpha > 0$ is not δ-exponential (*i.e.*, δ-exponential stability is a property that can only take place when $\alpha = 0$).

Remark 2.6. Let us note that if a vector field f is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then f is locally r^*-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i^* = r_0^*$, $\forall i \in \{1, \ldots, n\}$, for any $r_0^* > 0$. Indeed, observe that if, for every $x \in D$, $f(\varepsilon^{r_0}x) = \varepsilon^{r_0} f(x)$, $\forall \varepsilon \in (0, 1]$, then, by taking $\epsilon = \varepsilon^{r_0/r_0^*}$, we have that $f(\epsilon^{r_0^*}x) = \epsilon^{r_0^*} f(x)$, $\forall \epsilon \in (0, 1]$. Consequently, if f in (2.3) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then (under the consideration of Remark 2.4), the origin turns out to be exponentially stable if and only if it is δ-exponentially stable.

Consider and n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{2.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector fields such that $f(0_n) = \hat{f}(0_n) = 0_n$, The next result is an extended version of [16, Lemma 3.2].

Lemma 2.3. *Suppose that, for some $r \in \mathbb{R}^n_{>0}$, f in* (2.4) *is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a global asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system* (2.4) *if*

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta^r_\varepsilon(x))}{\varepsilon^{\alpha + r_i}} = 0 \tag{2.5}$$

$i = 1, \dots, n$, $\forall x \in S^{n-1}_c$, resp. $\forall x \in S^{n-1}_{r,c}$, for some $c > 0$ such that $S^{n-1}_c \subset D$, resp. $S^{n-1}_{r,c} \subset D$.

Remark 2.7. Notice that the condition required by Lemma 2.3 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\left[\varepsilon^{-r_1}, \dots, \varepsilon^{-r_n}\right] \hat{f}(\delta^r_\varepsilon(x)) \right\| = 0 \tag{2.6}$$

$\forall x \in S^{n-1}_c$ (resp. $\forall x \in S^{n-1}_{r,c}$). In other words, (2.5) is fulfilled for all $i = 1, \dots, n$ and all $x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$) if and only if (2.6) is satisfied for all $x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$).

2.4 Uniform finite-time stability

Consider an n-th order non-autonomous system

$$\dot{x} = f(t, x) \tag{2.7}$$

where $f : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$ is continuous, $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0_n$, and $f(t, 0_n) = 0_n$, $\forall t \geq 0$. We denote $x(t; t_0, x_0)$ —or simply $x(t)$ whenever convenient or clear from the context— a solution of (2.7) with initial condition $x(t_0; t_0, x_0) = x_0 \in \mathcal{D}$ at initial time $t_0 \geq 0$, and $\mathcal{S}(t_0, x_0)$ is the set of all solutions $x(t; t_0, x_0)$ starting from $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$.

Definition 2.8. *[17] The equilibrium point $x = 0_n$ of* (2.7) *is*

- *weakly finite-time stable if:*

  1) it is Lyapunov stable;

  2) for each $t_0 \geq 0$, there exists $\delta = \delta(t_0) > 0$ such that, if $\|x_0\| < \delta$ then, for all $x(t) \in \mathcal{S}(t_0, x_0)$:

  a) $x(t)$ is defined for all $t \geq t_0 \geq 0$;

  b) $\exists\, T \in [0, \infty)$ such that $x(t) = 0_n$, $\forall t \geq t_0 + T$.

  $T_0[x(t; t_0, x_0)] \triangleq \inf\{T \geq 0 : x(t; t_0, x_0) = 0_n, \forall t \geq t_0 + T\}$ *is called the settling time of $x(t; t_0, x_0)$.*

- *finite-time stable if, in addition to items 1) and 2) above:*

  3) $T_0(t_0, x_0) \triangleq \sup_{x(t) \in \mathcal{S}(t_0, x_0)} T_0[x(t)] < \infty$.

$T_0(t_0, x_0)$ is called the settling time with respect to initial conditions (at (t_0, x_0)).

Remark 2.8. If f in (2.7) is locally Lipschitz-continuous in x on $\mathcal{D} \setminus \{0_n\}$ (uniformly in t on $\mathbb{R}_{\geq 0}$) then, by uniqueness of solutions, the settling time of a solution $x(t; t_0, x_0)$ and the settling time with respect to initial conditions at (t_0, x_0) are the same, *i.e.*, $T_0(t_0, x_0) = T_0[x(t; t_0, x_0)]$. △

Definition 2.9. *[17] The equilibrium point $x = 0_n$ of (2.7) is uniformly finite-time stable if:*

1) it is uniformly asymptotically stable;

2) it is finite-time stable;

3) there exists a positive definite continuous function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the settling time with respect to initial conditions satisfies $T_0(t_0, x_0) \leq \varphi(\|x_0\|)$.

Theorem 2.2. *[17] Let $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that $W_1(x) \leq V(t, x) \leq W_2(x)$ and $\dot{V}(t, x) \leq -\nu(V(t, x))$, $\forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on $\mathcal{D}$, $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f$, and $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a positive definite continuous function such that $\int_0^{\varpi} \frac{dz}{\nu(z)} < \infty$, for some $\varpi > 0$. Then $x = 0_n$ is uniformly finite-time stable.*

Remark 2.9. With $\nu(z) = cz^{\alpha}$, for any $c > 0$ and $\alpha \in (0, 1)$, we have $\int_0^{\varpi} \frac{dz}{\nu(z)} = \frac{\varpi^{1-\alpha}}{c(1-\alpha)} < \infty$, for any $\varpi \in (0, \infty)$. This special case generates a natural or direct extension to time-varying vector fields of the celebrated Lyapunov-type criterion stated for autonomous systems in [14]. △

Remark 2.10. The stability properties stated through Definitions 2.8 and 2.9 are global if $\mathcal{D} = \mathbb{R}^n$ and items a)–b) in Definition 2.8 are satisfied for any $x(t_0) = x_0 \in \mathbb{R}^n$. Moreover, one notes from Definition 2.9 that an equilibrium may be concluded to be globally uniformly finite-time stable if it is globally uniformly asymptotically stable and uniformly finite-time stable. △

2.5 Passivity

Consider the following dynamical system

$$\dot{x} = f(x, u) \tag{2.8a}$$
$$y = h(x, u) \tag{2.8b}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ being continuous, $f(x, u)$ locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m \setminus (0_n, 0_m)$, $f(0_n, 0_m) = 0_n$ and $h(0_n, 0_m) = 0_m$. The following definitions were reproduced from [40].

Definition 2.10. *The system represented by the state model in Eqs.* (2.8) *is said to be passive if there exists a continuously differentiable positive semidefinite function* $V(x)$ *(called the storage function) such that*

$$\dot{V}(x,u) = \frac{\partial V}{\partial x} f(x,u) \leq u^T y$$

$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. *Moreover, it is said to be*

- *lossless if* $\dot{V}(x,u) = u^T y$;
- *input strictly passive if* $\dot{V}(x,u) \leq u^T y - u^T \phi(u)$ *for some function* $\phi : \mathbb{R}^m \to \mathbb{R}^m$ *such that* $u^T \phi(u) > 0$, $\forall u \neq 0_m$;
- *output strictly passive if* $\dot{V}(x,u) \leq u^T y - y^T \rho(y)$ *for some function* $\rho : \mathbb{R}^m \to \mathbb{R}^m$ *such that* $y^T \rho(y) > 0$, $\forall y \neq 0_m$;
- *strictly passive if* $\dot{V}(x,u) \leq u^T y - \psi(x)$ *for some positive definite function* $\psi : \mathbb{R}^n \to \mathbb{R}$.

Definition 2.11. *The system represented by the state model in Eqs.* (2.8) *is said to be zero-state observable, if no solution of* $\dot{x} = f(x, 0_m)$ *can stay identically in* $S = \{x \in \mathbb{R}^n : h(x, 0_m) = 0_m\}$, *other than the trivial solution* $x(t) \equiv 0_n$ *(or equivalently* $u(t) \equiv y(t) \equiv 0_m \Rightarrow x(t) \equiv 0_n$*).*

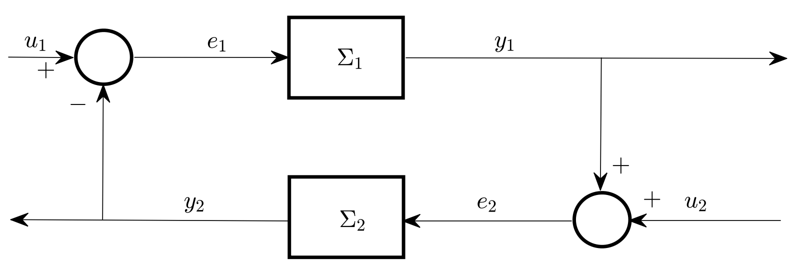

Figure 2.1: Feedback connection.

Consider the feedback system in Figure 2.1, where each feedback component Σ_i, $i = 1, 2$, is represented by the state model

$$\dot{x}_i = f_i(x_i, e_i)$$
$$y_i = h_i(x_i, e_i)$$

with $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ and $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^m$ being continuous, $f_i(x_i, e_i)$ locally Lipschitz on $\mathbb{R}^{n_i} \times \mathbb{R}^m \setminus (0_{n_i}, 0_m)$, $f_i(0_{n_i}, 0_m) = 0_{n_i}$ and $h_i(0_{n_i}, 0_m) = 0_m$. We will consider that the feedback connection is well-defined. We state the following feedback-system passivity theorem.

Theorem 2.3. *For the considered feedback connection with $u_1 = u_2 = 0_m$, the origin of the consequent closed-loop system, $(x_1, x_2) = (0_{n_1}, 0_{n_2})$, is asymptotically stable if, for some $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$ (or equivalently $j = i - (-1)^i$), Σ_i is zero-state observable and passive with positive definite storage function, Σ_j is strictly passive and*

$$f_j(0_{n_j}, e_j) = 0_{n_j} \Rightarrow e_j = 0_m$$

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable.

2.6 Uniform ultimate boundedness

The concept of (global) uniform ultimate boundedness, as stated and characterized for instance in [41] or [40], will be involved in this work. In particular, the following alternative version of the Lyapunov-function-candidate-based criterion presented as Theorem 10.4 in [41], stating an analog global result that requires the local characterization of Theorem 4.18 from [40] to get an estimate of the ultimate bound, will be involved in this work.

Let us consider an n-th order non-autonomous system

$$\dot{x} = f(t, x) \tag{2.10}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We denote $x(t; t_0, x_0)$ —or simply $x(t)$ whenever convenient or clear from the context— a solution of (2.10) with initial condition $x(t_0; t_0, x_0) = x_0 \in \mathbb{R}^n$ at initial time $t_0 \geq 0$.

Corollary 2.1. *Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that, for all $t \geq 0$ and some $\mu > 0$:*

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$\forall x \in \mathbb{R}^n$, and

$$\dot{V}(t, x) \leq -W_3(x)$$

$\forall \|x\| \geq \mu$, where $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f$, and $W_i(x)$, $i = 1, 2, 3$, are continuous positive definite functions, with W_1 (and consequently W_2) being additionally radially unbounded, and let us further suppose that there exist class $\mathcal{K}$ functions $\alpha_i : [0, r] \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, such that $W_1(x) \geq \alpha_1(\|x\|)$ and $W_2(x) \leq \alpha_2(\|x\|)$, $\forall x \in \mathcal{B}_r^n$, for some $r \geq \alpha_1^{-1}(\alpha_2(\mu))$. Then, for any $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, every solution of (2.10), $x(t; t_0, x_0)$, is uniformly ultimately bounded with ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$, i.e., such that $\|x(t; t_0, x_0)\| \leq \alpha_1^{-1}(\alpha_2(\mu))$, $\forall t \geq t_0 + T$, for some $T \in [0, \infty)$.

By noting that $\|x\| \leq \mu \Rightarrow V(t, x) \leq \alpha_2(\mu)$, $\forall t \geq 0$, one sees that $\mathbb{R}_{\geq 0} \times \mathcal{B}_\mu^n \subset \{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n : V(t, x) \leq \alpha_2(\mu)\}$, while the consideration of the positive definite and decrescent characters of V and negativity of $\dot{V}$ on $\{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n : \|x\| \geq \mu\}$ ensures that, for any $t_0 \geq 0$ and $\|x_0\| \geq \mu$, every solution instantaneously evolves in a decreasing direction of $V(t, x(t))$ (as long as $\|x(t; t_0, x_0)\| \geq \mu$), and consequently, for any $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, there is a finite time $t_0 + T \geq t_0$ such that $V(t, x(t)) < \alpha_2(\mu)\ \forall t \geq t_0 + T$, which implies

$$\alpha_1(\|x(t)\|) < \alpha_2(\mu) \Longleftrightarrow \|x(t)\| < \alpha_1^{-1}(\alpha_2(\mu))$$

$\forall t \geq t_0 + T$, and consequently $\{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n : V(t,x) \leq \alpha_2(\mu)\} \subset \mathbb{R}_{\geq 0} \times \mathcal{B}^n_{\alpha_1^{-1}(\alpha_2(\mu))}$, whence one sees that r shall be greater than or equal to $\alpha_1^{-1}(\alpha_2(\mu))$.

Corollary 2.1 relaxes the estimation of the ultimate bound when explicit class $\mathcal{K}_\infty$ functions $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, (as required by Theorem 4.18 from [40] in the global case) are difficult to be obtained, while getting explicit class $\mathcal{K}$ functions $\alpha_i : [0, r] \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, proves to be easier. It is worth further adding that V in Corollary 2.1 does not really need to be continuously differentiable but that, as in [41], it can be a locally Lipschitz continuous function and $\dot{V}$ be obtained using the upper-right (Dini) derivative. Finally, from the analytical framework of uniform ultimate boundedness [41], [40] it is evident that uniformly ultimately bounded system solutions $x(t; t_0, x_0)$ are defined for all $t \geq t_0$.

2.7 Scalar functions with particular properties

Definition 2.12. *A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ is said to be:*

1) *positively upper bounded (by M^+) if $\sigma(\varsigma) \leq M^+$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M^+;*

2) *negatively lower bounded (by $-M^-$) if $\sigma(\varsigma) \geq -M^-$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M^-;*

3) *bounded (by M) if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;*

4) *strictly passive if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;*

5) *strongly passive —for (κ, a, b)— if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa |b \operatorname{sat}(\varsigma/b)|^a = \kappa(\min\{|\varsigma|, b\})^a$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b;*

6) *bounded strongly passive —for $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$— if it is a strongly passive function for (κ, a, b) such that $|\sigma(\varsigma)| \leq \bar{\kappa} |\bar{b} \operatorname{sat}(\varsigma/\bar{b})|^{\bar{a}} = \bar{\kappa}(\min\{|\varsigma|, \bar{b}\})^{\bar{a}}$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a, b, $\bar{\kappa}$, $\bar{a}$ and $\bar{b}$.*

Remark 2.11. Equivalent characterizations of strictly passive functions are:

$$\varsigma\sigma(\varsigma) > 0 \iff \operatorname{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \operatorname{sign}(\sigma(\varsigma)) = \operatorname{sign}(\varsigma) \quad \forall \varsigma$$

Remark 2.12. Let us note that a non-decreasing strictly passive function σ is strongly passive. Indeed, notice that the striclty passive character of σ implies the existence of a sufficiently small $a > 0$ such that $|\sigma(\varsigma)| \geq \kappa |\varsigma|^b$, $\forall |\varsigma| \leq a$, for some positive constants κ and b, while from its non-decreasing character, we have that $|\sigma(\varsigma)| \geq |\sigma(\operatorname{sign}(\varsigma) a)| \geq \kappa a^b$, $\forall |\varsigma| \geq a$, and thus $|\sigma(\varsigma)| \geq \kappa(\min\{|\varsigma|, a\})^b = \kappa |a \operatorname{sat}(\varsigma/a)|^b$, $\forall \varsigma \in \mathbb{R}$.

Lemma 2.4. *Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then,*

1) $\int_0^\varsigma \sigma(k\nu)d\nu > 0$, $\forall \varsigma \neq 0$*;*

2) $\int_0^\varsigma \sigma(k\nu)d\nu \to \infty$ *as* $|\varsigma| \to \infty$*;*

3) $\sigma_0 \circ \sigma_1$ *is strongly passive.*

Lemma 2.5. *Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function, $\sigma_2 : \mathbb{R} \to \mathbb{R}$ be strictly passive, and k be a positive constant. Then,*

$$\varsigma_2[\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1)] > 0 \quad \forall \varsigma_2 \neq 0, \forall \varsigma_1 \in \mathbb{R}$$

Proof. Let ς_0, ς_1, $\varsigma_2 \in \mathbb{R}$. Since σ_0 is strictly increasing, we have that

$$\sigma_0(\varsigma_0) > \sigma_0(\varsigma_1) \Longleftrightarrow \varsigma_0 > \varsigma_1 \quad \wedge \quad \sigma_0(\varsigma_0) < \sigma_0(\varsigma_1) \Longleftrightarrow \varsigma_0 < \varsigma_1$$

From this and the strictly passive character of σ_2, we have, by letting $\varsigma_0 = \varsigma_1 + \sigma_2(k\varsigma_2)$, that

$$\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1) > 0 \Longleftrightarrow \sigma_2(k\varsigma_2) > 0 \Longleftrightarrow \varsigma_2 > 0$$

and

$$\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1) < 0 \Longleftrightarrow \sigma_2(k\varsigma_2) < 0 \Longleftrightarrow \varsigma_2 < 0$$

$\forall \varsigma_1 \in \mathbb{R}$, whence it follows that

$$\varsigma_2[\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1)] > 0 \quad \forall \varsigma_2 \neq 0,\ \forall \varsigma_1 \in \mathbb{R}$$

■

Lemma 2.6. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a strongly passive function for (κ, a, b) and k be a positive constant. Then, for all $\varsigma \in \mathbb{R}$:*

$$\int_0^\varsigma \sigma(kz)dz \geq S(\varsigma) = \begin{cases} \frac{\kappa k^a}{1+a}|\varsigma|^{1+a} & \forall |\varsigma| \leq \frac{b}{k} \\ \kappa b^a\left(|\varsigma| - \frac{ab}{k(1+a)}\right) & \forall |\varsigma| > \frac{b}{k} \end{cases}$$

Lemma 2.6 arises from Definition 2.12 whence, since $|\sigma(\varsigma)| \geq \kappa |b \operatorname{sat} \varsigma/b|^a$, we have that $\int_0^\varsigma \sigma(kz)dz \geq \int_0^\varsigma \operatorname{sign}(z)\kappa|b \operatorname{sat} \varsigma/b|^a dz = S(\varsigma)$.

Lemma 2.7. *For every $j = 1, \ldots, n$, let σ_j be a strongly passive function for (κ, a, b), k_j be a positive constant, $k_m = \min_j\{k_j\}$, $k_M = \max_j\{k_j\}$ and, for any $x \in \mathbb{R}^n$ and $c > 0$,*

$$S_0(x; a, c) = \begin{cases} \|x\|^{1+a} & \forall \|x\| \leq c \\ c^a\|x\| & \forall \|x\| > c \end{cases} \tag{2.11}$$

Then

1. $\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j) dz_j \geq \frac{\kappa k_m^a}{1+a} S_0(x; a, b/k_M)$, $\forall x \in \mathbb{R}^n$;
2. $\sum_{j=1}^n x_j \sigma_j(k_j x_j) \geq \kappa k_m^a S_0(x; a, b/k_M)$, $\forall x \in \mathbb{R}^n$.

Proof. Item 1. Departing from Lemma 2.6, we have that

$$\sum_{j=1}^{n} \int_0^{x_j} \sigma_j(k_j z_j) dz_j \geq \sum_{j=1}^{n} S(x_j) \quad \forall x \in \mathbb{R}^n$$

From this and Lemma 1.1 we get, for all $\|x\| \leq b/k_M$, that

$$\begin{aligned}\sum_{j=1}^{n} S(x_j) = \frac{\kappa}{1+a} \sum_{j=1}^{n} k_j^a |x_j|^{1+a} \geq \frac{\kappa k_m^a}{1+a} \sum_{j=1}^{n} |x_j|^{1+a} &= \frac{\kappa k_m^a}{1+a} \|x\|_{1+a}^{1+a} \\ &\geq \frac{\kappa k_m^a}{1+a} \|x\|^{1+a} = \frac{\kappa k_m^a}{1+a} S_0(x; a, b/k_M)\end{aligned}$$

and for all $\|x\| > b/k_M$ we have, for every $j = 1, \ldots, n$, that

$$\begin{aligned}\|x\| \geq |x_j| &\iff \|x\|^{1-a} \geq |x_j|^{1-a} \\ &\implies |x_j|^a \geq \|x\|^a \frac{|x_j|}{\|x\|} \geq \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|} \\ &\implies \kappa k_j^a |x_j|^a \geq \kappa k_j^a \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|} \geq \kappa k_m^a \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|} \geq \frac{\kappa k_m^a}{1+a} \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|}\end{aligned}$$

and, on the other hand that

$$\kappa b^a \geq \kappa b^a \left(\frac{k_m}{k_M}\right)^a \frac{|x_j|}{\|x\|} \geq \frac{\kappa k_m^a}{1+a} \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|}$$

consequently $\min\{\kappa k_j^a |x_j|^a,\ \kappa b^a\} \geq \frac{\kappa k_m^a}{1+a} (b/k_M)^a \frac{|x_j|}{\|x\|}$, whence we get that

$$\begin{aligned}D_x\left[\sum_{j=1}^{n} S(x_j)\right] &= \sum_{j=1}^{n} |x_j| \min\{\kappa k_j^a |x_j|^a,\ \kappa b^a\} \\ &\geq \sum_{j=1}^{n} |x_j| \frac{\kappa k_m^a}{1+a} \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|} = \frac{\kappa k_m^a}{1+a} \left(\frac{b}{k_M}\right)^a \|x\| = D_x\left[\frac{\kappa k_m^a}{1+a} S_0(x; a, b/k_M)\right] \\ &\implies \sum_{j=1}^{n} S(x_j) \geq \frac{\kappa k_m^a}{1+a} S_0(x; a, b/k_M)\end{aligned}$$

Item 2. Departing from Definition 2.12, we have that

$$\sum_{j=1}^{n} x_j\sigma_j(k_jx_j) \geq \sum_{j=1}^{n} |x_j|\kappa\big(\min\{|k_jx_j|,\ b\}\big)^a = \kappa\sum_{j=1}^{n} |x_j|k_j^a \min\left\{|x_j|^a,\ \left(\frac{b}{k_j}\right)^a\right\}$$

$$\geq \kappa k_m^a \sum_{j=1}^{n} |x_j| \min\left\{|x_j|^a,\ \left(\frac{b}{k_M}\right)^a\right\}$$

$\forall x \in \mathbb{R}^n$. From this and Lemma 1.1 we get, for all $\|x\| \leq b/k_M$, that

$$\kappa k_m^a \sum_{j=1}^{n} |x_j| \min\left\{|x_j|^a,\ \left(\frac{b}{k_M}\right)^a\right\} = \kappa k_m^a \sum_{j=1}^{n} |x_j|^{1+a} = \kappa k_m^a \|x_j\|_{1+a}^{1+a}$$

$$\geq \kappa k_m^a \|x_j\|^{1+a} = \kappa k_m^a S_0(x; a, b/k_M)$$

and for all $\|x\| > b/k_M$ we have, for every $j = 1, \ldots, n$, that

$$\|x\| \geq |x_j| \iff \|x\|^{1-a} \geq |x_j|^{1-a}$$

$$\implies |x_j|^a \geq \|x\|^a \frac{|x_j|}{\|x\|} \geq \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|}$$

$$\implies (1+a)|x_j|^a \geq |x_j|^a \geq \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|}$$

and, on the other hand that $(b/k_M)^a \geq (b/k_M)^a \frac{|x_j|}{\|x\|}$, under the consideration of this last we obtain that $\min\{(1+a)|x_j|^a, (b/k_M)^a\} \geq (b/k_M)^a \frac{|x_j|}{\|x\|}$, whence we get that

$$D_x\left[\kappa k_m^a \sum_{j=1}^{n} |x_j| \min\left\{|x_j|^a, \left(\frac{b}{k_M}\right)^a\right\}\right] = D_x\left[\kappa k_m^a \sum_{j=1}^{n} \min\left\{|x_j|^{1+a}, \left(\frac{b}{k_M}\right)^a |x_j|\right\}\right]$$

$$= \kappa k_m^a \sum_{j=1}^{n} |x_j| \min\left\{(1+a)|x_j|^a, \left(\frac{b}{k_M}\right)^a\right\}$$

$$\geq \kappa k_m^a \sum_{j=1}^{n} |x_j| \left(\frac{b}{k_M}\right)^a \frac{|x_j|}{\|x\|} = \kappa k_m^a \left(\frac{b}{k_M}\right)^a \|x\|$$

$$= D_x\big[\kappa k_m^a S_0(x; a, b/k_M)\big]$$

$$\implies \kappa k_m^a \sum_{j=1}^{n} |x_j| \min\left\{|x_j|^a, \left(\frac{b}{k_M}\right)^a\right\} \geq \kappa k_m^a S_0(x; a, b/k_M)$$

■

Remark 2.13. Note that for a bounded strongly passive function σ for some $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$, we have:

1. $\kappa(\min\{|\varsigma|, b\})^a \le |\sigma(\varsigma)| \le \bar{\kappa}(\min\{|\varsigma|, \bar{b}\})^{\bar{a}} \le \bar{\kappa}|\varsigma|^{\bar{a}}$, $\forall \varsigma \in \mathbb{R}$;
2. $\int_0^\varsigma \sigma(kz)dz \le \int_0^\varsigma \operatorname{sign}(z)\bar{\kappa}|\bar{b}\operatorname{sat} kz/\bar{b}|^{\bar{a}}dz \triangleq \bar{S}(\varsigma)$, and consequently, in addition to (2.6),

$$\int_0^\varsigma \sigma(kz)dz \le \bar{S}(\varsigma) = \begin{cases} \frac{\bar{\kappa}k^{\bar{a}}}{1+\bar{a}}|\varsigma|^{1+\bar{a}} & \forall |\varsigma| \le \frac{\bar{b}}{k} \\ \bar{\kappa}\bar{b}^{\bar{a}}\left(|\varsigma| - \frac{\bar{a}\bar{b}}{k(1+\bar{a})}\right) & \forall |\varsigma| > \frac{\bar{b}}{k} \end{cases}$$

Lemma 2.8. *For every $j = 1, \ldots, n$, let σ_j be a bounded strongly passive function for $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$, k_j be a positive constant, $k_m = \min_j\{k_j\}$, $k_M = \max_j\{k_j\}$ and S_0 as defined through* (2.11)*. Then, in addition to item 1 of Lemma 2.7,*

$$\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j)dz_j \le \bar{\kappa}k_M^{\bar{a}} n S_0(x; \bar{a}, \bar{b}/k_M) \qquad \forall x \in \mathbb{R}^n$$

Proof. Departing from (item 2 of) Remark 2.13 we have that

$$\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j)dz_j \le \sum_{j=1}^n \bar{S}(x_j)$$

$\forall x \in \mathbb{R}^n$. From this and Remark 1.1 we get, for all $\|x\| \le \bar{b}/k_M$, that

$$\begin{aligned}\sum_{j=1}^n \bar{S}(x_j) = \frac{\bar{\kappa}}{1+\bar{a}} \sum_{j=1}^n k_j^{\bar{a}} |x_j|^{1+\bar{a}} &\le \bar{\kappa}k_M^{\bar{a}} \sum_{j=1}^n |x_j|^{1+\bar{a}} = \bar{\kappa}k_M^{\bar{a}} \|x\|_{1+\bar{a}}^{1+\bar{a}} \\ &\le \bar{\kappa}k_M^{\bar{a}} n \|x\|^{1+\bar{a}} = \bar{\kappa}k_M^{\bar{a}} n S_0(x; \bar{a}, \bar{b}/k_M)\end{aligned}$$

and for all $\|x\| > \bar{b}/k_M$ we have, for every $j = 1, \ldots, n$ that

$$\min\{\bar{\kappa}k_j^{\bar{a}}|x_j|^{\bar{a}}, \bar{\kappa}\bar{b}^{\bar{a}}\} \le \bar{\kappa}\bar{b}^{\bar{a}} \Rightarrow |x_j| \min\{\bar{\kappa}k_j^{\bar{a}}|x_j|^{\bar{a}}, \bar{\kappa}\bar{b}^{\bar{a}}\} \le \bar{\kappa}k_M^{\bar{a}}(\bar{b}/k_M)^{\bar{a}}|x_j|$$

whence we get that

$$\begin{aligned} D_x\left[\sum_{j=1}^n \bar{S}(x_j)\right] &= \sum_{j=1}^n |x_j| \min\{\bar{\kappa}k_j^{\bar{a}}|x_j|^{\bar{a}}, \bar{\kappa}\bar{b}^{\bar{a}}\} \\ &\le \sum_{j=1}^n \bar{\kappa}k_M^{\bar{a}}(\bar{b}/k_M)^{\bar{a}}|x_j| = \bar{\kappa}k_M^{\bar{a}}(\bar{b}/k_M)^{\bar{a}}\|x\|_1 \\ &\le \bar{\kappa}k_M^{\bar{a}}(\bar{b}/k_M)^{\bar{a}} n\|x\| = D_x\left[\bar{\kappa}k_M^{\bar{a}} n S_0(x; \bar{a}, \bar{b}/k_M)\right] \\ &\Rightarrow \sum_{j=1}^n \bar{S}(x_j) \le \bar{\kappa}k_M^{\bar{a}} n S_0(x; \bar{a}, \bar{b}/k_M) \end{aligned}$$

∎

CHAPTER 3

Proposed finite-time/exponential controllers

Throughout this chapter the proposed control schemes will be presented.

3.1 Regulation with on-line conservative force compensation

In the regulation problem, the mechanical system model described in (1.1) will be taken disregarding the linear damping term, *i.e.*, with $F = 0$. In this case, such a consideration is possible due to the passive nature of the controlled system. For this reason, damping injected via control turns out to be sufficient in order to achieve the control objectives. Thus, such system model takes the form

$$H(q)q + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{3.1}$$

The properties of $H(q)$, $C(q, \dot{q})$ and $g(q)$ are as described in Section 1.4. Let us recall that the realistic bounded input case is considered, *i.e.*, the i-th element of τ keeps a non-linear relation with the i-th element of the controller output u of the form

$$\tau_i = T_i \operatorname{sat}\left(\frac{u_i}{T_i}\right) \tag{3.2}$$

An on-line conservative-force compensation term is involved in the following control schemes, *i.e.*, the conservative-force vector $g(q)$ is continuously calculated for every position configuration, q, resulting from the system dynamics. Further, the approaches are presented in accordance to the kind of feedback taken into account: we begin by presenting the state-feedback controller and subsequently the output-feedback scheme.

3.1.1 State-feedback control scheme

Consider the following SPD (Saturating-Proportional-Derivative) type state-feedback controller:

$$u(q,\dot q) = -s_0\big(s_1(K_1\bar q) + s_2(K_2\dot q)\big) + g(q) \tag{3.3}$$

where $\bar q = q - q_d$, for any constant (desired equilibrium position) $q_d \in \mathbb{R}^n$; $K_i = \mathrm{diag}[k_{i1},\dots,k_{in}]$ with $k_{ij} > 0$, $i = 1,2$, $\forall j \in \{1,\dots,n\}$; $g(q)$ is the conservative force compensation term; and for any $x \in \mathbb{R}^n$, $s_i(x) = (\sigma_{i1}(x_1),\dots,\sigma_{in}(x_n))^T$, $i = 0,1,2$, with, for each $j \in \{1,\dots,n\}$, σ_{0j} being a strictly increasing strictly passive function, σ_{1j} a strongly passive function and σ_{2j} a strictly passive function, all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$ and such that

$$B_j \triangleq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{0j}\big(\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big)\big| < T_j - B_{gj} \tag{3.4}$$

(recall Assumptions 1.3.1 and 1.4).

Remark 3.1. Note that by (3.4), we have that —for each $j \in \{1,\dots,n\}$— either:

(a) σ_{0j} is bounded (whether σ_{1j} and/or σ_{2j} are/is bounded or not), or

(b) σ_{1j} and σ_{2j} are both bounded (whether σ_{0j} is bounded or not), or

(c) σ_{0j} is positively upper bounded, resp. negatively lower bounded, and σ_{ij}, $i = 1,2$, are both negatively lower bounded, resp. positively upper bounded (wether σ_{ij}, $i = 0,1,2$, are bounded or not).

Proposition 3.1. *Consider system* (3.1)–(3.2) *in closed loop with the proposed control law* (3.3)*. Thus, for any positive definite diagonal matrices K_1 and K_2: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\dots,n$, $\forall t \geq 0$, and global asymptotic stability of the closed-loop trivial solution $\bar q(t) \equiv 0_n$ is guaranteed.*

Proof. Notice that —for every $j \in \{1,\dots,n\}$— by (3.4), we have that, for any $(q,\dot q) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned}
|u_j(q,\dot q)| &= \big| -\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar q_j) + \sigma_{2j}(k_{2j}\dot q_j)\big) + g_j(q)\big| \\
&\leq \big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar q_j) + \sigma_{2j}(k_{2j}\dot q_j)\big)\big| + |g_j(q)| \\
&\leq B_j + B_{gj} < T_j
\end{aligned}$$

From this and (3.2), one sees that $T_j > |u_j(q,\dot q)| = |u_j| = |\tau_j|$, $\forall (q,\dot q) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the systems trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\dots,n$, $\forall t \geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never reached. Hence, the closed-loop dynamics takes the form

$$H(q)\ddot q + C(q,\dot q)\dot q = -s_0\big(s_1(K_1\bar q) + s_2\big(K_2\dot q\big)\big)$$

By defining $x_1 = \bar q$ and $x_2 = \dot q$, the closed-loop dynamics adopts the $2n$-order state-space representation

$$\begin{aligned}
\dot x_1 &= x_2 \\
\dot x_2 &= H^{-1}(x_1 + q_d)\big[-C(x_1 + q_d, x_2)x_2 - s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big]
\end{aligned}$$

Additionally, taking $x = (x_1^T, x_2^T)^T$, these state equations may be rewritten in the form of system (2.4) (recalling Section 2.3) with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{3.5a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1+q_d)C(x_1+q_d, x_2)x_2 - \mathcal{H}(x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{3.5b}$$

where

$$\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d) \tag{3.6}$$

Thus, the closed-loop stability property stated through Proposition 3.1 is corroborated by showing that $x = 0_{2n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem (whose formulation proves to be convenient for subsequent developments and proofs).

Theorem 3.1. *Under the stated design requirements, the origin is a globally asymptotically stable equilibrium of* $\dot{x} = f(x) + \ell\hat{f}(x)$, $\forall \ell \in \{0, 1\}$ —i.e., *of both the state equation* $\dot{x} = f(x)$ *and the (closed-loop) system* $\dot{x} = f(x) + \hat{f}(x)$—, *with* $f(x)$ *and* $\hat{f}(x)$ *defined through Eqs.* (3.5).

Proof. For every $\ell \in \{0, 1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1, x_2) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \int_{0_n}^{x_1} s_0^T(s_1(K_1r))\, dr$$

where $\int_{0_n}^{x_1} s_0^T(s_1(K_1r))\, dr = \sum_{j=1}^n \int_0^{x_{1j}} \sigma_{0j}(\sigma_{1j}(k_{1j}r_j))\, dr_j$. Since $H(q) \geq \mu_m I_n$ (as stated in Property 1.1), we have that

$$V_\ell(x_1, x_2) \geq \frac{\mu_m}{2}\|x_2\|^2 + \int_{0_n}^{x_1} s_0^T(s_1(K_1r))\, dr \tag{3.7}$$

for a positive constant μ_m, $\ell = 0, 1$. One sees from (3.7), Lemma 2.4 and Remark 2.12 that $V_\ell(x_1, x_2)$, $\ell = 0, 1$, are concluded to be positive definite and radially unbounded. Furthermore, for every $\ell \in \{0, 1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$ is obtained as

$$\begin{aligned} \dot{V}_\ell(x_1, x_2) &= x_2^T H(\ell x_1 + qd)\dot{x}_2 + \frac{\ell}{2}x_2^T \dot{H}(\ell x_1 + q_d, x_2)x_2 + s_0^T(s_1(K_1x_1))\dot{x}_1 \\ &= -x_2^T[\ell C(x_1 + q_d, x_2)x_2 + s_0(s_1(K_1x_1) + s_2(K_2x_2))] \\ &\quad + \frac{\ell}{2}x_2^T \dot{H}(\ell x_1 + q_d, x_2)x_2 + x_2^T s_0(s_1(K_1x_1)) \\ &= -x_2^T[s_0(s_1(K_1x_1) + s_2(K_2x_2)) - s_0(s_1(K_1x_1))] \\ &= -\sum_{j=1}^n x_{2j}[\sigma_{0j}(\sigma_{1j}(k_{1j}x_{1j}) + \sigma_{2j}(k_{2j}x_{2j})) - \sigma_{0j}(\sigma_{1j}(k_{1j}x_{1j}))] \end{aligned}$$

where, in the case of $\ell = 1$, Property 1.2.1 has been applied. Note, from Lemma 2.5, that $\dot{V}_\ell(x_1, x_2) \leq 0$, $\forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, with

$$Z_\ell \triangleq \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell(x_1, x_2) = 0\} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : x_2 = 0_n\}$$

Furthermore, from the system dynamics $\dot{x} = f(x) + \ell \hat{f}(x)$ —under the consideration of the strongly passive character of σ_{1j}, $j = 1, \ldots, n$, Property 1.1 and the positive definiteness of K_1— one sees that

$$x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n$$

and

$$x_2(t) \equiv \dot{x}_2(t) \equiv 0_n \implies s_0(s_1(K_1 x_1(t))) \equiv 0_n \iff s_1(K_1 x_1(t)) \equiv 0_n \iff x_1(t) \equiv 0_n$$

which shows that $(x_1, x_2)(t) \equiv (0_n, 0_n)$ is the only system solution completely remaining in Z_ℓ, and corroborates that at any $(x_1, x_2) \in \{(\bar{q}, \dot{q}) \in Z_\ell : \bar{q} \neq 0_n\}$, the resulting unbalanced force term $-s_0(s_1(K_1 x_1))$ acts on the closed-loop dynamics, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n, 0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0, 1$. Therefore, by the invariance theory [42, § 7.2] —more precisely by [42, Corollary 7.2.1]—, $x = 0_{2n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. ■

Through Proposition 3.1 and Theorem 3.1 global asymptotic stability of the closed-loop trivial solution and input saturation avoidance are concluded. The results obtained so far will prove to be helpful in further developments.

Finite-time and exponential stabilization

In view of the last results, all that remains to be proven is finite-time, respectively exponential, stability.

Proposition 3.2. *Consider the proposed control scheme in* (3.3) *under the additional consideration that, for every $j \in \{1, \ldots, n\}$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_j > 0$ —i.e., $r_{1j} = r_1$, $r_{2j} = r_2$ and $\alpha_{1j} = \alpha_{2j} = \alpha_j > 0$ for all $j \in \{1, \ldots, n\}$— with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0, \infty]\}$ and σ_{0j} is locally α_j-homogeneous of degree $\alpha_0 = 2r_2 - r_1$ —i.e., $\alpha_{0j} = \alpha_0 = 2r_2 - r_1$ for all $j \in \{1, \ldots, n\}$— with domain of homogeneity $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j} \in (0, \infty]\}$, for some dilation coefficients $r_i > 0$, $i = 1, 2$, such that $\alpha_0 = 2r_2 - r_1 > 0$. Thus, for any positive definite diagonal matrices K_1 and K_2, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:*

1) globally finite-time stable if $r_2 < r_1$;

2) globally asymptotically stable and (locally) exponentially stable if $r_2 = r_1$.

Proof. Since the proposed control scheme is applied —with all its previously stated specifications— $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$ holds as a result of Proposition 3.2. Then, the proof is just focused on the specific stability properties claimed in items 1) and 2) of the statement. In this direction, let $\hat{r}_i = (r_{i1}, \ldots, r_{in})^T$, $i = 1, 2$, $r = (\hat{r}_1^T, \hat{r}_2^T)^T$, $\hat{r}_0 = (\alpha_1, \ldots, \alpha_n)^T$, $\hat{\alpha}_0 = (\alpha_{01}, \ldots, \alpha_{0n})^T$,

$$\begin{aligned} D \triangleq \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : K_i x_i \in D_{i1} \times \cdots \times D_{in},\ i = 1, 2, \\ s_1(K_1 x_1) + s_2(K_2 x_2) \in D_{01} \times \cdots \times D_{0n}\} \\ = \Big\{ (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : |x_{1j}| < \frac{L_{1j}}{k_{1j}},\ |x_{2j}| < \frac{L_{2j}}{k_{2j}}, \\ |\sigma_{1j}(k_{1j} x_{1j}) + \sigma_2(k_{2j} x_{2j})| < L_{0j},\ j = 1, \ldots, n \Big\} \end{aligned}$$

and consider the closed-loop state-space representation $\dot{x} = f(x) + \hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (3.5). Since D defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{2n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0, 1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall \varepsilon \in (0, 1)$), and, for every $j \in \{1, \ldots, n\}$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2 - r_1) + r_1} x_{2j} = \varepsilon^{(r_2 - r_1) + r_{1j}} f_j(x) \tag{3.8}$$

Further, for every $x \in B_\rho$ and all $\varepsilon \in (0,\, 1]$, we have that

$$\sigma_{ij}(k_{ij} \varepsilon^{r_{ij}} x_{ij}) = \sigma_{ij}(\varepsilon^{r_i} k_{ij} x_{ij}) = \varepsilon^{\alpha_j} \sigma_{ij}(k_{ij} x_{ij}) \iff s_i(K_i \delta_\varepsilon^{\hat{r}_i}(x_i)) = s_i(\varepsilon^{r_i} K_i x_i) = \delta_\varepsilon^{\hat{r}_0}\big(s_i(K_i x_i)\big)$$

for $i = 1, 2$, $j = 1, \ldots, n$, and, $\sigma_0(\varepsilon^{\alpha_j} \cdot) = \varepsilon^{\alpha_{0j}} \sigma_0(\cdot) = \varepsilon^{\alpha_0} \sigma_0(\cdot) \iff s_0(\delta_\varepsilon^{\hat{r}_0}(\cdot)) = \delta_\varepsilon^{\hat{\alpha}_0}\big(s_0(\cdot)\big) = \varepsilon^{\alpha_0} s_0(\cdot)$ for $j = 1, \ldots, n$, wherefrom

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d) s_0(s_1(K_1 \delta_\varepsilon^{\hat{r}_1}(x_1)) + s_2(K_2 \delta_\varepsilon^{\hat{r}_2}(x_2))) \\ &= -H_j^{-1}(q_d) s_0(s_1(\varepsilon^{r_1} K_1 x_1) + s_2(\varepsilon^{r_2} K_2 x_2)) \\ &= -H_j^{-1}(q_d) s_0(\delta_\varepsilon^{\hat{r}_0}(s_1(K_1 x_1)) + \delta_\varepsilon^{\hat{r}_0}(s_2(K_2 x_2))) \\ &= -H_j^{-1}(q_d) s_0(\delta_\varepsilon^{\hat{r}_0}(s_1(K_1 x_1) + s_2(K_2 x_2))) \\ &= -H_j^{-1}(q_d) \delta_\varepsilon^{\hat{\alpha}_0}(s_0(s_1(K_1 x_1) + s_2(K_2 x_2))) \\ &= -\varepsilon^{\alpha_0} H_j^{-1}(q_d) s_0(s_1(K_1 x_1) + s_2(K_2 x_2)) \\ &= \varepsilon^{(r_2 - r_1) + r_{2j}} f_{n+j}(x) \end{aligned} \tag{3.9}$$

From (3.8)–(3.9) one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium if $r_2 < r_1$ (by Theorems 2.1 and 3.1), and a globally asymptotically stable equilibrium (from Theorem 3.1) with (local) exponential stability if $r_2 = r_1$ (by Lemma 2.2 and Remark 2.6). Thus, under the additional consideration of Lemma 2.3 and Remark 2.7, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability provided that $r_2 = r_1$, if

$$\begin{aligned}
\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{11}}, \ldots, \varepsilon^{-r_{1n}}, \varepsilon^{-r_{21}}, \ldots, \varepsilon^{-r_{2n}}] \hat{f}\big(\delta_\varepsilon^r(x)\big) \right\| \\
&= \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{21}}, \ldots, \varepsilon^{-r_{2n}}] \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha-r_2} \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2} \left\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= 0
\end{aligned} \tag{3.10}$$

for all $x \in S_c^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\| = c\}$ (resp. $x \in S_{r,c}^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\|_r = c\}$), for some $c > 0$ such that $S_c^{2n-1} \subset D$ (resp. $S_{r,c}^{2n-1} \subset D$). Hence, from (3.5b), under the consideration of Property 1.2.3 (related to $C(q, \dot{q})$), we have, for all $x \in S_c^{2n-1}$ (resp. $x \in S_{r,c}^{2n-1}$):

$$\begin{aligned}
\left\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| &= \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 \\
&\qquad + \mathcal{H}(\varepsilon^{r_1}x_1) s_0\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big) \big\| \\
&\le \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2 \big\| \\
&\quad + \big\| \mathcal{H}(\varepsilon^{r_1}x_1) s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big) \big\|
\end{aligned}$$

whence, through a procedure similar to the one developed to obtain (3.9), we get

$$\begin{aligned}
\left\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| &\le \varepsilon^{2r_2} \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \big\| \\
&\quad + \varepsilon^{2r_2-r_1} \big\| \mathcal{H}(\varepsilon^{r_1}x_1) s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\|
\end{aligned}$$

and consequently, from (3.10), we get

$$\begin{aligned}
\mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \big\| \\
&\qquad + \lim_{\varepsilon\to 0^+} \big\| \mathcal{H}(\varepsilon^{r_1}x_1) s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \\
&\le \big\| H^{-1}(q_d) C(q_d, x_2)x_2 \big\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} + \big\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \lim_{\varepsilon\to 0^+} \big\| \mathcal{H}(\varepsilon^{r_1}x_1) \big\| \\
&\le \big\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \cdot \big\| \mathcal{H}(0_n) \big\| = 0
\end{aligned}$$

(note, from (3.6), that $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$), which completes the proof. ■

Corollary 3.1. *Consider the proposed control scheme in* (3.3) *taking* σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$, *such that*

$$\sigma_{ij}(\varsigma) = \text{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall|\varsigma| \leq L_{ij} \in (0.\infty] \tag{3.11}$$

with constants β_{ij} *such that*

$$\beta_{1j} > 0, \quad \beta_{2j} = \gamma\beta_{1j}, \quad \beta_{0j} = \frac{2-\gamma}{\gamma\beta_{1j}} \tag{3.12}$$

for a constant $\gamma \in (0, 2)$*. Thus, for any positive definite diagonal matrices* K_1 *and* K_2, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, *and the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is:*

(1) globally finite-time stable if $1 < \gamma < 2$*;*

(2) globally asymptotically stable and (locally) exponentially stable if $\gamma = 1$.

Proof. Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij}) : \varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and

$$\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \text{sign}(\varepsilon^{r_{ij}}\varsigma)|\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \,\text{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma)$$

$\forall\varepsilon \in (0, 1]$. Hence, under the consideration of expressions (3.12), for every $j \in \{1, \ldots, n\}$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = r_1/\gamma$ and $r_{0j} = r_1\beta_{1j} : \sigma_{ij}$, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{2j} = r_2\beta_{2j} = r_1\beta_{1j} = \alpha_{1j} = \alpha_j$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{0j} is locally α_j-homogeneous of degree $\alpha_{0j} = \alpha_0 = (2-\gamma)r_1/\gamma$ with domain of homogeneity $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j}\}$. Moreover, under the additional condition on γ, we have that $0 < \gamma < 2 \iff (0 < \gamma) \wedge (0 < 2-\gamma) \implies 0 < (2-\gamma)r_1/\gamma = \alpha_0 \iff 0 < 2r_2 - r_1 = \alpha_0$. The requirements of Proposition 3.2 are thus satisfied with $r_2 < r_1 \iff 1 < \gamma < 2$ and $r_2 = r_1 \iff \gamma = 1$. ∎

Corollary 3.1 states a useful particular way to define the functions involved in the control scheme (3.3). Such particular way to defined the referred functions permits, via the parameters γ and β, to get either finite-time or exponential convergence.

3.1.2 Output-feedback control scheme

Suppose that the velocity measurements have poor quality or are unavailable. This brings to the fore the need for the design of output-feedback control schemes. Under this consideration, the following SP-SD type controller is proposed:

$$u(q, \vartheta) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q) \tag{3.13}$$

where $\bar{q}$, K_1, and K_2 were already defined in the previous section; $\vartheta \in \mathbb{R}^n$ is the output vector variable of an auxiliary subsystem defined as

$$\dot{\vartheta}_c = -As_3(\vartheta_c + B\bar{q}) \tag{3.14a}$$

$$\vartheta = \vartheta_c + B\bar{q} \tag{3.14b}$$

with A and B positive definite diagonal matrices, *i.e.*, $A = \text{diag}[a_1, \ldots, a_n]$, $B = \text{diag}[b_1, \ldots, b_n]$, $a_j > 0$ and $b_j > 0$, $\forall j \in \{1, \ldots, n\}$; and for any $x \in \mathbb{R}^n$, $s_i(x) = (\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n))^T$, $i = 1, 2, 3$, with, for each $j \in \{1, \ldots, n\}$, $\sigma_{3j}(\cdot)$ being a strictly passive function, while $\sigma_{ij}(\cdot)$, $i = 1, 2$, are strongly passive functions such that

$$B_j \triangleq \sup_{(\varsigma_1, \varsigma_2) \in \mathbb{R}^2} \left|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\right| < T_j - B_{gj} \tag{3.15}$$

(recalling Assumptions 1.3.1 and 1.4 with $\eta = 1$), all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$. Note that by (3.15), we have that —for each $j \in \{1, \ldots, n\}$— σ_{1j} and σ_{2j} shall both be bounded, while σ_{3j} may be bounded or not.

Remark 3.2. Observe that the auxiliary subsystem in Eqs. (3.14) is a nonlinear version of the dirty derivative operator, applied to the position error vector variable. Indeed, note that if s_3 were chosen to be the identity function, i.e., $s_3(x) \equiv x$, the conventional linear dynamics of the dirty derivative (applied to $\bar{q}$) [43] is obtained. The output variable of the (non-linear) dirty-derivative-type subsystem, ϑ, may thus be seen as an approximated dirty derivative of $\bar{q}$ (or an approximated *dirty* calculation of $\dot{q}$). even though a more appropriate insight on the role played by the auxiliary subsystem in Eq. (3.14) will be brought to the fore (later, in Remark 10) under an energy-related optics.

Proposition 3.3. *Consider system* (3.1)–(3.2) *in closed loop with the proposed control scheme in Eqs.* (3.13) *and* (3.14). *Thus, for any positive definite diagonal matrices K_1, K_2, A and B: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$ and global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed.*

Proof. Observe that —for every $j \in \{1, \ldots, n\}$— by (3.15), we have that, for any $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} |u_j(q, \vartheta)| &= \left| -\sigma_{1j}(k_{1j}\bar{q}_j) - \sigma_{2j}(k_{2j}\vartheta_j) + g_j(q)\right| \\ &\leq \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\vartheta_j)\right| + |g_j(q)| \\ &\leq B_j + B_{gj} < T_j \end{aligned}$$

From this and (3.2), one sees that $T_j > |u_j(q, \vartheta)| = |u_j| = |\tau_j|$, $\forall (q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the systems trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never attained. Hence, the closed-loop dynamics takes the form

$$\begin{aligned} H(q)\ddot{q} + C(q, \dot{q})\dot{q} &= -s_1(K_1\bar{q}) - s_2(K_2\vartheta) \\ \dot{\vartheta} &= -As_3(\vartheta) + B\dot{q} \end{aligned}$$

By defining $x_1 = \bar{q}$, $x_2 = \dot{q}$ and $x_3 = \vartheta$, the closed-loop dynamics adopts the $3n$-order state-space representation

$$\dot{x}_1 = x_2 \tag{3.16a}$$
$$\dot{x}_2 = -H^{-1}(x_1 + q_d)\big[C(x_1 + q_d, x_2)x_2 + s_1(K_1x_1) + s_2(K_2x_3)\big] \tag{3.16b}$$
$$\dot{x}_3 = -As_3(x_3) + Bx_2 \tag{3.16c}$$

Furthermore, by taking $x = (x_1^T, x_2^T, x_3^T)^T$, these state equations may be rewritten in the form of system (2.4), with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)\big[s_1(K_1x_1) + s_2(K_2x_3)\big] \\ -As_3(x_3) + Bx_2 \end{pmatrix} \tag{3.17a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1 + q_d)C(x_1 + q_d, x_2)x_2 - \mathcal{H}(x_1)\big[s_1(K_1x_1) + s_2(K_2x_3)\big] \\ 0_n \end{pmatrix} \tag{3.17b}$$

with $\mathcal{H}(x_1)$ as defined in (3.6). Thus, the closed-loop stability property stated through Proposition 3.3 is corroborated by showing that $x = 0_{3n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem. (whose formulation proves to be convenient for subsequent developments and proofs).

Theorem 3.2. *Considering the stated design specifications, the origin is a globally asymptotically stable equilibrium of $\dot{x} = f(x) + \ell\hat{f}(x)$, $\forall \ell \in \{0, 1\}$ —i.e., of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$—, with $f(x)$ and $\hat{f}(x)$ defined through Eqs.* (3.17).

Proof. For every $\ell \in \{0, 1\}$, the following continuously differentiable scalar function is defined

$$V_\ell(x_1, x_2, x_3) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \int_{0_n}^{x_1} s_1^T(K_1r)\, dr + \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}\, dr$$

where

$$\int_{0_n}^{x_1} s_1^T(K_1r)\, dr = \sum_{j=1}^{n} \int_0^{x_{1j}} \sigma_{1j}(k_{1j}r_j)\, dr_j \quad , \quad \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}\, dr = \sum_{j=1}^{n} \int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}r_j)}{b_j}\, dr_j$$

From Property 1.1 (whence $H(q) \geq \mu_m I_n$) and Lemma 2.4 (where strongly passive function characteristics are stated), $V_\ell(x_1, x_2, x_3)$, $\ell = 0, 1$, are concluded to be positive definite and radially unbounded. Further, for every $\ell \in \{0, 1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$ is obtained as

$$\begin{aligned}
\dot{V}_\ell(x_1,x_2,x_3) &= x_2^T H(\ell x_1 + qd)\dot{x}_2 + \frac{\ell}{2}x_2^T\dot{H}(x_1+qd,x_2)x_2 + s_1^T(K_1x_1)\dot{x}_1 + s_2^T(K_2x_3)B^{-1}\dot{x}_3 \\
&= x_2^T[-\ell C(x_1+qd,x_2)x_2 - s_1(K_1x_1) - s_2(K_2x_3)] + \frac{\ell}{2}x_2^T\dot{H}(x_1+qd,x_2)x_2 \\
&\quad + s_1^T(K_1x_1)x_2 + s_2^T(K_2x_3)B^{-1}[-As_3(x_3)+Bx_2] \\
&= -s_2^T(K_2x_3)B^{-1}As_3(x_3) \\
&= -\sum_{j=1}^{n}\frac{a_j}{b_j}\sigma_{2j}(k_{2j}x_{3j})\sigma_{3j}(x_{3j})
\end{aligned}$$

where, in the case of $\ell = 1$, Property 1.2.1 has been applied. Note, from the strictly passive character of σ_{2j} and σ_{3j}, $j = 1,\ldots,n$, that $\dot{V}_\ell(x_1,x_2,x_3) \le 0$, $\forall(x_1,x_2,x_3) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, with

$$Z_\ell \triangleq \{(x_1,x_2,x_3)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : \dot{V}_\ell(x_1,x_2,x_3)=0\} = \{(x_1,x_2,x_3)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : x_3 = 0_n\}$$

Moreover, from the system dynamics $\dot{x} = f(x) + \ell\hat{f}(x)$ —under the consideration of the strictly passive character of σ_{1j}, $j = 1,\ldots,n$, Property 1.1 and the positive definiteness of K_1— one sees that

$$x_3(t) \equiv 0_n \implies \dot{x}_3(t) \equiv 0_n$$

while

$$x_3(t) \equiv \dot{x}_3(t) \equiv 0_n \implies x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n$$

and

$$x_3(t) \equiv \dot{x}_3(t) \equiv x_2(t) \equiv \dot{x}_2(t) \equiv 0_n \implies s_1(K_1x_1(t)) \equiv 0_n \iff x_1(t) \equiv 0_n$$

which shows that $(x_1,x_2,x_3)(t) \equiv (0_n,0_n,0_n)$ is the only system solution completely remaining in Z_ℓ and corroborates that at any $(x_1,x_2,x_3) \in Z_\ell\setminus\{(0_n,0_n,0_n)\}$, the resulting unbalanced force terms act on the closed-loop dynamics $[\dot{x} = f(x_1,x_2,0_n) + \ell\hat{f}(x_1,x_2,0_n)$ with $(x_1,x_2) \neq (0_n,0_n)]$, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0,1$. Therefore, by the invariance theory [42, Corollary 7.2.1], $x = 0_{3n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. ■

Until now, input saturation avoidance and global asymptotic stability of the closed-loop trivial solution are guaranteed through Proposition 3.3 and Theorem 3.2. These results will prove to be helpful in further developments.

Remark 3.3. Consider the closed-loop system in Eqs. (3.16). Let $e_1 = -y_2 = -s_2(K_2x_3)$, $e_2 = y_1 = x_2$, $\psi(x_3) = s_2^T(K_2x_3)B^{-1}As_3(x_3)$,

$$V_{11}(x_1,x_2) = \frac{1}{2}x_2^T H(x_1+q_d)x_2 + \int_{0_n}^{x_1} s_1^T(K_1r)dr$$

and

$$V_{12}(x_3) = \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}dr$$

By previous arguments and developments, V_{11} and V_{12} are radially unbounded positive definite functions in their respective arguments. Following an analysis analog to that of the proof of Theorem 3.2, one obtains

$$\begin{aligned}\dot{V}_{11} &= x_2^T H(x_1+q_d)\dot{x}_2 + \frac{1}{2}x_2^T \dot{H}(x_1+q_d,x_2)x_2 + s_1^T(K_1x_1)\dot{x}_1\\ &= x_2^T[-C(x_1+qd,x_2)x_2 - s_1(K_1x_1) - s_2(K_2x_3)] + \frac{1}{2}x_2^T\dot{H}(x_1+qd,x_2)x_2 + s_1^T(K_1x_1)x_2\\ &= -s_2^T(K_2x_3)x_2 = e_1^T y_1\end{aligned}$$

and

$$\dot{V}_{12} = s_2^T(K_2x_3)B^{-1}\dot{x}_3 = s_2^T(K_2x_3)B^{-1}[-As_3(x_3)+Bx_2] = e_2^T y_2 - \psi(x_3)$$

with $\psi(x_3)$ being positive definite (in its argument). Hence, the closed-loop system in Eqs. (3.16) may be seen as a (negative) feedback system connection among a passive —actually lossless— subsystem Σ_1 with dynamic model

$$\Sigma_1 : \begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = H^{-1}(x_1+q_d)\big[-C(x_1+q_d,x_2)x_2 - s_1(K_1x_1) + e_1\big]\\ y_1 = x_2\end{cases}$$

and a positive definite storage function $V_{11}(x_1,x_2)$, and a strictly passive subsystem Σ_2 with state model

$$\Sigma_2 : \begin{cases}\dot{x}_3 = -As_3(x_3) + Be_2 \triangleq f_2(x_3,e_2)\\ y_2 = s_2(K_2x_3)\end{cases} \tag{3.18}$$

and storage function $V_{12}(x_3)$. Moreover, one sees from (3.18) that $f_2(0_n,e_2) = Be_2 = 0_n \Rightarrow e_2 = 0_n$, completing the requirements of Theorem 2.3. This formulation actually brings to the fore the damping-injection role that subsystem (3.16c) —or, equivalently, in Eqs. (3.14)— plays in the closed loop. Indeed, through its x_3-dependent term, subsystem (3.16c) in fact acts as a dynamic damper in charge to dissipate the feedback system stored energy, thus leading the closed-loop trajectories to the (unique) minimum-energy configuration, located (by feedback) at the desired position.

Finite-time and exponential stabilization

Since input-saturation avoidance and global asymptotic stability have been concluded in the previous developments, all that remains to be proven is finite-time, respectively exponential, stability.

Proposition 3.4. *Consider the proposed control scheme in* (3.13)–(3.14)*, taking into account that, for every* $j \in \{1,\dots,n\}$, σ_{ij}, $i=1,2$, *are locally* r_i*-homogeneous of (common) degree* $\alpha_i = 2r_2 - r_1 > 0$ *—i.e.,* $r_{1j} = r_1$, $r_{2j} = r_2$ *and* $\alpha_{1j} = \alpha_1 = 2r_2 - r_1 = \alpha_2 = \alpha_{2j} > 0$ *for all* $j \in \{1,\dots,n\}$*— with domain of homogeneity* $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0,\infty]\}$ *and* σ_{3j} *is locally* r_1*-homogeneous of degree* $\alpha_3 = r_2$ *—i.e.,* $r_{3j} = r_3 = r_1$ *and* $\alpha_{3j} = \alpha_3 = r_2$ *for all* $j \in \{1,\dots,n\}$*— with domain of homogeneity*

$D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j} \in (0, \infty]\}$. *Thus, for any positive definite diagonal matrices* K_1, K_2, A *and* B: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, *and the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is:*

1. *globally finite-time stable if* $r_2 < r_1$;

2. *globally asymptotically stable and (locally) exponentially stable if* $r_2 = r_1$.

Proof. Observe that Proposition 3.3 holds since the proposed control scheme is applied. Consequently $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. Subsequently, all that remains to prove are the specific stability properties claimed in items 1 and 2 of the statement. In this direction, let $\hat{r}_i = (r_{i1}, \ldots, r_{in})^T, i = 1, 2, 3$; $r = (\hat{r}_1^T, \hat{r}_2^T, \hat{r}_3^T)^T$; $K_3 = \text{diag}[k_{31}, \ldots, k_{3n}]$, with $k_{3j} = 1$, $\forall j = 1, \ldots, n$,

$$\begin{aligned} D_o &\triangleq \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : K_i x_i \in D_{i1} \times \cdots \times D_{in},\ i = 1, 2, 3\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x_{ij}| < L_{ij}/k_{ij},\ i = 1, 2, 3,\ j = 1, \ldots, n\} \end{aligned}$$

and consider the closed-loop state-space representation $\dot{x} = f(x) + \hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (3.16). Since D_o defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{3n} : \|x\| < \rho\} \subset D_o$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0, 1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall \varepsilon \in (0, 1)$), and, for every $j \in \{1, \ldots, n\}$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2 - r_1) + r_1} x_{2j} = \varepsilon^{(r_2 - r_1) + r_{1j}} f_j(x)$$

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)[s_1(K_1 \delta_\varepsilon^{\hat{r}_1}(x_1)) + s_2(K_2 \delta_\varepsilon^{\hat{r}_3}(x_3))] \\ &= -H_j^{-1}(q_d)[s_1(\varepsilon^{r_1} K_1 x_1) + s_2(\varepsilon^{r_3} K_2 x_3)] \\ &= -H_j^{-1}(q_d)[\varepsilon^{\alpha_1} s_1(K_1 x_1) + \varepsilon^{\alpha_2} s_2(K_2 x_3)] \\ &= -H_j^{-1}(q_d)\varepsilon^{2r_2 - r_1}[s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= -\varepsilon^{(r_2 - r_1) + r_2} H_j^{-1}(q_d)[s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= \varepsilon^{(r_2 - r_1) + r_{2j}} f_{n+j}(x) \end{aligned}$$

$$\begin{aligned} f_{2n+j}(\delta_\varepsilon^r(x)) &= -A s_3(\delta_\varepsilon^{\hat{r}_3}(x_3)) + B \delta_\varepsilon^{\hat{r}_2}(x_2) \\ &= -A s_3(\varepsilon^{r_3} x_3) + \varepsilon^{r_2} B x_2 \\ &= -A \varepsilon^{\alpha_3} s_3(x_3) + \varepsilon^{r_2} B x_2 \\ &= \varepsilon^{r_2}[-A s_3(x_3) + B x_2] \\ &= \varepsilon^{(r_2 - r_3) + r_3}[-A s_3(x_3) + B x_2] \\ &= \varepsilon^{(r_2 - r_1) + r_{3j}} f_{2n+j}(x) \end{aligned}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium if $r_2 < r_1$ (by Theorems 2.1

and 3.2), and a globally asymptotically stable equilibrium (Theorem 3.2) with (local) exponential stability if $r_2 = r_1$ (recalling Lemma 2.2 and Remark 2.6) . Thus, appealing to Lemma 2.3, as well as Remarks 2.3 and 2.7, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability provided that $r_2 = r_1$, if

$$\begin{aligned}
\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{11}},\ldots,\varepsilon^{-r_{1n}},\varepsilon^{-r_{21}},\ldots,\varepsilon^{-r_{2n}},\varepsilon^{-r_{31}},\ldots,\varepsilon^{-r_{3n}}]\hat{f}\big(\delta_\varepsilon^r(x)\big)\right\| \\
&= \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{21}},\ldots,\varepsilon^{-r_{2n}}]\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\
&= \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha-r_2}\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2}\left\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| = 0
\end{aligned} \tag{3.19}$$

for all $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$, resp. $x \in S_{r,c}^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\|_r = c\}$, for some $c > 0$ such that $S_c^{3n-1} \subset D$, resp. $S_{r,c}^{3n-1} \subset D$. Hence, from (3.17b) and the application of Property 1.2.3, we have, for all $x \in S_c^{3n-1}$, resp. $x \in S_{r,c}^{3n-1}$:

$$\begin{aligned}
\left\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| &= \Big\| - H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,\varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 \\
&\qquad - \mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(\varepsilon^{r_1}K_1x_1)+s_2(\varepsilon^{r_3}K_2x_3)\big]\Big\| \\
&\le \left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)\varepsilon^{2r_2}x_2\right\| \\
&\quad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[\varepsilon^{\alpha_1}s_1(K_1x_1)+\varepsilon^{\alpha_2}s_2(K_2x_3)\big]\right\| \\
&\le \left\|\varepsilon^{2r_2}H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\
&\quad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)\varepsilon^{2r_2-r_1}\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\| \\
&\le \varepsilon^{2r_2}\left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\
&\quad + \varepsilon^{2r_2-r_1}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\|
\end{aligned}$$

and consequently, from (3.19), we get

$$\begin{aligned}
\mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+} \varepsilon^{r_1}\left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\
&\quad + \lim_{\varepsilon\to 0^+}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\| \\
&\le \left\|H^{-1}(q_d)C(q_d,x_2)x_2\right\| \lim_{\varepsilon\to 0^+}\varepsilon^{r_1} + \left\|s_1(K_1x_1)+s_2(K_2x_3)\right\| \lim_{\varepsilon\to 0^+}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\right\| \\
&\le \left\|s_1(K_1x_1)+s_2(K_2x_3)\right\|\cdot\left\|\mathcal{H}(0_n)\right\| = 0
\end{aligned}$$

This is concluded from the definition of $\mathcal{H}(x_1)$ in (3.6), which completes the proof. ■

Corollary 3.2. *Consider σ_{ij}, $i = 1, 2, 3$, $j = 1, \ldots, n$, in the control scheme* (3.13)–(3.14)*, such that*

$$\sigma_{ij}(\varsigma) = \operatorname{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall|\varsigma| \leq L_{ij} \in (0.\infty] \tag{3.20}$$

with —for every $j = 1, \ldots, n$— constants $\beta_{ij} = \beta_i$, $i = 1, 2, 3$, such that

$$0 < \beta_1, \quad \beta_2 = \beta_1, \quad \beta_3 = \frac{1 + \beta_1}{2} \tag{3.21}$$

Thus, for any positive definite diagonal matrices K_1, K_2, A and B, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

1. *globally finite-time stable if* $0 < \beta_1 < 1$*;*
2. *globally asymptotically stable and (locally) exponentially stable if $\beta_1 = 1$.*

Proof. Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and

$$\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \operatorname{sign}(\varepsilon^{r_{ij}}\varsigma)|\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \operatorname{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma)$$

$\forall\varepsilon \in (0, 1]$. Hence, under the consideration of expressions (3.21), for every $j \in \{1, \ldots, n\}$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = (1 + \beta_1)r_1/2$ and $r_{3j} = r_3 = r_1$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{1j} = \alpha_1 = r_1\beta_1 = r_3\beta_2 = \alpha_2 = \alpha_{2j}$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{3j} is locally r_1-homogeneous of degree $\alpha_{3j} = \alpha_3 = (1 + \beta_1)r_3/2 = (1 + \beta_1)r_1/2 = r_2$ with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j}\}$, while

$$\begin{aligned}
0 < \beta_1 \leq 1 \implies \beta_1 > 0 \geq \beta_1 - 1 \geq \frac{(\beta_1 - 1)}{2} \quad &\implies \quad \frac{(\beta_1 + 1)}{2} \leq 1 < \beta_1 + 1 \\
&\iff \quad \frac{(\beta_1 + 1)r_1}{2} \leq r_1 < (\beta_1 + 1)r_1 \\
&\iff \quad r_2 \leq r_1 < 2r_2 \\
&\iff \quad r_2 - r_1 \leq 0 < 2r_2 - r_1
\end{aligned}$$

The requirements of Proposition 3.4 are thus concluded to be satisfied with $0 < \beta_1 < 1 \Rightarrow r_2 < r_1$ and $\beta_1 = 1 \Rightarrow r_2 = r_1$. ■

Corollary 3.2 states a useful particular way to define the functions involved in the control scheme (3.13)–(3.14). Such particular way to define the referred functions permits, via the parameters β, to get either finite-time or exponential convergence.

Remark 3.4. Since the results exposed in this section (the proposed state-feedback and output-feedback control schemes) depart from the application of the control law (3.3), resp. (3.13), the cases of Proposition 3.2 with $r_2 > r_1$ and Corollary 3.1 with $\gamma \in (0, 1)$, as well as Proposition 3.4 with $r_2 > r_1$ and Corollary 3.2 with $\beta_1 > 1$, are particular cases of Proposition 3.1, resp. 3.3, where the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically stable but not (locally) exponentially stable (in accordance to Remark 2.5).

3.2 Regulation with desired conservative force compensation

In the following, the study is focused on control schemes involving desired conservative force compensation, *i.e.*, the controller includes a term for the compensation of the conservative force evaluated at the desired equilibrium position vector, which constitutes a simplification improvement in the implementation of the control law by reducing the computations required to obtain the conservative force; however, such a compensation implies a more complex and non-trivial analytical development. Analogously to the preceding section, a state-feedback controller, and subsequently, an output feedback scheme are proposed.

In this section Assumption 1.4 is modified as follows,

Assumption 3.1. *$T_i > \eta B_{gi}$, $\forall i \in \{1, \dots, n\}$, for some scalar $\eta \geq 1$.*

The sense of Assumption 3.1 will be clarified later on through Remarks 3.5 and 3.12.

3.2.1 State-feedback control scheme

Consider the following SPD-type controller with desired conservative force compensation

$$u(q, \dot{q}) = -s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) + g(q_d) \tag{3.22}$$

where $\bar{q} = q - q_d$ and K_i, $i = 1, 2$, are as defined in the previous subsection; for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \dots, \sigma_{in}(x_n)\big)^T$, $i = 0, 1, 2$, with —for each $j = 1, \dots, n$— σ_{0j} being a strictly increasing strictly passive function, σ_{1j} being strongly passive and σ_{2j} being strictly passive, all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$, and such that

$$B_j \triangleq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} |\sigma_{0j}\big(\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big)| < T_j - B_{gj} \tag{3.23}$$

(recall Assumption 1.3.1) with —for each $j = 1, \dots, n$— k_{1j}, σ_{0j} and σ_{1j} additionally required to be such that

$$|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)| > \min\big\{k_g|\varsigma|, \, 2B_{gj}\big\} \tag{3.24}$$

$\forall\varsigma \neq 0$ (recall Assumption 1.3.2).

Remark 3.5. From the formulation of the proposed scheme, one can verify that the proper satisfaction of the stated requirements entails that

$$2B_{gj} < \big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\big| \leq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} |\sigma_{0j}\big(\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big)| < T_j - B_{gj}$$

$\forall|\varsigma| \geq 2B_{gj}/k_g$, whence one sees that Assumption 3.1 with $\eta = 3$ is a necessary condition for the feasibility of simultaneous fulfilment of (3.23) and (3.24). A similar condition on the control input bounds has been required by other approaches where input constraints have been considered [44], [27], generally arising from the worst-case procedure followed to ensure that the analytical requirements that guarantee the result are fulfilled.

Remark 3.6. Observe that (3.24) could have been alternatively stated as requiring $|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)| \geq \min\{\hat{k}_{1j}|\varsigma|, b_j\}$ for some constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$. However, by stating (3.24), the existence of constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$ such that $|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)| \geq \min\{\hat{k}_{1j}|\varsigma|, b_j\} > \min\{k_g|\varsigma|, 2B_{gj}\}$, $\forall \varsigma \neq 0$, is implied.

Remark 3.7. Note that the control gains in K_2 are not at all restricted and are consequently free to take any positive value, while those in K_1 are the only ones whose choice remains restricted in accordance with the design requirement stated through (3.24) (where they are involved in).

Proposition 3.5. *Consider system* (3.1)–(3.2) *in closed-loop with the proposed control law* (3.22), *under Assumptions* (1.1)–(1.3) *and* (1.4) *with* $\eta = 3$, *and the above stated design specifications. Thus, the global asymptotic stability of the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is guaranteed with* $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.

Proof. Notice that —for every $j = 1, \ldots, n$— by (3.23), we have that, for any $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$

$$\begin{aligned}
|u_j(q,\dot{q})| &= |-\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j})\big) + g_j(q_d)| \\
&\leq |\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j})\big)| + |g_j(q_d)| \\
&\leq B_j + B_{gj} < T_j
\end{aligned}$$

From this and (3.2), one sees that $T_j > |u_j(q,\dot{q})| = |u_j| = |\tau_j|$, $\forall (q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, the input saturation values, T_j, are never reached. Hence, the closed-loop dynamics takes the form

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = -s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) + g(q_d)$$

By defining $x_1 = \bar{q}$ and $x_2 = \dot{q}$, the closed-loop dynamics adopts the $2n$ state-space representation

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= H^{-1}(x_1 + q_d)\Big[-s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) - C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d)\Big]
\end{aligned}$$

Additionally, taking $x = (x_1^T, x_2^T)^T$, these state equations may be rewritten in the form of system (2.4) with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{3.25a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)] \\ \quad - \mathcal{H}((x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{3.25b}$$

recalling that $\mathcal{H}(x_1)$ was defined in (3.6) as $\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d)$. Thus, the closed-loop stability property stated through Proposition 3.5 is corroborated by showing that $x = 0_{2n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem.

Theorem 3.3. *Under the stated design requirements, the origin is a globally asymptotically stable equilibrium of* $\dot{x} = f(x) + \ell\hat{f}(x)$, $\forall \ell \in \{0, 1\}$ *—i.e., of both the state equation* $\dot{x} = f(x)$ *and the (closed-loop) system* $\dot{x} = f(x) + \hat{f}(x)$*— with* $f(x)$ *and* $\hat{f}(x)$ *defined through Eqs.* (3.25).

Proof. For every $\ell \in \{0, 1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1, x_2) = \frac{1}{2} x_2^T H(\ell x_1 + q_d) x_2 + \mathcal{U}_\ell(x_1) \tag{3.26}$$

where

$$\mathcal{U}_\ell(x_1) \triangleq \int_{0_n}^{x_1} s_0^T\big(s_1(K_1 z)\big)\, dz + \ell\, \mathcal{U}(x_1) \tag{3.27}$$

with

$$\int_{0_n}^{x_1} s_0^T\big(s_1(K_1 z)\big)\, dz = \sum_{j=1}^{n} \int_0^{x_{1j}} \sigma_{0j}\big(\sigma_{1j}(k_{1j} z_j)\big)\, dz_j \tag{3.28}$$

and

$$\begin{aligned} \mathcal{U}(x_1) &\triangleq \mathcal{U}_{ol}(x_1 + q_d) - \mathcal{U}_{ol}(q_d) - g^T(q_d) x_1 &\text{(3.29a)}\\ &= \int_{0_n}^{x_1} [g(z + q_d) - g(q_d)]^T\, dz &\text{(3.29b)}\\ &= \int_{0_n}^{x_1} \left[\int_{0_n}^{z} \frac{\partial g}{\partial q}(\bar{z} + q_d) d\bar{z} \right]^T dz &\text{(3.29c)} \end{aligned}$$

where $\mathcal{U}_{ol}(q)$ is the potential energy function of the open-loop system. Observe from (3.29) and Assumption 1.3.2 that

$$\begin{aligned} \mathcal{U}(x_1) \leq \int_{0_n}^{x_1} \left[\int_{0_n}^{z} \left\| \frac{\partial g}{\partial q}(\bar{z} + q_d) \right\| d\bar{z} \right]^T dz \leq \int_{0_n}^{x_1} \left[\int_{0_n}^{z} k_g d\bar{z} \right]^T dz &= \int_{0_n}^{x_1} k_g z^T\, dz \\ &= \sum_{j=1}^{n} \int_0^{x_{1j}} k_g z_j dz_j \end{aligned} \tag{3.30}$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (3.29c)) and simultaneously that

$$\mathcal{U}(x_1) \leq \sum_{j=1}^{n} \int_0^{x_{1j}} \operatorname{sign}(z_j)|g_j(z+q_d) - g_j(q_d)|dz_j \leq \sum_{j=1}^{n} \int_0^{x_{1j}} \operatorname{sign}(z_j) 2B_{gj} dz_j \tag{3.31}$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (3.29b)). From these inequalities, the expressions defined in (3.27) and (3.28), the satisfaction of (3.24) and Remark 3.6, we have that

$$\mathcal{U}_\ell(x_1) \geq \sum_{j=1}^{n} \int_0^{x_{1j}} \operatorname{sign}(z_j) \min\{(\hat{k}_{1j} - \ell k_g)|z_j|,\, (b_j - 2\ell B_{gj})\} dz_j \tag{3.32a}$$

$$\geq \sum_{j=1}^{n} \int_0^{x_{1j}} \operatorname{sign}(z_j) \min\{\bar{k}_{\ell j}|z_j|,\, \bar{b}_{\ell j}\} dz_j \tag{3.32b}$$

$$= \sum_{j=1}^{n} w_{\ell j}(x_{1j}) \triangleq S_\ell(x_1) \tag{3.32c}$$

with

$$w_{\ell j}(x_{1j}) = \begin{cases} \frac{\bar{k}_{\ell j}}{2} x_{1j}^2 & \text{if } |x_{1j}| \leq \bar{b}_{\ell j}/\bar{k}_{\ell j} \\ \bar{b}_{\ell j}\left[|x_{1j}| - \bar{b}_{\ell j}/2\bar{k}_{\ell j}\right] & \text{if } |x_{1j}| > \bar{b}_{\ell j}/\bar{k}_{\ell j} \end{cases} \tag{3.32d}$$

for some $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$, and any positive constants $\bar{k}_{\ell j} \leq \hat{k}_{1j} - \ell k_g$ and $\bar{b}_{\ell j} \leq b_j - 2\ell B_{gj}$.

Remark 3.8. Notice, from expressions (3.32), that S_ℓ, $\ell = 0, 1$, are positive definite radially unbounded functions of x_1. Observe further that (evoking Remark 2.11 and previous arguments)

$$\begin{aligned} D_{x_1}\mathcal{U}_\ell(x_1) &= x_1^T \nabla_{x_1} \mathcal{U}_\ell(x_1) = x_1^T \Big[s_0\big(s_1(K_1 x_1)\big) + \ell\big(g(x_1 + q_d) - g(q_d)\big) \Big] \\ &= \sum_{j=1}^{n} |x_{1j}| \Big[\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j} x_{1j})\big)\big| + \ell \operatorname{sign}(x_{1j}) \big(g_j(x_1 + q_d) - g_j(q_d)\big) \Big] \\ &\geq \sum_{j=1}^{n} |x_{1j}| \Big[\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j} x_{1j})\big)\big| - \ell |g_j(x_1 + q_d) - g_j(q_d)| \Big] \\ &\geq \sum_{j=1}^{n} |x_{1j}| \min\{(\hat{k}_{1j} - \ell k_g)|x_{1j}|,\, (b_j - 2\ell B_{gj})\} \\ &\geq \sum_{j=1}^{n} |x_{1j}| \min\{(\bar{k}_{\ell j}|x_{1j}|,\, (\bar{b}_{\ell j})\} > 0 \end{aligned} \tag{3.33}$$

$\forall x_1 \neq 0_n$ [in any radial direction, $\mathcal{U}_\ell(x_1)$ is strictly increasing, and consequently $x_1 = 0_n$ is the unique stationary point of $\mathcal{U}_\ell(x_1)$], whence one sees that, for every $\ell = 0, 1$,

$$\nabla_{x_1}\mathcal{U}_\ell(x_1) = s_0\big(s_1(K_1x_1)\big) + \ell[g(x_1+q_d) - g(q_d)] = 0_n \iff x_1 = 0_n \tag{3.34}$$

Thus, from (3.26) and (3.32), and Property 1.1, we get that

$$V_\ell(x_1,x_2) \geq \frac{\mu_m}{2}\|x_2\|^2 + S_\ell(x_1) \tag{3.35}$$

with a positive constant μ_m. From this, positive definiteness and radial unboundedness of V_ℓ, $\ell = 0,1$, is concluded. Furthermore, for every $\ell \in \{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$, is obtained as

$$\begin{aligned}
\dot{V}_\ell(x_1,x_2) &= x_2^T H(\ell x_1 + q_d)\dot{x}_2 + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 \\
&\quad + \big[s_0\big(s_1(K_1x_1)\big) + \ell[g(x_1+q_d) - g(q_d)]\big]^T\dot{x}_1 \\
&= x_2^T\big[-\ell\big(C(x_1+q_d,x_2)x_2 + g(x_1+q_d) - g(q_d)\big) - s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big] \\
&\quad + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 + \big[s_0\big(s_1(K_1x_1)\big) + \ell[g(x_1+q_d) - g(q_d)]\big]^T x_2 \\
&= -x_2^T\big[s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) - s_0\big(s_1(K_1x_1)\big)\big] \\
&= -\sum_{j=1}^n x_{2j}\big[\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j}) + \sigma_{2j}(k_{2j}x_{2j})\big) - \sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})\big)\big]
\end{aligned}$$

where in the case of $\ell = 1$, Property 1.2.1 has been applied. Note, from Lemma 2.5, that $\dot{V}_\ell(x_1,x_2) \leq 0$, $\forall(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, with

$$Z_\ell \triangleq \{(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell(x_1,x_2) = 0\} = \{(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n : x_2 = 0\}$$

Furthermore, from the system dynamics $\dot{x} = f(x) + \ell\hat{f}(x)$ —under the consideration of Property 1.1 and Remark 3.8 (more precisely (3.34))— one sees that

$$x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n$$

and

$$x_2(t) \equiv \dot{x}_2(t) \equiv 0_n \implies s_0\big(s_1(K_1x_1)\big) + \ell[g(x_1+q_d) - g(q_d)] \equiv 0_n \iff x_1(t) \equiv 0_n$$

which corroborates that at any $(x_1,x_2) \in \{(x_1,x_2) \in Z_\ell : x_1 \neq 0_n\}$, the resulting unbalanced force term $-s_0\big(s_1(K_1x_1)\big) + \ell[g(x_1+q_d) - g(q_d)]$ acts on the closed-loop dynamics, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0,1$. Therefore, by the invariance theory [42, Corollary 7.2.1], $x = 0_{2n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. ■

Through Proposition 3.5 and Theorem 3.3 both global asymptotic stability of the closed-loop trivial solution and input saturation avoidance are guaranteed. The results obtained so far will prove to be helpful in further developments.

Remark 3.9. The proof of Theorem 3.3 brings to the fore how the proposed scheme shapes the closed-loop potential energy and injects damping to guarantee the stabilization goal. Indeed, one sees from the proof that, through the proposed scheme, the closed-loop potential energy is given the shape adopted from its generalized expression:

$$\mathcal{U}_1(\bar{q}) = \int_{0_n}^{\bar{q}} s_0^T\big(s_1(K_1 z)\big)\, dz + \mathcal{U}_{ol}(x_1 + q_d) - \mathcal{U}_{ol}(q_d) - g^T(q_d)\bar{q}$$

which —through the requirement stated by (3.24)— is guaranteed to be a positive definite radially unbounded function with a global minimum at the origin, giving rise to the closed-loop conservative force

$$u_c(\bar{q}) = \nabla_{\bar{q}}\mathcal{U}_1(\bar{q}) - \nabla_q \mathcal{U}_{ol}(q) = s_0\big(s_1(K_1\bar{q})\big) - g(q_d)$$

Furthermore, damping is injected through a force vector of the form

$$s_d(\bar{q}, \dot{q}) = s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) - s_0\big(s_1(K_1\bar{q})\big)$$

which —through the properties required for σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$— is proven to fulfil $\dot{q}^T s_d(\bar{q}, \dot{q}) > 0$, $\forall \dot{q} \neq 0_n$, $\forall \bar{q} \in \mathbb{R}^n$. Thus, the proposed control law proves to be the addition of a dissipative force opposing to motion, $-s_d(\bar{q}, \dot{q})$, and a restituting conservative force, $-u_c(\bar{q})$; more precisely $u(\bar{q}, \dot{q}) = -s_d(\bar{q}, \dot{q}) - u_c(\bar{q})$, giving rise to the expression in (3.22), which —through the additional requirement in (3.23)— is guaranteed to be suitably bounded.

Finite-time stabilization

Through this section finite-time stability will be proven, recalling that global asymptotic stability and input-saturation avoidance were already concluded.

Proposition 3.6. *Consider the control scheme in* (3.22), *under the additional consideration that, for every* $j = 1, \ldots, n$, σ_{ij}, $i = 1, 2$, *are locally* r_i*-homogeneous of degree* $\alpha_j > 0$ *—i.e.,* $r_{1j} = r_1$, $r_{2j} = r_2$ *and* $\alpha_{1j} = \alpha_{2j} = \alpha_j > 0$, $\forall j = 1, \ldots, n$*— with domain of homogeneity* $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0, \infty]\}$ *and* σ_{0j} *is locally* α_j*-homogeneous of degree* $\alpha_0 = 2r_2 - r_1$ *—i.e.,* $\alpha_{0j} = \alpha_0 = 2r_2 - r_1$ *for all* $j = 1, \ldots, n$*— with domain of homogeneity* $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j} \in (0, \infty]\}$, *for some dilation coefficients* $r_i > 0$, $i = 1, 2$, *such that* $\alpha_0 = 2r_2 - r_1 > 0 > r_2 - r_1$. *Thus, the global finite-time stability of the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is guaranteed with* $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.

Proof. Since the proposed control scheme is applied —with all its previously stated specifications— $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$ holds as a result of Proposition 3.5. Then, the proof is just focused on the finite-time stability property. In this direction, the first part of the proof follows (exactly) from the arguments exposed in the first part of the proof corresponding to Proposition 3.2 (in the on-line compensation case), where it is guaranteed that the origin of the state equation $\dot{x} = f(x)$ is globally finite-time stable. Thus, all that remains to be proven is the finite-time stability property of the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$. In this direction, recalling Theorem 3.3, Lemma 2.3 and Remark 2.7, the origin of $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time equilibrium, provided that $r_2 - r_1 < 0$, if

$$\begin{aligned}\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{11}},\ldots,\varepsilon^{-r_{1n}},\varepsilon^{-r_{21}},\ldots,\varepsilon^{-r_{2n}}]\hat{f}\big(\delta_\varepsilon^r(x)\big)\right\| \\ &= \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{21}},\ldots,\varepsilon^{-r_{2n}}]\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\ &= \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha-r_2}\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2}\left\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\ &= 0\end{aligned} \tag{3.36}$$

for all $x \in S_c^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{2n-1} \subset D$. Hence, from (3.25b) and Property 1.2.3 (related to $C(q,\dot{q})$) , we have, for all such $x \in S_c^{2n-1}$:

$$\begin{aligned}&\left\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \\ &\quad= \Big\| - H^{-1}(\varepsilon^{r_1}x_1 + q_d)[C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 + g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)] \\ &\qquad - \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big)\Big\| \\ &\quad\le \left\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2\right\| \\ &\qquad + \left\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\right\|\left\|g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\right\| \\ &\qquad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big)\right\|\end{aligned}$$

whence, under the consideration of Assumption 1.3.2, we get

$$\begin{aligned}\left\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\right\| \le{}& \varepsilon^{2r_2}\left\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\right\| \\ &+ k_g\varepsilon^{r_1}\left\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\right\|\|x_1\| \\ &+ \varepsilon^{2r_2-r_1}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\right\|\end{aligned}$$

and consequently, from (3.36), (recalling that by design specifications: $r_1 > r_2 > 0$), we get

$$
\begin{aligned}
\mathcal{L}_0 \leq & \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \big\| \\
& + k_g \|x_1\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)} \big\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) \big\| \\
& + \lim_{\varepsilon\to 0^+} \big\| \mathcal{H}(\varepsilon^{r_1}x_1) s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \\
\leq & \big\| H^{-1}(q_d) C(q_d, x_2)x_2 \big\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \\
& + k_g \|x_1\| \big\| H^{-1}(q_d) \big\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)} \\
& + \big\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \lim_{\varepsilon\to 0^+} \big\| \mathcal{H}(\varepsilon^{r_1}x_1) \big\| \\
\leq & \big\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \big\| \cdot \big\| \mathcal{H}(0_n) \big\| = 0
\end{aligned} \tag{3.37}
$$

Evoking (3.6) the proof is completed. ■

Thus, global finite-time stability of the closed-loop trivial solution has already been concluded.

Corollary 3.3. *Consider σ_{ij}, $i = 0, 1, 2$, $j = 1, \dots, n$, for the involved functions in the proposed control scheme* (3.22)*, such that*

$$
\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall |\varsigma| \leq L_{ij} \in (0, \infty)
$$

with constants β_{ij} such that

$$
\beta_{1j} > 0, \quad \beta_{2j} = \gamma\beta_{1j}, \quad \beta_{0j} = \frac{2-\gamma}{\gamma\beta_{1j}} \tag{3.38}
$$

for a constant $\gamma \in (1, 2)$. Thus, the global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$.

Proof. Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and

$$
\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \varepsilon^{r_{ij}\beta_{ij}} \,\mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \sigma_{ij}(\varsigma)
$$

$\forall \varepsilon \in (0, 1]$. Hence, under the consideration of expressions (3.38), for every $j = 1, \dots, n$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = r_1/\gamma$ and $r_{0j} = r_1\beta_{1j}, \sigma_{ij}$, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{2j} = r_2\beta_{2j} = r_1\beta_{1j} = \alpha_{1j} = \alpha_j$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{0j} is locally α_j-homogeneous of degree $\alpha_{0j} = \alpha_0 = (2-\gamma)r_1/\gamma$ with domain of homogeneity $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j}\}$. The requirements of Proposition 3.6 are thus concluded to be satisfied with

$$
1 < \gamma < 2 \iff r_2 < r_1 < 2r_2 \iff r_2 - r_1 < 0 < 2r_2 - r_1 = \alpha_0
$$

■

Remark 3.10. Since the result of this section departs from the application of the proposed control scheme, the case of Proposition 3.6 with $r_2 \geq r_1$, and Corollary 3.3 with $\gamma \in (0, 1]$, is a particular case of Proposition 3.5 where the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically (but not finite-time) stable. It is further worth pointing out that with $r_2 = r_1$ —or analogously $\gamma = 1$ in the case of Corollary 3.3— we have that $\varepsilon^{r_2 - r_1} = 1$, $\forall \varepsilon > 0$. Hence, in this case, analog developments to those giving rise to inequalities (3.37) lead to $\mathcal{L}_0 \leq k_g \|x_1\| \big\| H^{-1}(q_d) \big\|$, and consequently, Lemma 2.3 (under the consideration of Remark 2.6) cannot be applied to conclude (local) exponential stability (contrarly to the on-line gravity force compensation case). Nevertheless, exponential stability is next proven to be achieved (locally), through an alternative (strict-Lyapunov-function-based) analytical procedure.

Exponential stabilization

Exponential stability is proven through the following corollary, recalling what was mentioned in Remark 3.10.

Corollary 3.4. *Consider the proposed control scheme in* (3.22) *taking —for every* $i = 0, 1, 2$ *and* $j = 1, \ldots, n$— σ_{ij} *such that*

$$\sigma_{ij}(\varsigma) = \varsigma \quad \forall |\varsigma| \leq L_{ij} \in (0, \infty) \tag{3.39}$$

Thus $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, *and the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is globally asymptotically stable and (locally) exponentially stable.*

Proof. The global asymptotic stability follows from Proposition 3.5. Thus, all that remains to be proven is the (local) exponential stability property. In this direction, let us consider the scalar function

$$V_2(x_1, x_2) = V_1(x_1, x_2) + \epsilon x_1^T H(x_1 + q_d) x_2$$

with $V_1(x_1, x_2)$ as defined through (3.26) (with $\ell = 1$), *i.e.*,

$$\begin{aligned} V_2(x_1, x_2) = &\frac{1}{2} x_2^T H(x_1 + q_d) x_2 + \int_{0_n}^{x_1} s_0\big(s_1(K_1 z)\big) dz \\ &+ \mathcal{U}_{01}(x_1 + q_d) - \mathcal{U}_{01}(q_d) - g^T(q_d) x_1 + \epsilon x_1^T H(x_1 + q_d) x_2 \end{aligned}$$

where ϵ is a positive constant such that

$$\epsilon < \min\{\epsilon_1, \epsilon_2\} \tag{3.40}$$

with

$$\epsilon_1 = \frac{[\bar{k}_{1m} \mu_m]^{1/2}}{\mu_M} \quad , \quad \epsilon_2 = \frac{\bar{k}_{1m} k_{2m}}{\bar{k}_{1m} k_C \varrho + \bar{k}_{1m} \mu_M + k_{2M}^2 / 4}$$

$\bar{k}_{1m} = \min_j\{\bar{k}_{1j}\}$; $k_{2m} = \min_j\{k_{2j}\}$; $k_{2M} \max_j\{k_{2j}\}$; μ_m, μ_M and k_C as defined through Property 1.1 and Assumptions 1.1–1.2; and ϱ is a positive constant to be defined later on. From the proof of Theorem 3.3 (particularly, from inequality (3.35)), we have that

$$V_2(x_1, x_2) \geq \frac{\mu_m}{2} \|x_2\|^2 + S_1(x_1) - \epsilon |x_1^T H(x_1 + q_d) x_2|$$

with $S_1(x_1)$ as defined through (3.32) (with $\ell = 1$). More precisely, on $\mathcal{Q}_1 \times \mathbb{R}^n$, with $\mathcal{Q}_1 = \{x_1 \in \mathbb{R}^n : |x_{1j}| < \bar{b}_{1j}/\bar{k}_{1j},\ j = 1, \ldots, n\}$, we have that

$$\begin{aligned} V_2(x_1, x_2) &\geq \frac{\mu_m}{2}\|x_2\|^2 + \sum_{j=1}^{n} \frac{\bar{k}_{1j}}{2} x_{1j}^2 - \epsilon |x_1^T H(x_1 + q_d) x_2| \\ &\geq \frac{\mu_m}{2}\|x_2\|^2 + \frac{\bar{k}_{1m}}{2}\|x_1\|^2 - \epsilon \mu_M \|x_1\|\|x_2\| \\ &= \frac{1}{2}\begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix}^T Q_1 \begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix} \end{aligned}$$

with

$$Q_1 = \begin{pmatrix} \bar{k}_{1m} & -\epsilon\mu_M \\ -\epsilon\mu_M & \mu_m \end{pmatrix}$$

where Assumption 1.1 has been considered, and since (3.40) $\implies \epsilon < \epsilon_1 \implies Q_1 > 0$, we get

$$V_2(x) \geq c_1 \|x\|^2 \tag{3.41}$$

$\forall x \in \mathcal{Q}_1 \times \mathbb{R}^n$, with $c_1 = \lambda_m(Q_1)/2 > 0$. On the other hand, observe that in view of (3.39), we have, on

$$\begin{aligned} \mathcal{Q}_0 &= \{x_1 \in \mathbb{R}^n : |x_{1j}| \leq L_{1j}/k_{1j},\ |\sigma_{1j}(k_{1j}x_{1j})| = |k_{1j}x_{1j}| \leq L_{0j}, j = 1, \ldots, n\} \\ &= \{x_1 \in \mathbb{R}^n : |x_{1j}| \leq \min\{L_{1j}, L_{0j}\}/k_{1j},\ j = 1, \ldots, n\} \end{aligned}$$

that $s_0\big(s_1(K_1 x_1)\big) = K_1 x_1$. From this, Assumption 1.1 and (3.30) we get on $\mathcal{Q}_0 \times \mathbb{R}^n$:

$$\begin{aligned} V_2(x_1, x_2) &= \frac{1}{2} x_2^T H(x_1 + q_d) x_2 + \frac{1}{2} x_1^T K_1 x_1 \\ &\quad + \mathcal{U}_{o1}(x_1 + q_d) - \mathcal{U}_{o1}(q_d) - g^T(q_d) x_1 + \epsilon x_1^T H(x_1 + q_d) x_2 \\ &\leq \frac{\mu_M}{2}\|x_2\|^2 + \frac{k_{1M}}{2}\|x_1\|^2 + \frac{k_g}{2}\|x_1\|^2 + \epsilon\mu_M \|x_1\|\|x_2\| \\ &= \frac{1}{2}\begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix}^T Q_2 \begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix} \end{aligned}$$

with

$$Q_2 = \begin{pmatrix} k_{1M} + k_g & \epsilon\mu_M \\ \epsilon\mu_M & \mu_M \end{pmatrix}$$

and $k_{1M} = \max_j\{k_{1j}\}$. From simple developments, one can further verify that (3.40) $\implies \epsilon < \epsilon_1 \implies Q_2 > 0$, whence we get

$$V_2(x) \leq c_2 \|x\|^2 \tag{3.42}$$

$\forall x \in \mathcal{Q}_0 \times \mathbb{R}^n$, with $c_2 = \lambda_M(Q_2)/2 > 0$. Furthermore, the derivative of V_2 along the closed-loop system trajectories is given by

$$
\begin{aligned}
\dot{V}_2(x_1, x_2) &= x_2^T H(x_1+q_d)\dot{x}_2 + \frac{1}{2}x_2^T \dot{H}(x_1+q_d, x_2)x_2 + [s_0\big(s_1(K_1x_1)\big) + g(x_1+q_d) - g(q_d)]^T \dot{x}_1 \\
&\quad + \epsilon x_1^T H(x_1+q_d)\dot{x}_2 + \epsilon x_1^T \dot{H}(x_1+q_d)x_2 + \epsilon \dot{x}_1^T H(x_1+q_d)x_2 \\
&= x_2^T[-C(x_1+q_d, x_2)x_2 - g(x_1+q_d) + g(q_d) - s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)] \\
&\quad + \frac{1}{2}x_2^T \dot{H}(x_1+q_d, x_2)x_2 + [s_0\big(s_1(K_1x_1)\big) + g(x_1+q_d) - g(q_d)]^T x_2 \\
&\quad + \epsilon x_1^T[-C(x_1+q_d, x_2)x_2 - g(x_1+q_d) + g(q_d) - s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)] \\
&\quad + \epsilon x_1^T[C(x_1+q_d, x_2) + C^T(x_1+q_d, x_2)]x_2 + \epsilon x_2^T H(x_1+q_d)x_2 \\
&= -x_2^T[s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) - s_0\big(s_1(K_1x_1)\big)] \\
&\quad - \epsilon x_1^T[s_0\big(s_1(K_1x_1)\big) + g(x_1+q_d) - g(q_d)] \\
&\quad - \epsilon x_1^T[s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) - s_0\big(s_1(K_1x_1)\big)] \\
&\quad + \epsilon x_2^T C(x_1+q_d, x_2)x_1 + \epsilon x_2^T H(x_1+q_d)x_2
\end{aligned}
$$

where Property 1.2.1 has been applied. Notice that, in view of (3.39), we have, on

$$
\begin{aligned}
\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : &|x_{1j}| \leq L_{1j}/k_{1j},\ |x_{2j}| \leq L_{2j}/k_{2j}, \\
&\big|\sigma_{1j}(k_{1j}x_{1j}) + \sigma_{2j}(k_{2j}x_{2j})\big| = |k_{1j}x_{1j} + k_{2j}x_{2j}| \leq L_{0j}, j = 1, \ldots, n\}
\end{aligned}
$$

that $s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big) - s_0\big(s_1(K_1x_1)\big) = K_2x_2$. From this, (3.33), and Assumptions 1.1–1.2, we get

$$
\begin{aligned}
\dot{V}_2(x_1, x_2) &\leq -x_2^T K_2 x_2 - \epsilon \sum_{j=1}^{n} \bar{k}_{1j}x_{1j}^2 + \epsilon|x_1^T K_2 x_2| + \epsilon|x_2^T C(x_1+q_d, x_2)x_1| \\
&\quad + \epsilon|x_2^T H(x_1+q_d)x_2| \\
&\leq -k_{2m}\|x_2\|^2 - \epsilon\bar{k}_{1m}\|x_1\|^2 + \epsilon k_{2M}\|x_1\|\|x_2\| + \epsilon k_C \varrho \|x_2\|^2 + \epsilon\mu_M\|x_2\|^2 \\
&= -\begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix}^T Q_3 \begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix}
\end{aligned}
$$

$\forall (x_1, x_2) \in \mathcal{S} \cap (\mathcal{Q}_1 \times \mathbb{R}^n)$, with

$$
Q_3 = \begin{pmatrix} \epsilon\bar{k}_{1m} & -\frac{\epsilon k_{2M}}{2} \\ -\frac{\epsilon k_{2M}}{2} & k_{2m} - \epsilon(k_C\varrho + \mu_M) \end{pmatrix}
$$

and $\varrho = \max_{x_1 \in \mathcal{Q}_1} \|x_1\| = \big[\sum_{j=1}^{n}[\bar{b}_{1j}/\bar{k}_{1j}]^2\big]^{1/2}$, and since (3.40) $\implies \epsilon < \epsilon_2 \implies Q_3 > 0$, we get

$$
\dot{V}_2(x) \leq -c_3\|x\|^2 \tag{3.43}
$$

$\forall x \in \mathcal{S} \cap (\mathcal{Q}_1 \times \mathbb{R}^n)$, with $c_3 = \lambda_m(Q_3) > 0$. Thus, from the simultaneous satisfaction of inequalities (3.41), (3.42), and (3.43) on $\mathcal{S} \cap [(\mathcal{Q}_0 \cap \mathcal{Q}_1) \times \mathbb{R}^n]$, we conclude —by [40, Theorem 4.10]— that the origin $(x_1, x_2) = (0_n, 0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system, whence the proof is completed. ■

Notice that the functions $\sigma_{ij}(\cdot)$, $i = 0, 1, 2$, $j = 1, \ldots, n$, defined in (3.39) through Corollary 3.4 turn out to be a particular case of the functions $\sigma_{ij}(\cdot)$, $i = 0, 1, 2$, $j = 1, \ldots, n$, defined in Corollary 3.3 for $\beta_{ij} = 1$, $i = 0, 1, 2$, $j = 1, \ldots, n$. Thus, the functions involved in the control scheme (3.22) are characterized through Corollary 3.3, such a characterization permits, via the parameters γ and β, to get either finite-time or exponential convergence.

3.2.2 Output-feedback control scheme

Consider the following SP-SD type controller with desired conservative-force compensation

$$u(q, \vartheta) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d) \tag{3.44}$$

where $\bar{q}$, q_d, K_1, and K_2 are as defined in (3.13), with ϑ involved as the output vector variable of an auxiliary subsystem defined as (in Eqs.(3.14), *i.e.*,)

$$\dot{\vartheta}_c = -As_3(\vartheta_c + B\bar{q}) \tag{3.45a}$$

$$\vartheta = \vartheta_c + B\bar{q} \tag{3.45b}$$

For any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\big)^T$, $i = 1, 2, 3$, with —for each $j = 1, \ldots, n$— σ_{3j} being a strictly passive function, while σ_{ij}, $i = 1, 2$, are strongly passive functions such that

$$B_j \triangleq \sup_{(\varsigma_1, \varsigma_2) \in \mathbb{R}^2} |\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)| < T_j - B_{gj} \tag{3.46}$$

all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$; and with —for each $j = 1, \ldots, n$— k_{1j} and σ_{1j} additionally required to be such that

$$|\sigma_{1j}(k_{1j}\varsigma)| > \min\left\{k_g|\varsigma|,\, 2B_{gj}\right\} \tag{3.47}$$

$\forall \varsigma \neq 0$, with k_g as defined through Assumption 1.3.2.

Remark 3.11. Note that by (3.44), we have that —for every $j = 1, \ldots, n$— σ_{1j} and σ_{2j} shall both to be bounded, while σ_{3j} may be freely chosen to be bounded or not.

Remark 3.12. Analogously to what is exposed in Remark 3.5 for the state-feedback case, Assumption 3.1 with $\eta = 3$ is a necessary condition for the feasibility of the simultaneous fulfilment of (3.46) and (3.47). Further, Remark 3.6 is accomplished, with $\sigma_{0j}(\varsigma) = \varsigma$, and similarly to Remark 3.7, the control parameters in K_2, A, and B are not restricted, whereas those in K_1 are the only ones whose choice remains restricted in accordance with the design requirement stated through (3.47).

Proposition 3.7. *Consider system* (3.1)–(3.2) *in closed-loop with the proposed control law* (3.44)–(3.45)*, under the above stated design specifications. Thus,* $|\tau_j(t)| = |u_j(t)| < T_j$*,* $j = 1, \ldots, n$*,* $\forall t \geq 0$ *and global asymptotic stability of the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is guaranteed.*

Proof. Observe that —for every $j = 1, \ldots, n$— by (3.46), we have that, for any $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$:

$$|u_j(q,\vartheta)| = |-\sigma_{1j}(k_{1j}\bar{q}_j) - \sigma_{2j}(\vartheta_j) + g_j(q_d)| \leq |\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(\vartheta_j)| + |g_j(q_d)| \leq B_j + B_{gj} < T_j$$

From this and (3.2), one sees that $T_j > |u_j(q,\vartheta)| = |u_j| = |\tau_j|$, $\forall (q,\vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. Hence, the closed-loop dynamics takes the (equivalent) form

$$\begin{aligned} H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) &= -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d) \\ \dot{\vartheta} &= -As_3(\vartheta) + B\dot{q} \end{aligned}$$

Let $x_1 = \bar{q}$, $x_2 = \dot{q}$ and $x_3 = \vartheta$, then the closed-loop dynamics adopts the $3n$ state-space representation

$$\dot{x}_1 = x_2, \tag{3.48a}$$

$$\dot{x}_2 = -H^{-1}(x_1 + q_d)\Big[s_1(K_1x_1) + s_2(K_2x_3) + C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)\Big] \tag{3.48b}$$

$$\dot{x}_3 = -As_3(x_3) + Bx_2 \tag{3.48c}$$

By further defining $x = (x_1^T, x_2^T, x_3^T)^T$, these state equations may be rewritten in the form of system $\dot{x} = f(x) + \hat{f}(x)$ with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)[s_1(K_1x_1) + s_2(K_2x_3)] \\ -As_3(x_3) + Bx_2 \end{pmatrix} \tag{3.49a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)] \\ -\mathcal{H}((x_1)[s_1(K_1x_1) + s_2(K_2x_3)] \\ 0_n \end{pmatrix} \tag{3.49b}$$

with $\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d)$. Thus, the closed-loop stability property stated through Proposition 3.7 is corroborated by showing that $x = 0_{3n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem. ∎

Theorem 3.4. *Consider the above stated specifications, then the origin is a globally asymptotically stable equilibrium of* $\dot{x} = f(x) + \ell\hat{f}(x)$*,* $\forall \ell \in \{0, 1\}$ *—i.e., of both the state equation* $\dot{x} = f(x)$ *and the (closed-loop) system* $\dot{x} = f(x) + \hat{f}(x)$*— with* $f(x)$ *and* $\hat{f}(x)$ *defined through Eqs.* (3.49).

Proof. For every $\ell \in \{0,1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1,x_2,x_3) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \mathcal{U}_\ell(x_1) + \int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz \tag{3.50}$$

where

$$\int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz = \sum_{j=1}^{n}\int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}z_j)}{b_j}dz_j$$

and

$$\mathcal{U}_\ell(x_1) \triangleq \int_{0_n}^{x_1} s_1^T(K_1 z)\,dz + \ell\,\mathcal{U}(x_1) \tag{3.51}$$

with

$$\int_{0_n}^{x_1} s_1^T(K_1 z)\,dz = \sum_{j=1}^{n}\int_0^{x_{1j}} \sigma_{1j}(k_{1j}z_j)\,dz_j \tag{3.52}$$

and $\mathcal{U}(x_1) \triangleq \mathcal{U}_{o1}(x_1+q_d) - \mathcal{U}_{o1}(q_d) - g^T(q_d)x_1$ as was defined in (3.29) for the state-feedback case. Observe that inequalities (3.30)–(3.31) hold. From this, (3.51) and (3.52), the satisfaction of (3.47), and Remark 3.6, we have that

$$\begin{aligned}
\mathcal{U}_\ell(x_1) &\geq \sum_{j=1}^{n}\int_0^{x_{1j}} \operatorname{sign}(z_j)\min\{(\hat{k}_{1j}-\ell k_g)|z_j|,\,(b_j - 2\ell B_{gj})\}dz_j && \text{(3.53a)}\\
&\geq \sum_{j=1}^{n}\int_0^{x_{1j}} \operatorname{sign}(z_j)\min\{\bar{k}_{\ell j}|z_j|,\,\bar{b}_{\ell j}\}dz_j && \text{(3.53b)}\\
&= \sum_{j=1}^{n} w_{\ell j}(x_{1j}) \triangleq S_\ell(x_1) && \text{(3.53c)}
\end{aligned}$$

with

$$w_{\ell j}(x_{1j}) = \begin{cases} \frac{\bar{k}_{\ell j}}{2}x_{1j}^2 & \text{if } |x_{1j}| \leq \bar{b}_{\ell j}/\bar{k}_{\ell j} \\ \bar{b}_{\ell j}\left[|x_{1j}| - \bar{b}_{\ell j}/2\bar{k}_{\ell j}\right] & \text{if } |x_{1j}| > \bar{b}_{\ell j}/\bar{k}_{\ell j} \end{cases} \tag{3.53d}$$

for some $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$, and any positive constants $\bar{k}_{\ell j} \leq \hat{k}_{1j} - \ell k_g$ and $\bar{b}_{\ell j} \leq b_j - 2\ell B_{gj}$.

Remark 3.13. One sees from expressions (3.53) that S_ℓ, $\ell = 0,1$, are positive definite radially unbounded functions of x_1. Observe further that (analog to Remark 3.8)

$$
\begin{aligned}
D_{x1}\mathcal{U}_\ell(x_1) = x_1^T\nabla_{x_1}\mathcal{U}(x_1) &= x_1^T\big[s_1(K_1x_1) + \ell\big(g(x_1+q_d) - g(q_d)\big)\big] \\
&= \sum_{j=1}^n |x_{1j}|\big[|\sigma_{1j}(k_{1j}x_{1j})| + \ell\,\mathrm{sign}(x_{1j})\big(g_j(x_1+q_d) - g_j(q_d)\big)\big] \\
&\geq \sum_{j=1}^n |x_{1j}|\big[|\sigma_{1j}(k_{1j}x_{1j})| - \ell|g_j(x_1+q_d) - g_j(q_d)|\big] \\
&\geq \sum_{j=1}^n |x_{1j}|\min\{(\hat{k}_{1j} - \ell k_g)|x_{1j}|,\ (b_j - 2\ell B_{gj})\} \\
&\geq \sum_{j=1}^n |x_{1j}|\min\{(\bar{k}_{\ell j}|x_{1j}|,\ (\bar{b}_{\{}\ell j\}\} > 0
\end{aligned} \tag{3.54}
$$

$\forall x_1 \neq 0_n$, whence one sees that, for every $\ell = 0,1$, $s_1(K_1x_1) + \ell[g(x_1+q_d) - g(q_d)] = 0_n \iff x_1 = 0_n$.

Thus, from Eqs. (3.50) and (3.53), and Property 1.1, we get that

$$
V_\ell(x_1,x_2,x_3) \geq \frac{\mu_m}{2}\|x_2\|^2 + S_\ell(x_1) + \int_{0_n}^{x_3} s_2^T(K_2z)B^{-1}dz \tag{3.55}
$$

whence, under the consideration of the strongly passive character of $\sigma_{2j}(\cdot)$, $j=1,\ldots,n$, involved in $s_2(\cdot)$, positive definiteness and radial unboundedness of V_ℓ, $\ell = 0,1$, is concluded. Further, for every $\ell \in \{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$, is obtained as

$$
\begin{aligned}
\dot{V}_\ell(x_1,x_2,x_3) &= x_2^T H(\ell x_1+q_d)\dot{x}_2 + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 \\
&\quad + \big[s_1(K_1x_1) + \ell[g(x_1+q_d) - g(q_d)]\big]^T\dot{x}_1 + s_2^T(K_2x_3)B^{-1}\dot{x}_3 \\
&= -x_2^T\big[\ell[C(x_1+q_d,x_2)x_2 + g(x_1+q_d) - g(q_d)] + s_1(K_1x_1) + s_2(K_2x_3)\big] \\
&\quad + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d)x_2 + \big[s_1(K_1x_1) + \ell[g(x_1+q_d) - g(q_d)]\big]^T x_2 \\
&\quad + s_2^T(K_2x_3)B^{-1}[-As_3(x_3) + Bx_2] \\
&= -s_2^T(K_2x_3)B^{-1}As_3(x_3) \\
&= -\sum_{j=1}^n \frac{a_j}{b_j}\sigma_{2j}(k_{2j}x_{3j})\sigma_{3j}(x_{3j})
\end{aligned}
$$

where Property 1.2.1 has been applied for $\ell = 1$. Note, from the strictly passive character of σ_{2j} and σ_{3j} (recalling that a strongly passive function is strictly passive), $j=1,\ldots,n$, that $\dot{V}_\ell(x_1,x_2,x_3) \leq 0$, $\forall(x_1,x_2,x_3) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, with

$$
Z_\ell \triangleq \{(x_1,x_2,x_3) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : \dot{V}_\ell(x_1,x_2,x_3) = 0\} = \{(x_1,x_2,x_3) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : x_3 = 0\}
$$

Further, from the system dynamics $\dot{x} = f(x) + \ell\hat{f}(x)$ —under the consideration of Remark 3.13— one sees that

$$x_3(t) \equiv 0_n \implies \dot{x}_3(t) \equiv 0_n$$

while

$$x_3(t) \equiv \dot{x}_3(t) \equiv 0_n \implies x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n$$

and

$$x_3(t) \equiv \dot{x}_3(t) \equiv x_2(t) \equiv \dot{x}_2(t) \equiv 0_n \implies s_1(K_1x_1) + \ell[g(x_1 + q_d) - g(q_d)] \equiv 0_n \iff x_1(t) \equiv 0_n$$

which corroborates that at any $(x_1, x_2, x_3) \in Z_\ell \setminus \{(0_n, 0_n, 0_n)\}$, the resulting unbalanced force terms act on the closed-loop dynamics $[\dot{x} = f(x_1, x_2, 0_n) + \ell\hat{f}(x_1, x_2, 0_n)$ with $(x_1, x_2) \neq (0_n, 0_n)]$, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n, 0_n, 0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0, 1$. Therefore, by the invariance theory [42, Corollary 7.2.1], $x = 0_{3n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. ■

Through Proposition 3.7 and Theorem 3.4 both global asymptotic stability of the closed-loop trivial solution and input saturation avoidance are guaranteed. The results obtained so far will prove to be helpful in further developments.

Remark 3.14. As shown in the on-line compensation case developed in Section 3.1, it is the dirty-derivative-based auxiliary subsystem in Eqs. (3.14) which performs the energy dissipation in the closed-loop system (in the absence of the velocity variables in the feedback). This is analogously visualised through the feedback-system passivity approach of Remark 3.3 as follows: under the consideration of the closed-loop system in Eqs. (3.48), let $e_1 = -y_2 = -s_2(K_2x_3)$, $e_2 = y_1 = x_2$, $\psi(x_3) = s_2^T(K_2x_3)B^{-1}As_3(x_3)$,

$$V_{11}(x_1, x_2) = \frac{1}{2}x_2^T H(x_1 + q_d)x_2 + \mathcal{U}_1(x_1)$$

and

$$V_{12}(x_3) = \int_{0_n}^{x_3} s_2^T(K_2z)B^{-1}dz$$

By previous arguments and developments, V_{11} and V_{12} are radially unbounded positive definite functions in their respective arguments. Following an analysis analog to that of the proof of Theorem 3.4, one obtains $\dot{V}_{11} = e_1^Ty_1$ and $\dot{V}_{12} = e_2^Ty_2 - \psi(x_3)$, with $\psi(x_3)$ being positive definite (in its argument). Hence, the closed-loop system in Eqs. (3.48) may be seen as a (negative) feedback system connection among a passive —actually lossless— subsystem Σ_1 with dynamic model

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = H^{-1}(x_1 + q_d)\big[- C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) + e_1\big] \\ y_1 = x_2 \end{cases}$$

and a positive definite storage function $V_{11}(x_1, x_2)$, and a strictly passive subsystem Σ_2 with state model

$$\Sigma_2 : \begin{cases} \dot{x}_3 = -As_3(x_3) + Be_2 \triangleq f_2(x_3, e_2) \\ y_2 = s_2(K_2 x_3) \end{cases} \tag{3.56}$$

and storage function $V_{12}(x_3)$. Moreover, one sees from (3.56) that $f_2(0_n, e_2) = Be_2 = 0_n \implies e_2 = 0_n$, completing the requirements of Theorem 2.3.

Finite-time stabilization

Through this section finite-time stability will be proven, recalling that global asymptotic stability and input-saturation avoidance were already concluded.

Proposition 3.8. *Consider the proposed control scheme* (3.44)–(3.45) *under the additional consideration that, for every* $j = 1, \dots, n$, σ_{ij}, $i = 1, 2$, *are locally* r_i*-homogeneous of (common) degree* $\alpha_i = 2r_2 - r_1$, *with dilation coefficients such that* $2r_2 - r_1 > 0 > r_2 - r_1$ *and domain of homogeneity* $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0, \infty]\}$, *and* σ_{3j} *is locally* r_1*-homogeneous of degree* $\alpha_3 = r_2$, *with domain of homogeneity* $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j} \in (0, \infty]\}$. *Thus, the global finite-time stability of the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is guaranteed with* $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$.

Proof. The first part of the proof follows (exactly) from the arguments exposed (for the on-line compensation case) in the proof of Proposition 3.4 (specifically from item 1), whence $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$ and the global finite-time stability property of the origin of the state equation $\dot{x} = f(x)$ are concluded. Thus, all that remains to be proven is the finite-time stability property of the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$. In this direction, recalling Theorem 3.4, Lemma 2.3 and Remark 2.7, the origin of $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time equilibrium, provided that $r_2 - r_1 < 0$, if

$$\begin{aligned}
\mathcal{L}_0 &\triangleq \lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}[\varepsilon^{-r_{11}}, \dots, \varepsilon^{-r_{1n}}, \varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}, \varepsilon^{-r_{31}}, \dots, \varepsilon^{-r_{3n}}] \hat{f}\big(\delta_\varepsilon^r(x)\big) \right\| \\
&= \lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}[\varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}] \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= \lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha - r_2} \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= \lim_{\varepsilon \to 0^+} \varepsilon^{r_1 - 2r_2} \left\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \right\| \\
&= 0
\end{aligned} \tag{3.57}$$

for all $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{3n-1} \subset D$. Hence, from (3.49b), under the consideration of Property 1.2.3, we have, for all such $x \in S_c^{3n-1}$:

$$\begin{aligned}
&\big\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \big\| \\
&\qquad = \big\| - H^{-1}(\varepsilon^{r_1}x_1 + q_d)[C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 + g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)] \\
&\qquad\qquad - \mathcal{H}(\varepsilon^{r_1}x_1)[s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_3}K_2x_3)]\big\| \\
&\qquad \le \big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2\big\| \\
&\qquad\quad + \big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\big\|\big\|g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\big\| \\
&\qquad\quad + \big\|\mathcal{H}(\varepsilon^{r_1}x_1)[\varepsilon^{\alpha_1}s_1(K_1x_1) + \varepsilon^{\alpha_2}s_2(K_2x_3)]\big\|
\end{aligned}$$

whence, under the consideration of Assumption 1.3.2, we get

$$\begin{aligned}
\big\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \big\| &\le \varepsilon^{2r_2}\big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\big\| \\
&\quad + k_g\varepsilon^{r_1}\big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\big\|\|x_1\| \\
&\quad + \varepsilon^{2r_2 - r_1}\big\|\mathcal{H}(\varepsilon^{r_1}x_1)[s_1(K_1x_1) + s_2(K_2x_3)]\big\|
\end{aligned}$$

and consequently, from (3.57), (recalling that by design specifications: $r_1 > r_2 > 0$), we get

$$\begin{aligned}
\mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+} \varepsilon^{r_1}\big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\big\| \\
&\quad + k_g\|x_1\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1 - r_2)}\big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\big\| \\
&\quad + \lim_{\varepsilon\to 0^+} \big\|\mathcal{H}(\varepsilon^{r_1}x_1)[s_1(K_1x_1) + s_2(K_2x_3)]\big\| \\
&\le \big\|H^{-1}(q_d)C(q_d, x_2)x_2\big\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \\
&\quad + k_g\|x_1\|\big\|H^{-1}(q_d)\big\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1 - r_2)} \\
&\quad + \big\|s_1(K_1x_1) + s_2(K_2x_3)\big\| \lim_{\varepsilon\to 0^+} \big\|\mathcal{H}(\varepsilon^{r_1}x_1)\big\| \\
&\le \big\|s_1(K_1x_1) + s_2(K_2x_3)\big\| \cdot \big\|\mathcal{H}(0_n)\big\| = 0
\end{aligned}$$

since the definition of $\mathcal{H}(x_1)$ in (3.6), $\big\|\mathcal{H}(0_n)\big\| = 0$, which completes the proof. ■

Thus, global finite-time stability of the closed-loop trivial solution has already been concluded.

Corollary 3.5. *Consider the proposed control scheme in* (3.44)–(3.45) *taking* σ_{ij}, $i = 1, 2, 3$, $j = 1, \ldots, n$, *such that*

$$\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall |\varsigma| \le L_{ij} \in (0, \infty)$$

with constants $\beta_{ij} = \beta_i$ *such that*

$$0 < \beta_1 < 1, \quad \beta_2 = \beta_1, \quad \beta_3 = \frac{1 + \beta_1}{2} \tag{3.58}$$

Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ *is guaranteed with* $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \ge 0$.

Proof. Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and

$$\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \operatorname{sign}(\varepsilon^{r_{ij}}\varsigma)|\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \operatorname{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma), \quad \forall \varepsilon \in (0, 1]$$

Hence, under the consideration of expressions (3.58), for every $j = 1, \ldots, n$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = (1 + \beta_1)r_1/2$ and $r_{3j} = r_3 = r_1, \sigma_{ij}$, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{1j} = \alpha_1 = r_1\beta_1 = 2r_2 - r_1 = r_3\beta_2 = \alpha_2 = \alpha_{2j}$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{3j} is locally r_1-homogeneous of degree $\alpha_{3j} = \alpha_3 = (1 + \beta_1)r_3/2 = (1 + \beta_1)r_1/2 = r_2$ with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j}\}$, while

$$\begin{aligned} 0 < \beta_1 \leq 1 \implies \beta_1 > 0 \geq \beta_1 - 1 \geq \frac{(\beta_1 - 1)}{2} &\implies \frac{(\beta_1 + 1)}{2} \leq 1 < \beta_1 + 1 \\ &\iff \frac{(\beta_1 + 1)r_1}{2} \leq r_1 < (\beta_1 + 1)r_1 \\ &\iff r_2 \leq r_1 < 2r_2 \\ &\iff r_2 - r_1 \leq 0 < 2r_2 - r_1 \end{aligned}$$

The requirements of Proposition 3.8 are thus concluded to be satisfied with $0 < \beta_1 < 1 \implies r_2 - r_1 < 0 < 2r_2 - r_1$. ■

Analogously to the state-feedback case, Proposition 3.8 with $r_2 \geq r_1$ is a particular case of Proposition 3.7. Moreover, when $r_2 = r_1$ we have that $\varepsilon^{r_2 - r_1} = 1$, $\forall \varepsilon > 0$. Hence, in this case, Lemma 2.3 (under the consideration of Remark 2.6) cannot be applied to conclude (local) exponential stability (contrarily to the on-line conservative force compensation case). However, exponential stability is next proven to be achieved (locally), through an alternative (strict-Lyapunov-function-based) analytical procedure.

Exponential stabilization

Exponential stability is proven through the following corollary, recalling what was just mentioned.

Corollary 3.6. *Consider the proposed control scheme in* (3.44)–(3.45) *taking —for every $i = 1, 2, 3$ and $j = 1, \ldots, n$— σ_{ij} as*

$$\sigma_{ij}(\varsigma) = \varsigma \quad \forall |\varsigma| \leq L_{ij} \in (0, \infty). \tag{3.59}$$

Thus: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically stable and (locally) exponentially stable.

Proof. The global asymptotic stability follows from Proposition 3.7. Thus, all that remains to be proven is the (local) exponential stability property. In this direction, let us consider the scalar function

$$V_2(x_1,x_2,x_3) = \frac{1}{2}x_2^T H(x_1+q_d)x_2 + \int_{0_n}^{x_1} s_1^T(K_1 z)dz + \mathcal{U}_{01}(x_1+q_d) - \mathcal{U}_{01}(q_d) - g^T(q_d)x_1$$
$$+ \int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz + \epsilon x_1^T H(x_1+q_d)x_2 - \epsilon\epsilon_0 x_2^T B^{-1}x_3$$

where ϵ and ϵ_0 are positive constants such that

$$\epsilon < \min\{\epsilon_1, \epsilon_2\} \tag{3.60}$$

$$\epsilon_0 > k_C \varrho_1 + \mu_M \tag{3.61}$$

with

$$\epsilon_1 = \left[\frac{\bar{k}_{1m}\bar{k}_{2m}\mu_m}{\bar{k}_{2m}\mu_M^2 + \bar{k}_{1m}(\epsilon_0/b_m)^2}\right]^{1/2}, \qquad \epsilon_2 = \frac{\bar{k}_{1m}\gamma_{22}\tilde{k}_{2m}}{\bar{k}_{1m}\gamma_{22}\gamma_{33} + \bar{k}_{1m}(\gamma_{23}/2)^2 + \gamma_{22}(\gamma_{13}/2)^2}$$
$$\bar{k}_{1m} = \min_j\{\bar{k}_{1j}\}, \quad \bar{k}_{2m} = \min_j\{k_{2j}/b_j\}, \quad b_m = \min_j\{b_j\} \tag{3.62}$$

$$\begin{array}{ll} \gamma_{13} = k_{2M} + (k_{1M}+k_g)\epsilon_0/(b_m\mu_m) & \gamma_{23} = \epsilon_0[\bar{a}_M + k_C\varrho_2/(b_m\mu_m)] \\ \gamma_{22} = \epsilon_0 - k_C\varrho_1 - \mu_M & \gamma_{33} = \epsilon_0 k_{2M}/(b_m\mu_m) \end{array} \tag{3.63}$$

$$\tilde{k}_{2m} = \min_j\{k_{2j}a_j/b_j\}, \quad k_{1M} = \max_j\{k_{1j}\}, \quad k_{2M} = \max_j\{k_{2j}\}, \quad \bar{a}_M = \max_j\{a_j/b_j\} \tag{3.64}$$

μ_m, μ_M, k_C and k_g as defined through Property 1.1 and Assumptions 1.1–1.3; ϱ_2 is any positive constant;

$$\varrho_1 = \max_{x_1\in\mathcal{Q}_1}\|x_1\| = \left[\sum_{j=1}^n [\min\{\bar{b}_{1j}/\bar{k}_{1j},\ L_{1j}/k_{1j}\}]^2\right]^{1/2}$$

and

$$\mathcal{Q}_1 = \mathcal{Q}_{11}\cap\mathcal{Q}_{12} = \left\{x_1\in\mathbb{R}^n : |x_{1j}| \le \min\{\bar{b}_{1j}/\bar{k}_{1j},\ L_{1j}/k_{1j}\},\ j=1,\ldots,n\right\} \tag{3.65a}$$

$$\mathcal{Q}_{11} = \{x_1\in\mathbb{R}^n : |x_{1j}| \le \bar{b}_{1j}/\bar{k}_{1j},\ j=1,\ldots,n\} \tag{3.65b}$$
$$\mathcal{Q}_{12} = \{x_1\in\mathbb{R}^n : |x_{1j}| \le L_{1j}/k_{1j},\ j=1,\ldots,n\} \tag{3.65c}$$

From the proof of Theorem 3.4 (particularly, from inequality (3.55)), we have that

$$V_2(x_1,x_2,x_3) \geq \frac{\mu_m}{2}\|x_2\|^2 + S_1(x_1) + \int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz - \epsilon|x_1^T H(x_1+q_d)x_2| - \epsilon\epsilon_0|x_2^T B^{-1}x_3|$$

with $S_1(x_1)$ as defined through (3.53c) (with $\ell = 1$). More precisely, by observing that $S_1(x_1) = \sum_{j=1}^n \bar{k}_{1j}x_{1j}^2$ on $\mathcal{Q}_{11}$ (recall (3.65b)) and

$$s_2(K_2x_3) = K_2x_3 \quad \text{on} \quad \mathcal{Q}_{31} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{2j}/k_{2j},\ j = 1,\ldots,n\}$$

we have that, on $\mathcal{Q}_{11} \times \mathbb{R}^n \times \mathcal{Q}_{31}$:

$$\begin{aligned}
V_2(x_1,x_2,x_3) &\geq \frac{\mu_m}{2}\|x_2\|^2 + \sum_{j=1}^n \frac{\bar{k}_{1j}}{2}x_{1j}^2 + \sum_{j=1}^n \frac{k_{2j}}{2b_j}x_{3j}^2 - \epsilon|x_1^T H(x_1+q_d)x_2| - \epsilon\epsilon_0|x_2^T B^{-1}x_3| \\
&\geq \frac{\mu_m}{2}\|x_2\|^2 + \frac{\bar{k}_{1m}}{2}\|x_1\|^2 + \frac{\bar{k}_{2m}}{2}\|x_3\|^2 - \epsilon\mu_M\|x_1\|\,\|x_2\| - \frac{\epsilon\epsilon_0}{b_m}\|x_2\|\,\|x_3\| \\
&= \frac{1}{2}\begin{pmatrix}\|x_1\|\\ \|x_2\|\\ \|x_3\|\end{pmatrix}^T Q_1 \begin{pmatrix}\|x_1\|\\ \|x_2\|\\ \|x_3\|\end{pmatrix}
\end{aligned}$$

with

$$Q_1 = \begin{pmatrix} \bar{k}_{1m} & -\epsilon\mu_M & 0 \\ -\epsilon\mu_M & \mu_m & -\epsilon\epsilon_0/b_m \\ 0 & -\epsilon\epsilon_0/b_m & \bar{k}_{2m} \end{pmatrix}$$

($\bar{k}_{1m}$, $\bar{k}_{2m}$ and b_m as defined through expressions (3.62)) where Assumption 1.1 has been considered, and since (3.60) $\Longrightarrow$ $\epsilon < \epsilon_1$ $\Longrightarrow$ $Q_1 > 0$, we get

$$V_2(x) \geq c_1\|x\|^2 \tag{3.66}$$

$\forall x \in \mathcal{Q}_{11} \times \mathbb{R}^n \times \mathcal{Q}_{31}$, with $c_1 = \lambda_m(Q_1)/2 > 0$. On the other hand, by analogously observing that in view of (3.59), we have, $s_1(K_1x_1) = K_1x_1$ on $\mathcal{Q}_{12}$ (recall (3.65c)), we get, under the consideration of (3.30) and Assumption 1.1, that, on $\mathcal{Q}_{12} \times \mathbb{R}^n \times \mathcal{Q}_{31}$

$$
\begin{aligned}
V_2(x_1,x_2,x_3) = &\frac{1}{2}x_2^T H(x_1+q_d)x_2 + \frac{1}{2}x_1^T K_1 x_1 + \mathcal{U}_{o1}(x_1+q_d) - \mathcal{U}_{o1}(q_d) - g^T(q_d)x_1 \\
&+ \sum_{j=1}^{n}\frac{k_{2j}}{2b_j}x_{3j}^2 + \epsilon x_1^T H(x_1+q_d)x_2 - \epsilon\epsilon_0 x_2^T B^{-1}x_3 \\
\leq &\frac{\mu_M}{2}\|x_2\|^2 + \frac{k_{1M}}{2}\|x_1\|^2 + \frac{k_g}{2}\|x_1\|^2 + \frac{\bar{k}_{2M}}{2}\|x_3\|^2 \\
&+ \epsilon\mu_M\|x_1\|\|x_2\| + \frac{\epsilon\epsilon_0}{b_m}\|x_2\|\|x_3\| \\
= &\frac{1}{2}\begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}^T Q_2 \begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}
\end{aligned}
$$

where

$$
Q_2 = \begin{pmatrix} \bar{k}_{1M} + k_g & \epsilon\mu_M & 0 \\ \epsilon\mu_M & \mu_M & \epsilon\epsilon_0/b_m \\ 0 & \epsilon\epsilon_0/b_m & \bar{k}_{2M} \end{pmatrix}
$$

with $\bar{k}_{2M} = \max_j\{k_{2j}/b_j\}$ (and k_{1M} as defined through expressions (3.64)). From simple developments, one can further verify that (3.60) $\implies \epsilon < \epsilon_1 \implies Q_2 > 0$, whence we get $V_2(x) \leq c_2\|x\|^2$, $\forall x \in \mathcal{Q}_{12} \times \mathbb{R}^n \times \mathcal{Q}_{31}$, with $c_2 = \lambda_M(Q_2)/2 > 0$. Furthermore, the derivative of V_2 along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}_2(x_1, x_2, x_3) &= x_2^T H(x_1 + q_d)\dot{x}_2 + \frac{1}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 \\
&\quad + [s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)]^T \dot{x}_1 + s_2^T(K_2x_3)B^{-1}\dot{x}_3 \\
&\quad + \epsilon x_1^T H(x_1 + q_d)\dot{x}_2 + \epsilon x_1^T \dot{H}(x_1 + q_d, x_2)x_2 \\
&\quad + \epsilon \dot{x}_1^T H(x_1 + q_d)x_2 - \epsilon\epsilon_0 x_2^T B^{-1}\dot{x}_3 - \epsilon\epsilon_0 \dot{x}_2^T B^{-1}x_3 \\
&= -x_2^T[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d) + s_1(K_1x_1) + s_2(K_2x_3)] \\
&\quad + \frac{1}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 + [s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)]^T x_2 \\
&\quad + s_2^T(K_2x_3)B^{-1}[-As_3(x_3) + Bx_2] \\
&\quad - \epsilon x_1^T[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d) + s_1(K_1x_1) + s_2(K_2x_3)] \\
&\quad + \epsilon x_1^T[C(x_1 + q_d, x_2) + C^T(x_1 + q_d, x_2)]x_2 + \epsilon x_2^T H(x_1 + q_d)x_2 \\
&\quad - \epsilon\epsilon_0 x_2^T B^{-1}[-As_3(x_3) + Bx_2] \\
&\quad + \epsilon\epsilon_0 x_3^T B^{-1}H^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) \\
&\qquad\qquad - g(q_d) + s_1(K_1x_1) + s_2(K_2x_3)] \\
&= -s_2^T(K_2x_3)B^{-1}As_3(x_3) - \epsilon x_1^T[s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)] \\
&\quad - \epsilon x_1^T s_2(K_2x_3) + \epsilon x_2^T C(x_1 + q_d, x_2)x_1 + \epsilon x_2^T H(x_1 + q_d)x_2 \\
&\quad + \epsilon\epsilon_0 x_2^T B^{-1}As_3(x_3) - \epsilon\epsilon_0 x_2^T x_2 \\
&\quad + \epsilon\epsilon_0 x_3^T B^{-1}H^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) \\
&\qquad\qquad - g(q_d) + s_1(K_1x_1) + s_2(K_2x_3)]
\end{aligned}$$

where Property 1.2.1 has been applied. By (3.54), Property 1.1, Assumptions 1.1, 1.2 and 1.3.2, Remark 1.2, observing that —in view of (3.59)— $s_3(x_3) = x_3$ on $\mathcal{Q}_{32} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{3j},\, j = 1, \ldots, n\}$, and defining $\mathcal{Q}_3 = \mathcal{Q}_{31} \cap \mathcal{Q}_{32}$, $\mathcal{B}_2 = \{x \in \mathbb{R}^n : \|x_2\| \leq \varrho_2\}$ for any $\varrho_2 > 0$, and $\mathcal{Q}_1$ as defined in (3.65), we get that, on $\mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$:

$$
\begin{aligned}
\dot{V}_2(x_1,x_2,x_3) \leq & -x_3^T K_2 B^{-1} A x_3 - \epsilon \sum_{j=1}^{n} \bar{k}_{1j} x_{1j}^2 + \epsilon |x_1^T K_2 x_3| + \epsilon |x_2^T C(x_1+q_d,x_2) x_1| \\
& + \epsilon |x_2^T H(x_1+q_d) x_2| + \epsilon\epsilon_0 |x_2^T B^{-1} A x_3| \\
& - \epsilon\epsilon_0 x_2^T x_2 + \epsilon\epsilon_0 \left| x_3^T B^{-1} H^{-1}(x_1+q_d) C(x_1+q_d,x_2) x_2 \right| \\
& + \epsilon\epsilon_0 \left| x_3^T B^{-1} H^{-1}(x_1+q_d)[g(x_1+q_d) - g(q_d)] \right| \\
& + \epsilon\epsilon_0 \left| x_3^T B^{-1} H^{-1}(x_1+q_d) K_1 x_1 \right| + \epsilon\epsilon_0 \left| x_3^T B^{-1} H^{-1}(x_1+q_d) K_2 x_3 \right| \\
\leq & -\tilde{k}_{2m}\|x_3\|^2 - \epsilon \bar{k}_{1m}\|x_1\|^2 + \epsilon k_{2M}\|x_1\|\|x_3\| + \epsilon k_C \varrho_1 \|x_2\|^2 + \epsilon \mu_M \|x_2\|^2 \\
& + \epsilon\epsilon_0 \bar{a}_M \|x_2\|\|x_3\| - \epsilon\epsilon_0 \|x_2\|^2 \\
& + \frac{\epsilon\epsilon_0 k_C \varrho_2}{b_m \mu_m}\|x_2\|\|x_3\| + \frac{\epsilon\epsilon_0 k_g}{b_m \mu_m}\|x_1\|\|x_3\| + \frac{\epsilon\epsilon_0 k_{1M}}{b_m \mu_m}\|x_1\|\|x_3\| + \frac{\epsilon\epsilon_0 k_{2M}}{b_m \mu_m}\|x_3\|^2 \\
= & -\begin{pmatrix} \|x_1\| \\ \|x_2\| \\ \|x_3\| \end{pmatrix}^T Q_3 \begin{pmatrix} \|x_1\| \\ \|x_2\| \\ \|x_3\| \end{pmatrix}
\end{aligned}
$$

($\tilde{k}_{2m}$, k_{2M}, $\bar{a}_M$ and ϱ_1 as defined through expressions (3.64)) with

$$
Q_3 = \begin{pmatrix} \epsilon \bar{k}_{1m} & 0 & -\epsilon\gamma_{13}/2 \\ 0 & \epsilon\gamma_{22} & -\epsilon\gamma_{23}/2 \\ -\epsilon\gamma_{13}/2 & -\epsilon\gamma_{23}/2 & \tilde{k}_{2m} - \epsilon\gamma_{33} \end{pmatrix}
$$

[γ_{22} (being positive in view of (3.61)), γ_{13}, γ_{23} and γ_{33} as defined through expressions (3.63)] and since (3.60) $\Longrightarrow$ $\epsilon < \epsilon_2$ $\Longrightarrow$ $Q_3 > 0$, we get

$$
\dot{V}_2(x) \leq -c_3\|x\|^2 \tag{3.67}
$$

$\forall x \in \mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$, with $c_3 = \lambda_m(Q_3) > 0$. Thus, from the simultaneous satisfaction of inequalities (3.66)–(3.67) on $\mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$, we conclude —by [40, Theorem 4.10]— that the origin $(x_1, x_2, x_3) = (0_n, 0_n, 0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system, whence the proof is completed. ■

Notice that the functions $\sigma_{ij}(\cdot)$, $i = 1,2,3$, $j = 1,\ldots,n$, defined in (3.59) through Corollary 3.6 turn out to be a particular case of the functions $\sigma_{ij}(\cdot)$, $i = 1,2,3$, $j = 1,\ldots,n$, defined in Corollary 3.5 for $\beta_{ij} = 1$, $i = 1,2,3$, $j = 1,\ldots,n$. Thus, the functions involved in the control scheme (3.44)–(3.45) are characterized through Corollary 3.5, such a characterization permits, via the parameters β, to get either finite-time or exponential convergence.

3.3 Tracking problem

Throughout this section the mechanical system model described in (1.1) is taken into account, *i.e.*,

$$H(q)q + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau \tag{3.68}$$

where all the properties for $H(q)$, $C(q,\dot{q})$, and $g(q)$ are as described in Section 1.4, while F is additionally supposed to satisfy the following.

Assumption 3.2. *The damping effect matrix F is symmetric positive definite, and consequently $f_m\|x\|^2 \leq x^TFx \leq f_M\|x\|^2$, $\forall x \in \mathbb{R}^n$, with $f_M \geq \lambda_{\max}(F) \geq \lambda_{\min}(F) \geq f_m > 0$.*

Assumption 3.2 is coherent with the dissipative nature of the damping term $F\dot{q}$ in the —(realistically) assumed fully-damped— system dynamics (3.68) [29]. Furthermore, recall that the realistic bounded input case is considered, *i.e.*,

$$\tau_i = T_i \operatorname{sat}\left(\frac{u_i}{T_i}\right) \tag{3.69}$$

The goal in this part of the dissertation consists on the achievement of a tracking objective (with finite-time or exponential convergence) avoiding input saturation along the system trajectories. With this goal in mind, we begin by characterizing —based on Assumptions 1.1–1.4— a set of desired trajectories $q_d(t)$ for which the proposed scheme will prove to guarantee the considered tracking objective avoiding input saturation and for any initial condition.

Assumption 3.3. *$q_d(t) \in \mathcal{C}^2(\mathbb{R}_{\geq 0};\mathbb{R}^n)$ such that $\|\dot{q}_d(t)\| \leq B_{dv}$ and $\|\ddot{q}_d(t)\| \leq B_{da}$, $\forall t \geq 0$, for sufficiently small (positive) bound values B_{dv} and B_{da} such that $\mu_{Mj}B_{da} + k_{Cj}B_{dv}^2 + \|F_j\|B_{dv} < T_j - B_{gj}$, $\forall j \in \{1,\ldots,n\}$, and $d \triangleq f_m - k_CB_{dv} > 0$.*

The following control law is proposed:

$$u(t,q,\dot{q}) = -s_1(K_1\bar{q}) - s_2(K_2\dot{\bar{q}}) + H(q)\ddot{q}_d(t) + C(q,\dot{q}_d(t))\dot{q}_d(t) + F\dot{q}_d(t) + g(q) \tag{3.70}$$

where $\bar{q}$ is as defined in the previous section; $K_i = \operatorname{diag}[k_{i1},\ldots,k_{in}]$ with $k_{ij} > 0$, $\forall i \in \{1,2\}$, $\forall j \in \{1,\ldots,n\}$; and for any $x \in \mathbb{R}^n$, $s_i(x) = (\sigma_{i1}(x_1),\ldots,\sigma_{in}(x_n))^T$, $i = 1,2$, with, for each $j \in \{1,\ldots,n\}$, σ_{ij} being a bounded strongly passive function, for some $(\kappa_i, a_i, b_i, \bar{\kappa}_i, a_i, b_i) \in \mathbb{R}^6_{>0}$, both $(i = 1,2)$ being locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$ and such that

$$a_1 \in (0,1], \quad a_2 = \frac{2a_1}{1+a_1} \in (0,1] \tag{3.71}$$

and

$$B_j \triangleq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \left|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\right| < T_j - \mu_{Mj}B_{da} - k_{Cj}B_{dv}^2 - \|F_j\|B_{dv} - B_{gj} \tag{3.72}$$

Proposition 3.9. *Consider system* (3.68)–(3.69) *in closed loop with the proposed control law in* (3.70)*, under Assumptions 1.1–1.4 and 3.2–3.3. Thus, for any positive definite diagonal matrices* K_1 *and* K_2, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq t_0 \geq 0$, *and the closed-loop trivial solution* $\bar{q}(t) \equiv 0_n$ *is:*

1) *globally uniformly finite-time stable if* $a_1 \in (0, 1)$;

2) *globally uniformly asymptotically stable and (locally) exponentially stable if* $a_1 = 1$.

Proof. The proof is divided into four stages. The first stage shows that input saturation is avoided and obtains the consequent closed-loop dynamics in the tracking error variable space. Throughout the second stage an energy function is introduced, as well as its derivative along the system trajectories. In the third stage, the Lyapunov function candidate V is presented and the expressions of its derivative along the closed-loop system $\dot{V}$ and a suitable time-invariant upper bound are obtained, whence global asymptotic stability is concluded. Finally, the fourth stage develops an analysis of V and $\dot{V}$ in a suitable origin-centered ball, whence finite-time stability is concluded and the conclusion of the proof is obtained.

1st stage: input saturation avoidance and closed-loop dynamics. Observe that —for every $j \in \{1, \ldots, n\}$— by Assumptions 1.1–1.4 and 3.3, and the satisfaction of (3.72), we have, for any $(t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} |u_j(t, q, \dot{q})| &\leq \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{\bar{q}}_j)\right| + \|H_j(q)\|\|\ddot{q}_d(t)\| + \|C_j(q, \dot{q}_d(t))\|\|\dot{q}_d(t)\| \\ &\quad + \|F_j\|\|\dot{q}_d(t)\| + |g_j(q)| \\ &\leq B_j + \mu_{Mj}B_{da} + k_{Cj}B_{dv}^2 + \|F_j\|B_{dv} + B_{gj} < T_j \end{aligned}$$

From this and (3.69), one sees that $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq t_0 \geq 0$, which proves that, under the proposed scheme, the input saturation values, are never reached. Hence, the closed-loop dynamics becomes

$$H(q)\ddot{\bar{q}} + C(q, \dot{q})\dot{\bar{q}} + C(q, \dot{q}_d(t))\dot{\bar{q}} + F\dot{\bar{q}} = -s_1(K_1\bar{q}) - s_2(K_2\dot{\bar{q}}) \tag{3.73}$$

where Property 1.2.3 has been used.

2nd stage: energy function. Let us consider the continuously differentiable energy function

$$V_0(t, \bar{q}, \dot{\bar{q}}) = \frac{1}{2}\dot{\bar{q}}^T H(\bar{q} + q_d(t))\dot{\bar{q}} + \int_{0_n}^{\bar{q}} s_1^T(K_1 z)\, dz \tag{3.74}$$

where $\int_{0_n}^{\bar{q}} s_1^T(K_1 z)\, dz = \sum_{j=1}^{n} \int_0^{\bar{q}_j} \sigma_{1j}(k_{1j}z_j)\, dz_j$. From Property 1.1, Assumption 1.1, Remark 2.13, and Lemmas 2.7 and 2.8 (more precisely, from the first part of de proof of Lemma 2.8), we have that:

$$W_{01}(\bar{q}, \dot{\bar{q}}) \leq V_0(t, \bar{q}, \dot{\bar{q}}) \leq W_{02}(\bar{q}, \dot{\bar{q}}) \tag{3.75}$$

with

$$W_{01}(\bar{q},\dot{\bar{q}}) \triangleq \frac{\mu_m}{2}\|\dot{\bar{q}}\|^2 + \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1} S_1(\bar{q}) \tag{3.76a}$$

and

$$W_{02}(\bar{q},\dot{\bar{q}}) \triangleq \frac{\mu_M}{2}\|\dot{\bar{q}}\|^2 + \frac{\bar{\kappa}_1 k_{1M}^{a_1} n}{1+a_1} \|\bar{q}\|^{1+a_1} \tag{3.77a}$$

where $S_1(\bar{q}) = S_0(\bar{q}; a_1, b_1/k_{1M})$ (defined in Lemma 2.7), $k_{1m} = \min_j\{k_{1j}\}$, and $k_{1M} = \max_j\{k_{1j}\}$. The derivative of V_0 along the closed-loop system trajectories is obtained as

$$\begin{aligned}
\dot{V}_0(t,\bar{q},\dot{\bar{q}}) &= \dot{\bar{q}}^T H(q)\ddot{\bar{q}} + \frac{1}{2}\dot{\bar{q}}^T \dot{H}(q,\dot{q})\dot{\bar{q}} + s_1^T(K_1\bar{q})\dot{\bar{q}} \\
&= -\dot{\bar{q}}^T[C(q,\dot{q})\dot{\bar{q}} + C(q,\dot{q}_d(t))\dot{\bar{q}} + F\dot{\bar{q}} + s_1(K_1\bar{q}) + s_2(K_2\dot{\bar{q}})] \\
&\quad + \frac{1}{2}\dot{\bar{q}}^T \dot{H}(q,\dot{q})\dot{\bar{q}} + s_1^T(K_1\bar{q})\dot{\bar{q}} \\
&= -\dot{\bar{q}}^T s_2(K_2\dot{\bar{q}}) - \dot{\bar{q}}^T C(q,\dot{q}_d(t))\dot{\bar{q}} - \dot{\bar{q}}^T F\dot{\bar{q}}
\end{aligned}$$

where $H(q)\ddot{\bar{q}}$ has been replaced by its equivalent expression from the closed-loop dynamics (3.73) and Property 1.2.1 has been used. Further, by Assumptions 1.2, 3.2 and 3.3, as well as Lemma 2.7:

$$\dot{V}_0(t,\bar{q},\dot{\bar{q}}) \leq -\sum_{j=1}^{n} \dot{\bar{q}}_j \sigma_{2j}(k_{2j}\dot{\bar{q}}_j) - (f_m - k_C B_{dv})\|\dot{\bar{q}}\|^2 \leq -\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) - d\|\dot{\bar{q}}\|^2 \leq -\bar{\eta}\|\dot{\bar{q}}\|^{1+a_2} \tag{3.78}$$

where $S_2(\dot{\bar{q}}) = S_0(\dot{\bar{q}}; a_2, b_2/k_{2M})$, $k_{2m} = \min_j\{k_{2j}\}$, $k_{2M} = \max_j\{k_{2j}\}$, $d = f_m - k_C B_{dv} > 0$ (by Assumption 3.3) and $\bar{\eta} = \min\left\{\kappa_2 k_{2m}^{a_2}, d\left(\frac{b_2}{k_{2M}}\right)^{1-a_2}\right\}$. The right-most inequality in (3.78) arises by observing that if $\|\dot{\bar{q}}\| \leq b_2/k_{2M}$, we have that for all

$$\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq \kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) = \kappa_2 k_{2m}^{a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq \bar{\eta}\|\dot{\bar{q}}\|^{1+a_2}$$

and for all $\|\dot{\bar{q}}\| > b_2/k_{2M}$ that

$$\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq d\|\dot{\bar{q}}\|^2 = d\|\dot{\bar{q}}\|^{1-a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq d\left(\frac{b_2}{k_{2M}}\right)^{1-a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq \bar{\eta}\|\dot{\bar{q}}\|^{1+a_2}$$

The expressions obtained so far will prove to be useful next.

3rd stage: global uniform asymptotic stability. Let us now define the scalar function

$$V(t,\bar{q},\dot{\bar{q}}) = V_0^{\beta}(t,\bar{q},\dot{\bar{q}}) + \varepsilon\rho^T(\bar{q})H(\bar{q}+q_d(t))\dot{\bar{q}} \tag{3.79}$$

where V_0 is defined in (3.74), with

$$\beta = \frac{3+a_1}{2(1+a_1)} \tag{3.80}$$

ε is a positive constant, and $\rho(\bar{q}) = h(\bar{q}; b_1/k_{1M})\bar{q}$, with $h \in \mathcal{C}^0(\mathbb{R}^n \times \mathbb{R}_{>0}; (0,1])$ being continuously differentiable on $\mathbb{R}^n \setminus \{0_n\}$, uniformly on $\mathbb{R}_{>0}$, and such that, for any $c > 0$, ρ is a continuously differentiable function satisfying

$$\|\rho(x)\| = h(x;c)\|x\| \leq \min\{\|x\|, c\} \tag{3.81}$$

$\forall x \in \mathbb{R}^n$, and

$$-h(x;c) < D_x h(x;c) < 0 \tag{3.82}$$

$\forall x \neq 0$; an example of a family of functions h with such properties is $h(x;c) = c/[c^{\varpi} + \|x\|^{\varpi}]^{1/\varpi}$ for any $\varpi > 0$ [1].

Remark 3.15. In view of (3.82), h is decreasing on any radial direction, and consequently (since $h : \mathbb{R}^n \times \mathbb{R}_{>0} \to (0,1]$) $h(x;c) \to \omega$ as $\|x\| \to \infty$ for some non-negative constant ω, while, on any compact connected neighborhood of the origin $\Upsilon \subset \mathbb{R}^n$, h is lower-bounded by a positive value $h_{m,\Upsilon}$, or more precisely: $1 \geq h(0_n;c) \geq h(x;c) \geq \inf_{x\in\Upsilon} h(x;c) \triangleq h_{m,\Upsilon} = \inf_{x\in\partial\Upsilon} h(x;c) > \omega \geq 0$, $\forall x \in \Upsilon$.

Remark 3.16. Observe, that

$$\frac{\partial \rho}{\partial x}(x) = \frac{\partial}{\partial x}[h(x;c)x] = h(x;c)I_n + x\frac{\partial h}{\partial x}(x;c)$$

whence we get that

$$\begin{aligned} x^T\frac{\partial \rho}{\partial x}(x)x = x^T\Big[h(x;c)I_n + x\frac{\partial h}{\partial x}(x;c)\Big]x &= h(x;c)x^Tx + x^Tx\frac{\partial h}{\partial x}(x;c)x \\ &= \|x\|^2[h(x;c) + D_x h(x;c)] \end{aligned}$$

wherefrom, in view of (3.82) (whence we have that $0 < h(x;c) + D_x h(x;c) < h(x;c) < 1$, $\forall x \neq 0_n$), one sees that

$$0 < x^T\frac{\partial \rho}{\partial x}(x)x < \|x\|^2, \quad \forall x \neq 0_n$$

and consequently

$$0 < \frac{\partial \rho}{\partial x}(x) \leq I_n, \quad \forall x \in \mathbb{R}^n$$

which implies that

$$\left\|\frac{\partial \rho}{\partial x}\right\| \leq 1, \quad \forall x \in \mathbb{R}^n$$

[1] Letting $\rho_{\varpi}(x) = h_{\varpi}(x;c)x$, $\varpi > 0$, with $h_{\varpi}(x;c) \triangleq c/[c^{\varpi} + \|x\|^{\varpi}]^{1/\varpi}$, one verifies after basic developments that $D_x h_{\varpi}(x;c) = -h_{\varpi}(x;c)(\|\rho_{\varpi}(x)\|/c)^{\varpi}$, whence one corroborates that $-h_{\varpi}(x;c) < D_x h(x;c) < 0$, $\forall x \neq 0_n$.

Remark 3.17. Let $h_1(\bar{q}) = h(\bar{q}; b_1/k_{1M})$. Useful facts on ρ that will be subsequently invoked are

$$\|\rho(\bar{q})\|^{1+a_1} \leq h_1(\bar{q})S_1(\bar{q}) \leq S_1(\bar{q}) \tag{3.83}$$

$$\|\rho(\bar{q})\|^2 \leq \left(\frac{b_1}{k_{1M}}\right)^{1-a_1} h_1(\bar{q})S_1(\bar{q}) \leq \left(\frac{b_1}{k_{1M}}\right)^{1-a_1} S_1(\bar{q}) \tag{3.84}$$

$\forall \bar{q} \in \mathbb{R}^n$. Indeed, based on the properties of ρ and h_1 (particularly (3.81) and $h_1(\bar{q}) \in (0,1]$, $\forall \bar{q} \in \mathbb{R}^n$), we have, for all $\bar{q} \in \mathcal{B}^n_{b_1/k_{1M}}$, that

$$\|\rho(\bar{q})\|^{1+a_1} = [h_1(\bar{q})]^{1+a_1}\|\bar{q}\|^{1+a_1} = h_1(\bar{q})^{a_1} h_1(\bar{q})S_1(\bar{q}) \leq h_1(\bar{q})S_1(\bar{q}) \leq S_1(\bar{q})$$

and for all $\bar{q} \notin \mathcal{B}^n_{b_1/k_{1M}}$ that

$$\|\rho(\bar{q})\|^{1+a_1} = \|\rho(\bar{q})\|^{a_1}\|\rho(\bar{q})\| \leq \left(\frac{b_1}{k_{1M}}\right)^{a_1} h_1(\bar{q})\|\bar{q}\| = h_1(\bar{q})S_1(\bar{q}) \leq S_1(\bar{q})$$

corroborating (3.83). On the other hand, by (3.81) and (3.83), we have that

$$\|\rho(\bar{q})\|^2 = \|\rho(\bar{q})\|^{1-a_1}\|\rho(\bar{q})\|^{1+a_1} \leq \left(\frac{b_1}{k_{1M}}\right)^{1-a_1} h_1(\bar{q})S_1(\bar{q}) \leq \left(\frac{b_1}{k_{1M}}\right)^{1-a_1} S_1(\bar{q})$$

$\forall \bar{q} \in \mathbb{R}^n$, corroborating (3.84). △

We will show that, for a sufficiently small value of ε, V in (3.79) is a suitable Lyapunov function through which the proof will be completed; in particular, this will be proven with

$$\varepsilon < \varepsilon_0 \triangleq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \tag{3.85a}$$

where

$$\varepsilon_1 = \frac{1}{\mu_M}\left(\frac{3+a_1}{1+a_1} \cdot \frac{\mu_m}{2}\right)^{\beta} \quad , \quad \varepsilon_2 = \frac{1}{\mu_M}\left(\frac{3+a_1}{1+a_1} \cdot \frac{\kappa_1 k_{1m}^{a_1}}{2}\right)^{\beta} \tag{3.85b}$$

$$\varepsilon_3 = \frac{\beta\bar{\eta}(\frac{\mu_m}{2})^{\beta-1}}{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}\left[\frac{h_{1m}\kappa_1 k_{1m}^{a_1}(1+a_1)}{2\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right]^{1/a_1} < \frac{\beta\bar{\eta}(\frac{\mu_m}{2})^{\beta-1}}{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}\left[\frac{\kappa_1 k_{1m}^{a_1}(1+a_1)}{2\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right]^{1/a_1} \triangleq \bar{\varepsilon}_3 \tag{3.85c}$$

$$\varepsilon_4 = \frac{h_{1m}\kappa_1 k_{1m}^{a_1}\beta\bar{\eta}(\frac{\mu_m}{2})^{\beta-1}}{2h_{1m}\kappa_1 k_{1m}^{a_1}\upsilon_1 + \upsilon_2^2} < \frac{\kappa_1 k_{1m}^{a_1}\beta\bar{\eta}(\frac{\mu_m}{2})^{\beta-1}}{2\kappa_1 k_{1m}^{a_1}\upsilon_1 + \upsilon_2^2} \triangleq \bar{\varepsilon}_4 \tag{3.85d}$$

$$h_{1m} \triangleq \inf_{q \in \mathcal{B}^n_{b_1/k_{1M}}} h_1(\bar{q}) = \inf_{q \in \partial\mathcal{B}^n_{b_1/k_{1M}}} h_1(\bar{q}) \in (0,1) \tag{3.86}$$

(recall Remark 3.15) and

$$\bar{c} = n^{1/(1+a_1)} \tag{3.87a}$$

$$\upsilon_1 = \frac{k_C b_1}{k_{1M}} + \mu_M \quad , \quad \upsilon_2 = (2k_C B_{dv} + f_M)\left(\frac{b_1}{k_{1M}}\right)^{\frac{1-a_1}{2}} \tag{3.87b}$$

With such a goal in mind, let us begin by noting, from (3.79) and (3.75), that

$$V(t,\bar{q},\dot{\bar{q}}) \geq W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) - \varepsilon\mu_M\Big(\|\rho(\bar{q})\|^{1/\beta}\|\dot{\bar{q}}\|^{1/\beta}\Big)^{\beta} \tag{3.88}$$

$$\geq W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) - \varepsilon\mu_M\left(\frac{2}{3+a_1}\|\rho(\bar{q})\|^{1+a_1} + \frac{1+a_1}{3+a_1}\|\dot{\bar{q}}\|^2\right)^{\beta} \tag{3.89}$$

$$\geq W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) - \varepsilon\mu_M\left(\frac{2}{3+a_1}S_1(\bar{q}) + \frac{1+a_1}{3+a_1}\|\dot{\bar{q}}\|^2\right)^{\beta} \tag{3.90}$$

$$\geq W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) - W_{10}^{\beta}(\bar{q},\dot{\bar{q}}) \triangleq W_1(\bar{q},\dot{\bar{q}}) \tag{3.91}$$

with

$$W_{10}(\bar{q},\dot{\bar{q}}) = (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}S_1(\bar{q}) + \frac{1+a_1}{3+a_1}\|\dot{\bar{q}}\|^2\right) \tag{3.92}$$

where Assumption 1.1 has been applied to get (3.88), Young's inequality [with $\phi = (3+a_1)/2$ and $\psi = (3+a_1)/(1+a_1)$] to obtain (3.89), and Remark 3.17 (more specifically inequality (3.83)) to get (3.90). Notice further that

$$\begin{aligned} W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) - W_{10}^{\beta}(\bar{q},\dot{\bar{q}}) > 0 \iff W_{01}^{\beta}(\bar{q},\dot{\bar{q}}) > W_{10}^{\beta}(\bar{q},\dot{\bar{q}}) &\iff W_{01}(\bar{q},\dot{\bar{q}}) > W_{10}(\bar{q},\dot{\bar{q}}) \\ &\iff W_{01}(\bar{q},\dot{\bar{q}}) - W_{10}(\bar{q},\dot{\bar{q}}) > 0 \end{aligned}$$

Hence, by proving that $W_{01}(\bar{q},\dot{\bar{q}}) - W_{10}(\bar{q},\dot{\bar{q}}) > 0$, $\forall(\bar{q},\dot{\bar{q}}) \neq (0_n, 0_n)$, positive definiteness of $W_1(\bar{q},\dot{\bar{q}})$ in (3.91) is concluded. In this direction, let us define

$$\kappa_{mv} \triangleq \frac{\mu_m}{2} - (\varepsilon\mu_M)^{1/\beta}\left(\frac{1+a_1}{3+a_1}\right)$$

and

$$\kappa_{mp} \triangleq \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1} - (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}\right)$$

and let us further note that, from Eqs. (3.85), one may corroborate that

$$\varepsilon < \varepsilon_0 \leq \varepsilon_1 \implies \kappa_{mv} > 0 \quad \wedge \quad \varepsilon < \varepsilon_0 \leq \varepsilon_2 \implies \kappa_{mp} > 0$$

From this and expressions (3.76a) and (3.92), we have

$$W_{01}(\bar{q},\dot{\bar{q}}) - W_{10}(\bar{q},\dot{\bar{q}}) = \kappa_{mv}\|\dot{\bar{q}}\|^2 + \kappa_{mp}S_1(\bar{q}) > 0$$

$\forall(\bar{q},\dot{\bar{q}}) \neq (0_n, 0_n)$, whence positive definiteness of $W_1(\bar{q},\dot{\bar{q}})$ is concluded. Furthermore, from previous arguments, one sees that

$$\kappa_{mv} = \frac{\mu_m}{2} - (\varepsilon\mu_M)^{1/\beta}\left(\frac{1+a_1}{3+a_1}\right) > 0 \iff \bar{\kappa}_{mv} \triangleq \left(\frac{\mu_m}{2}\right)^{\beta} - \varepsilon\mu_M\left(\frac{1+a_1}{3+a_1}\right)^{\beta} > 0$$

and

$$\kappa_{mp} = \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1} - (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}\right) > 0 \iff \bar{\kappa}_{mp} \triangleq \left(\frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}\right)^{\beta} - \varepsilon\mu_M\left(\frac{2}{3+a_1}\right)^{\beta} > 0$$

From this and (3.91), one sees, for every $j = 1, \ldots, n$, that

$$\lim_{|\dot{\bar{q}}_j|\to\infty} W_1(0_n, \dot{\bar{q}}) = \lim_{|\dot{\bar{q}}_j|\to\infty} \bar{\kappa}_{mv}|\dot{\bar{q}}_j|^{2\beta} = \infty$$

on $\{\dot{\bar{q}} \in \mathbb{R}^n : \dot{\bar{q}}_\ell = 0\ \forall \ell \neq j\}$, and

$$\lim_{|\bar{q}_j|\to\infty} W_1(\bar{q}, 0_n) = \lim_{|\bar{q}_j|\to\infty} \bar{\kappa}_{mp}\left(\frac{b_1}{k_{1M}}\right)^{a_1\beta} |\bar{q}_j|^{\beta} = \infty$$

on $\{\bar{q} \in \mathbb{R}^n : \bar{q}_\ell = 0\ \forall \ell \neq j\}$, whence $W_1(\bar{q}, \dot{\bar{q}})$ is additionally concluded to be radially unbounded. Furthermore, from (3.75) and the properties of ρ (particularly (3.81)), one gets

$$V(t, \bar{q}, \dot{\bar{q}}) \leq \left(\frac{\mu_M}{2}\|\dot{\bar{q}}\|^2 + \frac{\bar{\kappa}_1 k_{1M}^{a_1} n}{1+a_1}\|\bar{q}\|^{1+a_1}\right)^{\beta} + \varepsilon\mu_M\|\bar{q}\|\|\dot{\bar{q}}\| \triangleq W_2(\bar{q}, \dot{\bar{q}}) \tag{3.93}$$

Since $W_2(\bar{q}, \dot{\bar{q}}) \geq W_1(\bar{q}, \dot{\bar{q}})$, $\forall (\bar{q}, \dot{\bar{q}}) \in \mathbb{R}^n \times \mathbb{R}^n$, and $W_2(0_n, 0_n) = W_1(0_n, 0_n) = 0$, the time-invariant function W_2 is corroborated to be a positive definite (radially unbounded) function. Therefore, V is concluded to be a positive definite, radially unbounded and decrescent function. Its derivative along the closed-loop system trajectories is obtained as

$$\begin{aligned}
\dot{V}(t, \bar{q}, \dot{\bar{q}}) &= \beta V_0^{\beta-1}(t, \bar{q}, \dot{\bar{q}})\dot{V}_0(t, \bar{q}, \dot{\bar{q}}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}} + \varepsilon\rho(\bar{q})^T \dot{H}\big(q, \dot{q}\big)\dot{\bar{q}} + \varepsilon\rho(\bar{q})^T H(q)\ddot{\bar{q}} \\
&= \beta V_0^{\beta-1}(t, \bar{q}, \dot{\bar{q}})\dot{V}_0(t, \bar{q}, \dot{\bar{q}}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}} + \varepsilon\rho(\bar{q})^T\left[C\big(q, \dot{q}\big) + C^T\big(q, \dot{q}\big)\right]\dot{\bar{q}} \\
&\quad - \varepsilon\rho(\bar{q})^T[C(q, \dot{q})\dot{\bar{q}} + C(q, \dot{q}_q(t))\dot{\bar{q}} + F\dot{\bar{q}} + s_1(K_1\bar{q}) + s_2(K_2\dot{\bar{q}})] \\
&= \beta V_0^{\beta-1}(t, \bar{q}, \dot{\bar{q}})\dot{V}_0(t, \bar{q}, \dot{\bar{q}}) - \varepsilon\rho^T(\bar{q})C(q, \dot{q}_d(t))\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) \\
&\quad - \varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) + \varepsilon\dot{\bar{q}}^T C(q, \dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T C(q, \dot{q}_d(t))\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}}
\end{aligned} \tag{3.94}$$

where $H(q)\ddot{\bar{q}}$ has been replaced by its equivalent expression from the closed-loop dynamics (3.73) and Properties 1.2.1–1.2.3 have been used. At this point, it is important to note that, from Eqs. (3.85), one may corroborate that

$$\varepsilon < \varepsilon_0 \leq \varepsilon_3 \implies \gamma_m < \gamma_M$$

with

$$\gamma_m \triangleq \frac{\varepsilon \bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}{\beta\bar{\eta}(\frac{\mu_m}{2})^{\beta-1}} \quad , \quad \gamma_M \triangleq \left(\frac{h_{1m}\kappa_1 k_{1m}^{a_2}(1+a_1)}{2\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right)^{\frac{1}{a_1}} < \left(\frac{\kappa_1 k_{1m}^{a_2}(1+a_1)}{2\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right)^{\frac{1}{a_1}} \triangleq \bar{\gamma}_M \tag{3.95}$$

From the analysis of $\dot{V}(t,\bar{q},\dot{\bar{q}})$ in (3.94), the following bound on the terms of $\dot{V}(t,\bar{q},\dot{\bar{q}})$ are obtained:

First term. From (3.75) and (3.78) (recalling (3.80) and (3.71)), we get:

$$\begin{aligned}\beta V_0^{\beta-1}(t,\bar{q},\dot{\bar{q}})\dot{V}_0(t,\bar{q},\dot{\bar{q}}) \leq -\beta W_{01}^{\beta-1}(\bar{q},\dot{\bar{q}})\bar{\eta}\|\dot{\bar{q}}\|^{1+a_2} &\leq -\beta\bar{\eta}\left(\frac{\mu_m}{2}\|\dot{\bar{q}}\|^2\right)^{\beta-1}\|\dot{\bar{q}}\|^{1+a_2}\\ &\leq -\beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1}\|\dot{\bar{q}}\|^2\end{aligned}$$

Second, third, sixth, seventh and eighth terms. From Assumptions 1.1, 1.2, 3.2, and 3.3, the properties of ρ (through inequality (3.81) and Remark 3.16), Remark 3.17 (particularly inequality (3.84)), and Eq. (3.87b), we get:

$$\begin{aligned}&-\varepsilon\rho^T(\bar{q})C(q,\dot{q}_d(t))\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} + \varepsilon\dot{\bar{q}}^T C(q,\dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T C(q,\dot{q}_d(t))\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}}\\ &\qquad \leq 2\varepsilon k_C B_{dv}\|\rho(\bar{q})\|\|\dot{\bar{q}}\| + \varepsilon f_M\|\rho(\bar{q})\|\|\dot{\bar{q}}\| + \varepsilon\frac{k_C b_1}{k_{1M}}\|\dot{\bar{q}}\|^2 + \varepsilon\mu_M\|\dot{\bar{q}}\|^2\\ &\qquad \leq \varepsilon\upsilon_2\left[h_1(\bar{q})S_1(\bar{q})\right]^{1/2}\|\dot{\bar{q}}\| + \varepsilon\upsilon_1\|\dot{\bar{q}}\|^2\end{aligned}$$

Fourth therm. From the definition of ρ and Lemma 2.7 we get:

$$-\varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) = -\varepsilon h_1(\bar{q})\bar{q}^T s_1(K_1\bar{q}) \leq -\varepsilon\kappa_1 k_{1m}^{a_1} h_1(\bar{q})S_1(\bar{q})$$

Fifth term. From Hölder and Young's inequalities (both with $\phi = 1+a_1$ and $\psi = 2/a_2$), the definition of s_2, Remarks 1.1, 2.13 and 3.17, in addition to the consideration of a positive constant $\gamma \in (\gamma_m, \gamma_M)$ (recall (3.95)), we have (recalling (3.71) and (3.87a)) that

$$\begin{aligned}-\varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) &\leq \varepsilon\left|\rho^T(\bar{q})s_2(K_2\dot{\bar{q}})\right|\\ &\leq \varepsilon\|\rho(\bar{q})\|_{1+a_1}\|s_2(K_2\dot{\bar{q}})\|_{2/a_2}\\ &\leq \varepsilon\bar{c}\bar{\kappa}_2\|\rho(\bar{q})\|\|K_2\dot{\bar{q}}\|^{a_2}\\ &\leq \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\left(\gamma^{a_2/2}\left[h_1(\bar{q})S_1(\bar{q})\right]^{1/(1+a_1)}\right)\left(\gamma^{-a_2/2}\|\dot{\bar{q}}\|^{a_2}\right)\\ &\leq \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\left(\frac{\gamma^{a_1}}{1+a_1}h_1(\bar{q})S_1(\bar{q}) + \frac{a_2}{2}\gamma^{-1}\|\dot{\bar{q}}\|^2\right)\end{aligned}$$

Thus, from the expressions obtained above, we get

$$\dot{V}(t,\bar{q},\dot{\bar{q}}) \le -\left[\beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1} - \varepsilon\upsilon_1 - \frac{\varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}{2}\gamma^{-1}\right]\|\dot{\bar{q}}\|^2 + \varepsilon\upsilon_2\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\|\dot{\bar{q}}\|$$
$$- \varepsilon\left(\kappa_1 k_{1m}^{a_1} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}{1+a_1}\gamma^{a_1}\right)h_1(\bar{q})S_1(\bar{q}) \tag{3.96}$$

which may be rewritten as

$$\dot{V}(t,\bar{q},\dot{\bar{q}}) \le -\frac{1}{2}\begin{pmatrix}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2} \\ \|\dot{\bar{q}}\|\end{pmatrix}^T Q_0 \begin{pmatrix}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2} \\ \|\dot{\bar{q}}\|\end{pmatrix}$$
$$- \varepsilon k_{mp} h_1(\bar{q})S_1(\bar{q}) - \frac{k_{mv}}{2}\|\dot{\bar{q}}\|^2 \triangleq W_3(\bar{q},\dot{\bar{q}}) \tag{3.97}$$

where

$$Q_0 = \begin{pmatrix}\varepsilon\kappa_1 k_{1m}^{a_1} & -\varepsilon\upsilon_2 \\ -\varepsilon\upsilon_2 & \beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1} - 2\varepsilon\upsilon_1\end{pmatrix}$$
$$k_{mp} \triangleq \frac{\kappa_1 k_{1m}^{a_1}}{2} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}{1+a_1}\gamma^{a_1} \quad , \quad k_{mv} \triangleq \beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1} - \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2 \gamma^{-1}$$

Furthermore, from (3.95), one may corroborate that $\gamma_m < \gamma < \gamma_M < \bar{\gamma}_M \implies k_{mp} > 0$ and $k_{mv} > 0$, and from Eqs. (3.85) that

$$\varepsilon < \varepsilon_0 < \varepsilon_4 < \bar{\varepsilon}_4 \implies Q_0 > 0$$

whence $W_3(\bar{q},\dot{\bar{q}})$ in (3.97) is concluded to be negative definite. Hence, V in (3.79) is a strict Lyapunov function proving that the trivial solution $\bar{q}(t) \equiv 0_n$ of the closed-loop system is globally uniformly asymptotically stable [40, Theorem 4.9].

4th stage: uniform finite-time/exponential stability. Thus, under the consideration of Remark 2.10, all that remains to be proven is that the trivial solution is uniformly finite-time stable if $a_1 \in (0,1)$, or (locally) exponentially stable if $a_1 = 1$. With this goal in mind, we retake V in (3.79) and analyze its derivative along the closed-loop system trajectories on $\mathbb{R}_{\ge 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$. More precisely, one sees from Remark 3.15 and (3.86) that, on $\mathbb{R}_{\ge 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, (3.96) takes the form

$$\dot{V}(t,\bar{q},\dot{\bar{q}}) \le -\left[\beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1} - \varepsilon\upsilon_1 - \frac{\varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2\gamma^{-1}}{2}\right]\|\dot{\bar{q}}\|^2 + \varepsilon\upsilon_2\|\bar{q}\|^{\frac{1+a_1}{2}}\|\dot{\bar{q}}\|$$
$$- \varepsilon\left(h_{1m}\kappa_1 k_{1m}^{a_1} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}\right)\|\bar{q}\|^{1+a_1}$$

which may be rewritten as

$$\dot{V}(t,\bar{q},\dot{\bar{q}}) \le -\frac{1}{2}\begin{pmatrix}\|\bar{q}\|^{(1+a_1)/2} \\ \|\dot{\bar{q}}\|\end{pmatrix}^T Q_1 \begin{pmatrix}\|\bar{q}\|^{(1+a_1)/2} \\ \|\dot{\bar{q}}\|\end{pmatrix} - \varepsilon\bar{k}_{mp}\|\bar{q}\|^{1+a_1} - \frac{k_{mv}}{2}\|\dot{\bar{q}}\|^2 \triangleq W_4(\bar{q},\dot{\bar{q}}) \tag{3.98}$$

where

$$Q_1 = \begin{pmatrix} \varepsilon h_{1m}\kappa_1 k_{1m}^{a_1} & -\varepsilon\upsilon_2 \\ -\varepsilon\upsilon_2 & \beta\bar{\eta}\left(\frac{\mu_m}{2}\right)^{\beta-1} - 2\varepsilon\upsilon_1 \end{pmatrix}$$

$$\bar{k}_{mp} \triangleq \frac{h_{1m}\kappa_1 k_{1m}^{a_1}}{2} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}$$

Furthermore, from (3.95), one may corroborate that $\gamma_m < \gamma < \gamma_M \implies \bar{k}_{mp} > 0$ and, from Eqs. (3.85), that $\varepsilon < \varepsilon_0 \leq \varepsilon_4 \implies Q_1 > 0$, whence $W_4(\bar{q}, \dot{\bar{q}})$ in (3.98) is concluded to be negative definite (on $\mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$). Furthermore, by defining $\bar{r}_i = (r_{i1}, \ldots, r_{in})^T$, $i = 1, 2$, with $r_{1j} = \alpha_0/(1 + a_1)$ and $r_{2j} = \alpha_0/2$ for all $j = 1, \ldots, n$, and any positive constant α_0, and $\bar{r} = (\bar{r}_1^T \; \bar{r}_2^T)^T$, one can see that, for every $(\bar{q}, \dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ and all $\epsilon \in (0, 1]$, we have on the one hand that $\|\delta_\epsilon^{\bar{r}_1}(\bar{q})\| \leq \|\bar{q}\| \leq b_1/k_{1M}$ and $\|\delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})\| \leq \|\dot{\bar{q}}\| \leq b_2/k_{2M}$, and consequently $\delta_\epsilon^{\bar{r}}(\bar{q}, \dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, and on the other hand, from (3.93) and (3.98), that

$$W_4(\delta_\epsilon^{\bar{r}}(\bar{q}, \dot{\bar{q}})) = W_4(\delta_\epsilon^{\bar{r}_1}(\bar{q}), \delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})) = W_4(\epsilon^{r_1}\bar{q}, \epsilon^{r_2}\dot{\bar{q}}) = \epsilon^{\alpha_0} W_4(\bar{q}, \dot{\bar{q}})$$

and

$$W_2(\delta_\epsilon^{\bar{r}}(\bar{q}, \dot{\bar{q}})) = W_2(\delta_\epsilon^{\bar{r}_1}(\bar{q}), \delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})) = W_2(\epsilon^{r_1}\bar{q}, \epsilon^{r_2}\dot{\bar{q}}) = \epsilon^{\alpha_0\beta} W_2(\bar{q}, \dot{\bar{q}})$$

i.e., that W_2 and W_4 are locally $\bar{r}$-homogeneous of degree $\alpha_2 = \alpha_0\beta$ and $\alpha_4 = \alpha_0$, respectively, with (common) domain of homogeneity $\mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$. Hence, by Lemma 2.1 and Remark 2.2, there exists a positive constant c such that

$$W_4(\bar{q}, \dot{\bar{q}}) \leq -c[W_2(\bar{q}, \dot{\bar{q}})]^{\frac{\alpha_4}{\alpha_2}}$$

$\forall(\bar{q}, \dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ and, consequently, from (3.93) and (3.98), we have that

$$\dot{V}(t, \bar{q}, \dot{\bar{q}}) \leq -c[V(t, \bar{q}, \dot{\bar{q}})]^{\frac{1}{\beta}}$$

$\forall(t, \bar{q}, \dot{\bar{q}}) \in \mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, with $1/\beta = \frac{2(1+a_1)}{3+a_1} \leq 1$. Moreover, since $a_1 \in (0, 1) \implies 1/\beta \in (0, 1)$, by Theorem 2.2 and Remark 2.9, item 1 of Proposition 3.9 is proven. On the other hand, since $a_1 = 1 \implies 1/\beta = 1$ item 2 of Proposition 3.9 follows from [40, Proof of Theorem 4.10]. ■

Remark 3.18. One notes from the second stage of the proof (see particularly (3.78)) that motion error dissipation is injected by the SD-type control term s_2, while the motion error damping term $F\dot{\bar{q}}$ is in charge to dominate the damping-indefinite residual third term (from left to right) in (3.73), $C(q, \dot{q}_d(t))\dot{\bar{q}}$, thus rendering a damping compound effect. The referred domination effect is included in the control strategy in view of the impossibility of the bounded term s_2 to dominate the referred unbounded residual term when this generates force/torque values beyond the limits of the SD-type control term. The motion error damping term $F\dot{\bar{q}}$ thus proves —in the third stage of the proof— to be useful to render the uniform asymptotic stability of the closed-loop trivial solution ($\bar{q}(t) \equiv 0_n$) global. Locally, s_2 actually suffices to provide damping enough to guarantee the finite-time/exponential tracking. Indeed, suppose that the last inequality in Assumption 3.3 ($B_{dv} < f_m/k_C$) is omitted, permitting further that $F \geq 0$; observe that this includes the naturally undamped case $F = 0$. One sees that if $d \geq 0$ then (3.78) holds on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ with $\bar{\eta} = \kappa_2 k_{2m}^{a_2}$ and consequently, the fourth stage of the proof holds, while if $d < 0$ then (3.78) holds on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ with $\bar{\eta} = \kappa_2 k_{2m}^{a_2} + d(b_2/k_{2M})^{1-a_2}$ and consequently, for suitable control parameters (for instance, sufficiently high control gains K_2) such that $\bar{\eta} > 0$, *i.e.*, $\kappa_2 k_{2m}^{a_2} k_{2M}^{1-a_2} > (k_C B_{dv} - f_m) b_2^{1-a_2}$ (or even $\kappa_2 k_{2m}^{a_2} k_{2M}^{1-a_2} > k_C B_{dv} b_2^{1-a_2} \geq (k_C B_{dv} - f_m) b_2^{1-a_2}$), the fourth stage of the proof holds as well. △

3.4 Robustness problem

This approach departs from the control law (3.70) proposed in the last section and the consideration of the realistic bounded-input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound T_i, *i.e.*, $|\tau_i| < T_i$, $i = 1, \ldots, n$. Additionally, an input-matching perturbation term, $\varrho = \varrho(t, q, \dot{q})$, is taken into account. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \operatorname{sat}\left(\frac{u_i + \varrho_i}{T_i}\right) \tag{3.100}$$

Further assumptions are stated.

Assumption 3.4. *The perturbation term* $\varrho = \varrho(t, q, \dot{q})$ *is bounded, i.e.,* $\|\varrho(t, q, \dot{q})\| \leq \bar{\varrho}$, $\forall (t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$, *for some known value* $\bar{\varrho} \geq 0$, *or equivalently* $|\varrho_i(t, q, \dot{q})| \leq \bar{\varrho}_i$, $\forall (t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, \ldots, n$, *for known bound values* $\bar{\varrho}_i \geq 0$.

Assumption 3.5. $T_i > B_{gi} + \bar{\varrho}_i$, $\forall i \in \{1, \ldots, n\}$.

Furthermore, Assumption 3.2 holds and Assumption 3.3 is modified as follows.

Assumption 3.6. $q_d(t) \in \mathcal{C}^2(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ *such that* $\|\dot{q}_d(t)\| \leq B_{dv}$ *and* $\|\ddot{q}_d(t)\| \leq B_{da}$, $\forall t \geq 0$, *for sufficiently small (positive) bound values* B_{dv} *and* B_{da} *such that* $\mu_{M_j} B_{da} + k_{C_j} B_{dv}^2 + \|F_j\| B_{dv} < T_j - B_{gj} - \bar{\varrho}_j$, $\forall j \in \{1, \ldots, n\}$, *and* $d \triangleq f_m - k_C B_{dv} > 0$.

Under the consideration of Assumption 3.6, the considered control scheme guarantees the tracking objective for any initial condition, under the absence of perturbation, and avoids input saturation in both analytical contexts of Section 3.3 and this section. In addition to the characteristics stated in Section 3.3, the involved σ_{ij}, $i = 1, 2$, $j = 1, \ldots, n$, functions in (3.70) satisfy

$$B_j \triangleq \sup_{(\varsigma_1, \varsigma_2) \in \mathbb{R}^2} \left|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\right| < T_j - \mu_{M_j} B_{da} - k_{C_j} B_{dv}^2 - \|F_j\| B_{dv} - B_{gj} - \bar{\varrho}_j \tag{3.101}$$

In the absence of the perturbation term, i.e., with $\varrho(t, q, \dot{q}) \equiv 0_n$ (or equivalently, $\bar{\varrho} = 0$) Proposition 3.9, stated in Section 3.3, holds.

The following result is stated and proven considering the presence of perturbation, *i.e.*, $\varrho(t, q, \dot{q}) \not\equiv 0_n$, $\forall (t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 3.10. *Consider system* (3.68), (3.100) *in closed loop with the control law in* (3.70), *under the stated design requirements with* $\bar{\varrho} \geq 0$ *(or equivalently* $\bar{\varrho}_j \geq 0$, $j = 1, \ldots, n$*). Thus, for any* $(t_0, \bar{q}(t_0), \dot{\bar{q}}(t_0)) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$*, we have* $|\tau_j(t)| = |u_j(t)| < T_j$, $\forall t \geq t_0 \geq 0$, $j = 1, \ldots, n$*, and, for a sufficiently small* $\bar{\varrho}$*, the close-loop solutions* $(\bar{q}, \dot{\bar{q}})(t)$ *are uniformly ultimately bounded with ultimate bound* $\varpi_0 (\bar{\varrho}/\varpi_1)^{\bar{\gamma}(a_1)}$*, i.e., such that*

$$\|(\bar{q}^T \; \dot{\bar{q}}^T)(t)\| \leq \varpi_0 \left(\frac{\bar{\varrho}}{\varpi_1} \right)^{\bar{\gamma}(a_1)}$$

$\forall t \geq t_0 + T$*, for some* ϖ_0, $\varpi_1 \in (0, \infty)$, $T \in [0, \infty)$*, and* $\bar{\gamma}(a_1) \geq 1$, $\forall a_1 \in (0, 1]$*, with* $\bar{\gamma}(a_1) = 1 \iff a_1 = 1$.

Proof. The proof is divided into five stages. The first stage shows that input saturation is avoided and obtains the consequent closed-loop dynamics in the tracking error variable space. In the second stage, the Lyapunov function candidate V is presented and the expressions of its derivative along the closed-loop system $\dot{V}$ and a suitable time-invariant upper bound are obtained. The third and fourth sections develop an analysis of V and $\dot{V}$ in and out from a suitable origin-centered ball, respectively. Based on these results, the fifth section develops the conclusion of the proof.

1st stage: input saturation avoidance and closed-loop dynamics. Observe that — for every $j \in \{1, \ldots, n\}$— by Assumptions 1.1–1.3, 3.2, and 3.4, in addition to the satisfaction of (3.101), we have that, for any $(t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} |u_j(t,q,\dot{q}) + \varrho_j(t,q,\dot{q})| &\leq \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{\bar{q}}_j)\right| + \|H_j(q)\|\|\ddot{q}_d(t)\| + \|C_j(q,\dot{q}_d(t))\|\|\dot{q}_d(t)\| \\ &\quad + \|F_j\|\|\dot{q}_d(t)\| + |g_j(q)| + |\varrho_j(t,q,\dot{q})| \\ &\leq B_j + \mu_{Mj}B_{da} + k_{Cj}B_{dv}^2 + \|F_j\|B_{dv} + B_{gj} + \bar{\varrho}_j < T_j \end{aligned}$$

From this and (3.100), one sees that $T_j > |u_j(t,q,\dot{q}) + \varrho_j(t,q,\dot{q})| = |u_j + \varrho_j| = |\tau_j|$, $\forall (t,q,\dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$, which shows that, under the proposed scheme, the input saturation values, T_j, are never reached. Thus, the closed-loop dynamics becomes

$$H(q)\ddot{\bar{q}} + C(q,\dot{q})\dot{\bar{q}} + C(q,\dot{q}_d(t))\dot{\bar{q}} + F\dot{\bar{q}} = -s_1(K_1\bar{q}) - s_2(K_2\dot{\bar{q}}) + \varrho(t,q,\dot{q}) \qquad (3.102)$$

where Property 1.2.3 has been used.

2nd stage: the Lyapunov function candidate and its derivative along the closed-loop trajectories. Let

$$V(t,\bar{q},\dot{\bar{q}}) = \frac{1}{2}\dot{\bar{q}}^T H(\bar{q} + q_d(t))\dot{\bar{q}} + \int_{0_n}^{\bar{q}} s_1^T(K_1 z)\, dz + \varepsilon\rho^T(\bar{q})H(\bar{q} + q_d(t))\dot{\bar{q}} \qquad (3.103)$$

with $\int_{0_n}^{\bar{q}} s_1^T(K_1 z)\, dz$ as defined in Eq. (3.74), ε a positive constant, and $\rho(\bar{q}) = h(\bar{q})\bar{q}$ (as defined in Section 3.3) is a continuously differentiable function satisfying Eqs. (3.81)–(3.82) and holding all the properties stated in Remarks 3.15–3.17. We will show that, for a small enough value of ε, $V(t,\bar{q},\dot{q})$ in (3.103) is a suitable Lyapunov function candidate through which the proof will be completed; in particular, this will be proven with

$$\varepsilon < \varepsilon_0 \triangleq \min\{\varepsilon_1,\, \varepsilon_2,\, \varepsilon_3,\, \varepsilon_4\} \tag{3.104a}$$

where

$$\varepsilon_1 = \frac{\kappa_1 \bar{k}_{1m}}{\mu_M \bar{r}_1} \leq \frac{2\kappa_1 k_{1m}^{a_1}}{(1+a_1)\mu_M}\left(\frac{k_{1M}}{b_1}\right)^{1-a_1} \quad , \quad \varepsilon_2 = \frac{\mu_m}{\mu_M} \tag{3.104b}$$

$$\varepsilon_3 = \frac{\eta_1}{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}\left[\frac{\kappa_1 k_{1m}^{a_1}(1+a_1)}{4\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right]^{1/a_1} \quad , \quad \varepsilon_4 = \frac{\eta_1 \kappa_1 k_{1m}^{a_1}}{2\kappa_1 k_{1m}^{a_1}\upsilon_1 + \upsilon_2^2} \tag{3.104c}$$

with $k_{1m} = \min_j\{k_{1j}\}$, $k_{1M} = \max_j\{k_{1j}\}$, $\bar{k}_{1m} = \min\{1, k_{1m}\}$, $\bar{r}_1 = \max\{1, b_1/k_{1M}\}$, $\bar{c}$ given in (3.87a),

$$\eta_1 = \min\left\{\frac{\kappa_2 k_{2m}^{a_2}}{2}\left(\frac{b_2}{k_{2M}}\right)^{a_2-1},\, \frac{d}{2}\right\} \tag{3.105}$$

d as defined in Assumption 3.6, while υ_1 and υ_2 are as defined in Eq. (3.87b). With such a goal in mind, let us begin by noting, from (3.103) that, by Property 1.1, Assumption 1.1, and Lemmas 2.7–2.8:

$$\frac{\mu_m}{2}\|\dot{q}\|^2 + \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}S_1(\bar{q}) - \varepsilon\mu_M\|\rho(\bar{q})\|\|\dot{q}\| \leq V(t,\bar{q},\dot{q}) \leq \frac{\mu_M}{2}\|\dot{q}\|^2 + \bar{\kappa}_1 k_{1M}^{a_1} n S_1(\bar{q}) + \varepsilon\mu_M\|\rho(\bar{q})\|\|\dot{q}\|$$

where $S_1(\bar{q}) = S_0(\bar{q}; a_1, b_1/k_{1M})$. Further, from Young's inequality (with $\phi = \psi = 2$) and Remark 3.17, we have that

$$\|\rho(\bar{q})\|\|\dot{q}\| \leq \frac{1}{2}(\|\rho(\bar{q})\|^2 + \|\dot{q}\|^2) \leq \frac{1}{2}\left[\left(\frac{b_1}{k_{1M}}\right)^{1-a_1} S_1(\bar{q}) + \|\dot{q}\|^2\right]$$

and consequently

$$W_1(\bar{q},\dot{q}) \leq V(t,\bar{q},\dot{q}) \leq W_2(\bar{q},\dot{q}) \tag{3.106a}$$

where, for every $\ell \in \{1,2\}$:

$$W_\ell(\bar{q},\dot{q}) = p_{\ell 1} S_1(\bar{q}) + p_{\ell 2}\|\dot{q}\|^2 \tag{3.106b}$$

with

$$p_{11} = \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1} - \frac{\varepsilon\mu_M}{2}\left(\frac{b_1}{k_{1M}}\right)^{1-a_1} \quad , \quad p_{12} = \frac{\mu_m - \varepsilon\mu_M}{2}$$
$$p_{21} = \bar{\kappa}_1 k_{1M}^{a_1} n + \frac{\varepsilon\mu_M}{2}\left(\frac{b_1}{k_{1M}}\right)^{1-a_1} \quad , \quad p_{22} = \frac{\mu_M(1+\varepsilon)}{2}$$

Furthermore, from expressions (3.104), one may corroborate that $\varepsilon < \varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2\} \implies \min\{p_{11}, p_{12}\} > 0$. From this, $W_1(\bar{q},\dot{q})$ is concluded to be positive definite and radially unbounded, while it is obvious that $W_2(\bar{q},\dot{\bar{q}})$ has the same properties too. Thus, $V(t,\bar{q},\dot{\bar{q}})$ is concluded to be positive definite, radially unbounded and decrescent. Its derivative along the closed-loop system trajectories is obtained as

$$
\begin{aligned}
\dot{V}(t,\bar{q},\dot{\bar{q}}) &= \dot{\bar{q}}^T H(q)\ddot{\bar{q}} + \frac{1}{2}\dot{\bar{q}}^T \dot{H}(q,\dot{q})\dot{\bar{q}} + s_1^T(K_1\bar{q})\dot{\bar{q}} + \varepsilon\left(\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}}\right)^T H(q)\dot{\bar{q}} \\
&\quad + \varepsilon\rho(\bar{q})^T \dot{H}(q,\dot{q})\dot{\bar{q}} + \varepsilon\rho(\bar{q})^T H(q)\ddot{\bar{q}} \\
&= -\dot{\bar{q}}^T[C(q,\dot{q})\dot{\bar{q}} + C(q,\dot{q}_d(t))\dot{\bar{q}} + F\dot{\bar{q}} + s_1(K_1\bar{q}) + s_2(K_2\dot{\bar{q}}) - \varrho(t,\bar{q},\dot{\bar{q}})] \\
&\quad + \frac{1}{2}\dot{\bar{q}}^T \dot{H}(q,\dot{q})\dot{\bar{q}} + s_1^T(K_1\bar{q})\dot{\bar{q}} + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial q}(\bar{q})\dot{\bar{q}} \\
&\quad + \varepsilon\rho(\bar{q})^T[C(q,\dot{q}) + C^T(q,\dot{q})]\dot{\bar{q}} \\
&\quad - \varepsilon\rho(\bar{q})^T[C(q,\dot{q})\dot{\bar{q}} + C(q,\dot{q}_d(t))\dot{\bar{q}} + F\dot{\bar{q}} + s_1(K_1\bar{q}) + s_2(K_2\dot{\bar{q}}) - \varrho(t,\bar{q},\dot{\bar{q}})] \\
&= -\dot{\bar{q}}^T s_2(K_2\dot{\bar{q}}) - \dot{\bar{q}}^T C(q,\dot{q}_d(t))\dot{\bar{q}} - \dot{\bar{q}}^T F\dot{\bar{q}} + \dot{\bar{q}}^T\varrho(t,\bar{q},\dot{\bar{q}}) - \varepsilon\rho^T(\bar{q})C(q,\dot{q}_d(t))\dot{\bar{q}} \\
&\quad - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) - \varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) + \varepsilon\rho^T(\bar{q})\varrho(t,\bar{q},\dot{\bar{q}}) \\
&\quad + \varepsilon\dot{\bar{q}}^T C(q,\dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T C(q,\dot{q}_d(t))\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial\bar{q}}(\bar{q})\dot{\bar{q}}
\end{aligned}
$$

where $H(q)\ddot{\bar{q}}$ has been replaced by its equivalent expression from the closed-loop dynamics (3.102) and Properties 1.2.1–1.2.3 have been used. At this point, it is important to note (for its subsequent use in the analysis) that, from expressions (3.104), one may corroborate that $\varepsilon < \varepsilon_0 \leq \varepsilon_3 \Longrightarrow \gamma_m < \gamma_M$, with

$$\gamma_m \triangleq \frac{\varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2}{\eta_1} \quad , \quad \gamma_M \triangleq \left(\frac{\kappa_1 k_{1m}^{a_1}(1+a_1)}{4\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}}\right)^{1/a_1} \tag{3.107}$$

We proceed to analyze the terms of $\dot{V}(t,\bar{q},\dot{\bar{q}})$.

First, second and third terms. From Lemma 2.7, we have that $-\dot{\bar{q}}^T s_2(K_2\dot{\bar{q}}) \leq -\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}})$, where $S_2(\dot{\bar{q}}) = S_0(\dot{\bar{q}}; a_2, b_2/k_{2M})$, $k_{2m} = \min_j\{k_{2j}\}$ and $k_{2M} = \max_j\{k_{2j}\}$, and consequently, under the additional consideration of Property 1.2.4 as well as Assumptions 1.2, 3.2 and 3.6 (recalling that $d = f_m - k_C B_{dv} > 0$), we get:

$$-\dot{\bar{q}}^T s_2(K_2\dot{\bar{q}}) - \dot{\bar{q}}^T C(q,\dot{q}_d(t))\dot{\bar{q}} - \dot{\bar{q}}^T F\dot{\bar{q}} \leq -\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) - d\|\dot{\bar{q}}\|^2 \leq -\eta_1\|\dot{\bar{q}}\|^2 - \eta_2\|\dot{\bar{q}}\|^{1+a_2}$$

with η_1 as defined in Eq. (3.105) and $\eta_2 = \min\left\{\frac{\kappa_2 k_{2m}^{a_2}}{2}, \frac{d}{2}\left(\frac{b_2}{k_{2M}}\right)^{1-a_2}\right\}$. The right-most inequality is corroborated by observing that for all $\|\dot{\bar{q}}\| \leq b_2/k_{2M}$, we have that

$$
\begin{aligned}
\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq \kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) &= \kappa_2 k_{2m}^{a_2}\|\dot{\bar{q}}\|^{1+a_2} \\
&\geq \frac{\kappa_2 k_{2m}^{a_2}}{2}\|\dot{\bar{q}}\|^{1+a_2} + \frac{\kappa_2 k_{2m}^{a_2}}{2}\left(\frac{b_2}{k_{2M}}\right)^{a_2-1}\|\dot{\bar{q}}\|^2 \\
&\geq \eta_2\|\dot{\bar{q}}\|^{1+a_2} + \eta_1\|\dot{\bar{q}}\|^2
\end{aligned}
$$

and for all $\|\dot{\bar{q}}\| > b_2/k_{2M}$ that

$$\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq d\|\dot{\bar{q}}\|^2 \geq \frac{d}{2}\|\dot{\bar{q}}\|^2 + \frac{d}{2}\left(\frac{b_2}{k_{2M}}\right)^{1-a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq \eta_2\|\dot{\bar{q}}\|^{1+a_2} + \eta_1\|\dot{\bar{q}}\|^2$$

Fourth and ninth terms. From Assumption 3.4, we have that

$$\dot{\bar{q}}^T \varrho(t,\bar{q},\dot{\bar{q}}) + \varepsilon\rho^T(\bar{q})\varrho(t,\bar{q},\dot{\bar{q}}) \leq \bar{\varrho}\|\dot{\bar{q}}\| + \varepsilon\bar{\varrho}\|\rho(\bar{q})\|$$

Fifth, sixth, tenth, eleventh and twelveth terms. From Assumptions 1.1, 1.2, 3.2, and 3.6, the properties of ρ (through inequality (3.81)), Remarks 3.16–3.17, and Eq. (3.87b), we get:

$$\begin{aligned}
-\varepsilon\rho^T(\bar{q})C(q,\dot{q}_d(t))\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} + \varepsilon\dot{\bar{q}}^T C(q,\dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T C(q,\dot{q}_d(t))\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^T H(q)\frac{\partial\rho}{\partial\bar{q}}(\bar{q})\dot{\bar{q}} \\
\leq 2\varepsilon k_C B_{dv}\|\rho(\bar{q})\|\|\dot{\bar{q}}\| + \varepsilon f_M\|\rho(\bar{q})\|\|\dot{\bar{q}}\| \\
+ \varepsilon\frac{k_C b_1}{k_{1M}}\|\dot{\bar{q}}\|^2 + \varepsilon\mu_M\|\dot{\bar{q}}\|^2 \\
\leq \varepsilon\upsilon_2\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\|\dot{\bar{q}}\| + \varepsilon\upsilon_1\|\dot{\bar{q}}\|^2
\end{aligned}$$

Seventh therm. From the definition of ρ and Lemma 2.7 we get:

$$-\varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) = -\varepsilon h_1(\bar{q})\bar{q}^T s_1(K_1\bar{q}) \leq -\varepsilon\kappa_1 k_{1m}^{a_1} h_1(\bar{q})S_1(\bar{q})$$

Eighth term. From Hölder and Young's inequalities (both with $\phi = 1 + a_1$ and $\psi = 2/a_2$), the definition of s_2, Remarks 1.1, 2.13 and 3.17, and the consideration of a positive constant $\gamma \in (\gamma_m, \gamma_M)$ (recall (3.107)), we have (recalling Eqs. (3.71) and (3.87a)) that

$$\begin{aligned}
-\varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) &\leq \varepsilon\big|\rho^T(\bar{q})s_2(K_2\dot{\bar{q}})\big| \leq \varepsilon\|\rho(\bar{q})\|_{1+a_1}\|s_2(K_2\dot{\bar{q}})\|_{2/a_2} \leq \varepsilon\bar{c}\bar{\kappa}_2\|\rho(\bar{q})\|\|K_2\dot{\bar{q}}\|^{a_2} \\
&\leq \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\left(\gamma^{a_2/2}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/(1+a_1)}\right)\left(\gamma^{-a_2/2}\|\dot{\bar{q}}\|^{a_2}\right) \\
&\leq \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\left(\frac{\gamma^{a_1}}{1+a_1}h_1(\bar{q})S_1(\bar{q}) + \frac{a_2}{2}\gamma^{-1}\|\dot{\bar{q}}\|^2\right)
\end{aligned}$$

Thus, from the expressions obtained above, we get

$$\begin{aligned}
\dot{V}(t,\bar{q},\dot{\bar{q}}) \leq &-\left[\eta_1 - \varepsilon\upsilon_1 - \frac{\varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2\gamma^{-1}}{2}\right]\|\dot{\bar{q}}\|^2 + \varepsilon\upsilon_2\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\|\dot{\bar{q}}\| \\
&- \varepsilon\left(\kappa_1 k_{1m}^{a_1} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}\right)h_1(\bar{q})S_1(\bar{q}) - \eta_2\|\dot{\bar{q}}\|^{1+a_2} + \varepsilon\bar{\varrho}\|\rho(\bar{q})\| + \bar{\varrho}\|\dot{\bar{q}}\|
\end{aligned}$$

which may be rewritten as

$$\dot{V}(t,\bar{q},\dot{\bar{q}}) \leq -W_3(\bar{q},\dot{\bar{q}}) + W_{41}(\bar{q}) + W_{42}(\dot{\bar{q}}) \tag{3.108}$$

where

$$W_3(\bar{q},\dot{\bar{q}}) = \frac{1}{2}\begin{pmatrix}\left[h_1(\bar{q})S_1(\bar{q})\right]^{1/2}\\ \|\dot{\bar{q}}\|\end{pmatrix}^T Q \begin{pmatrix}\left[h_1(\bar{q})S_1(\bar{q})\right]^{1/2}\\ \|\dot{\bar{q}}\|\end{pmatrix} + \varepsilon k_{mp} h_1(\bar{q})S_1(\bar{q}) + \frac{k_{mv}}{2}\|\dot{\bar{q}}\|^2$$

$$W_{41}(\bar{q}) = -\frac{\varepsilon\kappa_1 k_{1m}^{a_1}}{4} h_1(\bar{q})S_1(\bar{q}) + \varepsilon\varrho\|\rho(\bar{q})\|$$
$$W_{42}(\dot{\bar{q}}) = -\eta_2\|\dot{\bar{q}}\|^{1+a_2} + \varrho\|\dot{\bar{q}}\|$$

with

$$Q = \begin{pmatrix}\varepsilon\kappa_1 k_{1m}^{a_1} & -\varepsilon\upsilon_2\\ -\varepsilon\upsilon_2 & \eta_1 - 2\varepsilon\upsilon_1\end{pmatrix}$$

$$k_{mp} \triangleq \frac{\kappa_1 k_{1m}^{a_1}}{4} - \frac{\bar{c}\bar{\kappa}_2 k_{2M}^{a_2}\gamma^{a_1}}{1+a_1} \quad , \quad k_{mv} \triangleq \eta_1 - \varepsilon\bar{c}\bar{\kappa}_2 k_{2M}^{a_2} a_2\gamma^{-1}$$

Furthermore, from (3.107), one corroborates that $\gamma_m < \gamma < \gamma_M \implies k_{mp} > 0$ and $k_{mv} > 0$, and from expressions (3.104) that $\varepsilon < \varepsilon_0 < \varepsilon_4 \implies Q > 0$, whence $W_3(\bar{q},\dot{\bar{q}})$ is concluded to be positive definite.

3rd stage: analysis on $\left\|(\bar{q}^T,\dot{\bar{q}}^T)^T\right\| \leq r \triangleq b_1/k_{1M}$. Letting $x = (\bar{q}^T,\dot{\bar{q}}^T)^T$ and noting that $\|x\| \leq r \implies \max\{\|\bar{q}\|,\|\dot{\bar{q}}\|\} \leq r$, we have, from Eqs. (3.106), that on $\mathcal{B}_r^{2n}$:

$$\begin{aligned} W_1(x) = p_{11}\|\bar{q}\|^{1+a_1} + p_{12}\|\dot{\bar{q}}\|^2 &\geq p_{11}r^{a_1-1}\|\bar{q}\|^2 + p_{12}\|\dot{\bar{q}}\|^2 \\ &\geq \min\{p_{11}r^{a_1-1}, p_{12}\}\|x\|^2 \geq \bar{\alpha}_1\|x\|^2 \triangleq \alpha_1(\|x\|) \end{aligned} \tag{3.110a}$$

with $\bar{\alpha}_1 = \min\{\kappa_1\bar{k}_{1m}/\bar{r}_1 - \varepsilon\mu_M,\ \mu_m - \varepsilon\mu_M\}/2$ $[> 0 \Leftarrow \varepsilon \leq \varepsilon_0 < \min\{\varepsilon_1,\varepsilon_2\}]$ (recall that $\bar{k}_{1m} = \min\{1,k_{1m}\}$ and $\bar{r}_1 = \max\{1, b_1/k_{1M}\}$), and

$$\begin{aligned} W_2(x) = p_{21}\|\bar{q}\|^{1+a_1} + p_{22}\|\dot{\bar{q}}\|^2 &\leq p_{21}\|\bar{q}\|^{1+a_1} + p_{22}r^{1-a_1}\|\dot{\bar{q}}\|^{1+a_1} \\ &\leq (p_{21} + p_{22}r^{1-a_1})\|x\|^{1+a_1} \leq \bar{\alpha}_2\|x\|^{1+a_1} \triangleq \alpha_2(\|x\|) \end{aligned} \tag{3.110b}$$

with $\bar{\alpha}_2 = \bar{\kappa}_1\bar{k}_{1M}n + (\varepsilon + 1/2)\mu_M\bar{r}_1$, $\bar{k}_{1M} = \max\{1,k_{1M}\}$.

On the other hand, we have that $W_{42}(\dot{\bar{q}}) = W_{52}(\|\dot{\bar{q}}\|)$ and —by considering the definition of S_1 as well as the properties of ρ and h_1 (basically (3.81) and Remark 3.15)— on $\mathcal{B}_r^n:\ W_{41}(\bar{q}) \leq W_{51}(\|\bar{q}\|)$, with —for every $\ell \in \{1,2\}$— $W_{5\ell}$ defined on $\mathbb{R}_{\geq 0}$ as

$$W_{5\ell}(\varsigma) = -\bar{p}_{\ell 1}\varsigma^{1+a_\ell} + \bar{p}_{\ell 2}\varsigma \tag{3.111}$$

where

$$\bar{p}_{\ell 1} = \left(\frac{\varepsilon\kappa_1 k_{1m}^{a_1} h_{1m}}{4}\right)^{2-\ell}\eta_2^{\ell-1} \quad , \quad \bar{p}_{\ell 2} = \varepsilon^{2-\ell}\varrho$$

with h_{1m} as defined in Section 3.3. Further, with

$$\varsigma_{0\ell} \triangleq \left[\frac{\bar{\varrho}}{(\kappa_1 k_{1m}^{a_1} h_{1m}/4)^{2-\ell}\eta_2^{\ell-1}}\right]^{1/a_\ell} \quad , \quad \varsigma_{c\ell,\,\bar{\varrho}} \triangleq \left[\frac{\bar{\varrho}}{(\kappa_1 k_{1m}^{a_1} h_{1m}/4)^{2-\ell}\eta_2^{\ell-1}(1+a_\ell)}\right]^{1/a_\ell}$$

and

$$\bar{\omega}_\ell \triangleq W_{5\ell}(\varsigma_{c\ell,\,\bar{\varrho}}) = \varsigma_{c\ell,\,\bar{\varrho}} \cdot \frac{\varepsilon^{2-\ell} a_\ell \bar{\varrho}}{1+a_\ell} > 0$$

a simple analysis shows that:

- $W_{5\ell}(\varsigma_{0\ell}) = W_{5\ell}(0) = 0 = \frac{dW_{5\ell}}{d\varsigma}(\varsigma_{c\ell,\,\bar{\varrho}})$;
- $\frac{dW_{5\ell}}{d\varsigma}(\varsigma) > 0,\ \forall\, 0 < \varsigma < \varsigma_{c\ell,\,\bar{\varrho}}$ and $\frac{dW_{5\ell}}{d\varsigma}(\varsigma) < 0,\ \forall \varsigma > \varsigma_{c\ell,\,\bar{\varrho}}$;
- $W_{5\ell}(\varsigma) \leq \bar{\omega}_\ell,\ \forall \varsigma \geq 0$;
- $W_{5\ell}(\varsigma) > 0,\ \forall 0 < \varsigma < \varsigma_{0\ell}$ and $W_{5\ell}(\varsigma) < 0,\ \forall \varsigma > \varsigma_{0\ell}$; and
- $W_{5\ell}(\varsigma) \to -\infty$ as $\varsigma \to \infty$

i.e., $W_{5\ell}$ has a global maximum at $\varsigma_{c\ell,\,\bar{\varrho}}$ —with maximum value $\bar{\omega}_\ell$— and it is strictly decreasing thereafter, taking —all— negative values (exclusively) from $\varsigma_{0\ell}$ on. Hence, there exists $\varsigma_{*\ell} > \varsigma_{0\ell}$ such that

$$W_{5\ell}(\varsigma) \leq -\bar{\omega}_{3-\ell} \quad \forall \varsigma > \varsigma_{*\ell}$$

or equivalently, there is $\theta_\ell > 1$, such that

$$W_{5\ell}(\varsigma) \leq -\bar{\omega}_{3-\ell} \quad \forall \varsigma > \theta_\ell \varsigma_{0\ell} = \varsigma_{*\ell}$$

Thus, by noting that $\|x\| \geq [\varsigma_{*1}^2 + \varsigma_{*2}^2]^{1/2}$ implies that $\|\bar{q}\| \geq \varsigma_{*1},\ \forall \|\dot{\bar{q}}\| \leq \varsigma_{*2}$, and $\|\dot{\bar{q}}\| \geq \varsigma_{*2},\ \forall \|\bar{q}\| \leq \varsigma_{*1}$, we have —with $\theta_M = \max\{\theta_1, \theta_2\}$— that for all

$$\|x\| \geq \mu_0 \triangleq \theta_M [\varsigma_{01}^2 + \varsigma_{02}^2]^{1/2} \geq [\varsigma_{*1}^2 + \varsigma_{*2}^2]^{1/2}$$

either $\|\bar{q}\| \geq \varsigma_{*1}$ and consequently

$$W_{51}(\bar{q}) + W_{52}(\dot{\bar{q}}) \leq W_{51}(\bar{q}) + \bar{\omega}_2 \leq 0$$

or $\|\bar{q}\| \leq \varsigma_{*1} \implies \|\dot{\bar{q}}\| \geq \varsigma_{*2}$ and consequently

$$W_{51}(\bar{q}) + W_{52}(\dot{\bar{q}}) \leq \bar{\omega}_1 + W_{52}(\dot{\bar{q}}) \leq 0$$

[or analogously, either $\|\dot{\bar{q}}\| \geq \varsigma_{*2} \implies W_{51}(\bar{q}) + W_{52}(\dot{\bar{q}}) \leq \bar{\omega}_1 + W_{52}(\dot{\bar{q}}) \leq 0$, or $\|\dot{\bar{q}}\| \leq \varsigma_{*2} \implies \|\bar{q}\| \geq \varsigma_{*1} \implies W_{51}(\bar{q}) + W_{52}(\dot{\bar{q}}) \leq W_{51}(\bar{q}) + \bar{\omega}_2 \leq 0$], *i.e.*,

$$\|x\| \geq \mu_0 \implies W_{51}(\bar{q}) + W_{52}(\dot{\bar{q}}) \leq 0$$

Thus, for a sufficiently small value of $\bar{\varrho}$, such that (recalling expressions (3.110))

$$\bar{\varrho} < \bar{\varrho}_1 \triangleq \varpi_1 \min\left\{ \left(\frac{\bar{\alpha}_1 r^2}{\bar{\alpha}_2}\right)^{\frac{a_2}{1+a_1}},\ 1 \right\} \tag{3.112}$$

with

$$\varpi_1 \triangleq \frac{1}{\sqrt{2}\theta_M} \cdot \min\left\{ \frac{\kappa_1 \bar{k}_{1m} h_{1m}}{4},\ \bar{\eta}_2 \right\} \tag{3.113}$$

$\bar{\eta}_2 = \min\{\kappa_2 \bar{k}_{2m},\ d\bar{r}_2\}/2$, $\bar{k}_{2m} = \min\{1,\ k_{2m}\}$, $\bar{r}_2 = \min\{1,\ b_2/k_{2M}\}$, which implies

$$\mu \triangleq \left(\frac{\bar{\varrho}}{\varpi_1}\right)^{1/a_2} < \min\left\{ \left(\frac{\bar{\alpha}_1 r^2}{\bar{\alpha}_2}\right)^{\frac{1}{1+a_1}},\ 1 \right\} \tag{3.114}$$

and consequently

$$\mu < \left(\frac{\bar{\alpha}_1 r^2}{\bar{\alpha}_2}\right)^{\frac{1}{1+a_1}} = \alpha_2^{-1}(\alpha_1(r)) \tag{3.115}$$

and (recalling that $\theta_M = \max\{\theta_1, \theta_2\} \geq \min\{\theta_1, \theta_2\} > 1$)

$$\max\left\{ \frac{\bar{\varrho}}{\kappa_1 k_{1m}^{a_1} h_{1m}/4},\ \frac{\bar{\varrho}}{\eta_2} \right\} \leq \frac{\bar{\varrho}}{\min\{\kappa_1 \bar{k}_{1m} h_{1m}/4,\ \bar{\eta}_2\}} \leq \frac{\bar{\varrho}}{\varpi_1} < 1 \tag{3.116}$$

we have that

$$\begin{aligned}
\|x\| \geq \mu &\geq \left[\frac{(\sqrt{2}\theta_M)^{a_2} \bar{\varrho}}{\min\{\kappa_1 k_{1m}^{a_1} h_{1m}/4,\ \eta_2\}} \right]^{1/a_2} = \theta_M \left[2 \left(\frac{\bar{\varrho}}{\min\{\kappa_1 k_{1m}^{a_1} h_{1m}/4,\ \eta_2\}} \right)^{2/a_2} \right]^{1/2} \\
&\geq \theta_M \left[\left(\frac{\bar{\varrho}}{\kappa_1 k_{1m}^{a_1} h_{1m}/4} \right)^{2/a_2} + \left(\frac{\bar{\varrho}}{\eta_2} \right)^{2/a_2} \right]^{1/2} \\
&\geq \theta_M \left[\left(\frac{\bar{\varrho}}{\kappa_1 k_{1m}^{a_1} h_{1m}/4} \right)^{2/a_1} + \left(\frac{\bar{\varrho}}{\eta_2} \right)^{2/a_2} \right]^{1/2} = \mu_0 \\
&\implies W_{41}(\bar{q}) + W_{42}(\dot{\bar{q}}) \leq W_{51}(\|\bar{q}\|) + W_{52}(\|\dot{\bar{q}}\|) \leq 0
\end{aligned}$$

and consequently (recalling (3.108))

$$\dot{V}(t,x) \leq -W_3(x), \quad \forall (t,x) \in \mathbb{R}_{\geq 0} \times \{\mu \leq \|x\| \leq r\}$$

4th stage: analysis on $\left\|(\bar{q}^T\ \dot{\bar{q}}^T)^T\right\| \geq r \triangleq b_1/k_{1M}$.

In this stage, we consider two possible cases.

1) $\|\bar{q}\| \leq r$. In this case we recover the expressions of the 3rd stage, *i.e.*, $W_{4\ell} \leq W_{5\ell}$, $\ell = 1, 2$, with $W_{5\ell}$ given in Eq. (3.111). Thus, through an analog analysis, we get that for a sufficiently small value of $\bar{\varrho}$, satisfying (3.112) which implies (3.114) and consequently (3.115) and (3.116), we have that

$$\|x\| \geq r > \mu \geq \mu_0 \implies W_{41}(\bar{q}) + W_{42}(\dot{\bar{q}}) \leq W_{51}(\|\bar{q}\|) + W_{52}(\|\dot{\bar{q}}\|) \leq 0$$

and consequently

$$\dot{V}(t,x) \leq -W_3(x), \quad \forall (t,x) \in \mathbb{R}_{\geq 0} \times \{\|(\bar{q}^T\, \dot{\bar{q}}^T)^T\| \geq r : \ \|\bar{q}\| \leq r\}$$

2) $\|\bar{q}\| \geq r$. While W_{42} keeps the same expression as in the previous case, *i.e.*, $W_{42}(\dot{\bar{q}}) = W_{52}(\|\dot{\bar{q}}\|)$, by considering the definition of S_1 as well as the properties of ρ (basically (3.81)), we have that, in this case:

$$\begin{aligned} W_{41}(\bar{q}) &= -\left(\frac{\varepsilon \kappa_1 k_{1m}^{a_1}}{4}\right) h_1(\bar{q}) \left(\frac{b_1}{k_{1M}}\right)^{a_1} \|\bar{q}\| + \varepsilon \bar{\varrho} \|\rho(\bar{q})\| \\ &= -\varepsilon \left[\left(\frac{\kappa_1 k_{1m}^{a_1}}{4}\right)\left(\frac{b_1}{k_{1M}}\right)^{a_1} - \bar{\varrho}\right] \|\rho(\bar{q})\| \triangleq W_{61}(\bar{q}) \end{aligned} \tag{3.117}$$

Hence, for a sufficiently small value of $\bar{\varrho}$, such that

$$\bar{\varrho} < \bar{\varrho}_0 \triangleq \left(\frac{\kappa_1 k_{1m}^{a_1}}{4}\right)\left(\frac{b_1}{k_{1M}}\right)^{a_1}$$

we have that W_{61} in Eq. (3.117) is negative. Moreover, with $\bar{\varrho} < \bar{\varrho}_0$:

$$W_{52}(\|\dot{\bar{q}}\|) \leq \bar{\omega}_2 = \varsigma_{c2,\,\bar{\varrho}} \cdot \frac{a_2 \bar{\varrho}}{(1+a_2)} < \varsigma_{c2,\,\bar{\varrho}_0} \cdot \frac{a_2 \bar{\varrho}}{(1+a_2)}$$

and, since (by (3.82))

$$D_{\bar{q}} \|\rho(\bar{q})\| = \Big[h(\bar{q}; b_1/k_{1M}) + D_{\bar{q}} h(\bar{q}; b_1/k_{1M})\Big] \|\bar{q}\| > 0$$

$\forall \bar{q} \neq 0_n$ (*i.e.*, $\rho(\bar{q})$ is increasing in any radial direction), we have (recalling (3.81) and (3.86)) that

$$\inf_{\|\bar{q}\| \geq r} \|\rho(\bar{q})\| = \inf_{\|\bar{q}\| = b_1/k_{1M}} \|\rho(\bar{q})\| = \frac{h_{1m} b_1}{k_{1M}}$$

and consequently

$$W_{61}(\bar{q}) \leq -\varepsilon \left[\left(\frac{\kappa_1 k_{1m}^{a_1}}{4}\right)\left(\frac{b_1}{k_{1M}}\right)^{a_1} - \bar{\varrho}\right] \frac{h_{1m} b_1}{k_{1M}}$$

whence

$$\begin{aligned} W_{41}(\bar{q}) + W_{42}(\dot{\bar{q}}) &= W_{61}(\bar{q}) + W_{52}(\|\dot{\bar{q}}\|) \\ &< \frac{\varsigma_{c2,\,\bar{\varrho}_0} a_2 \bar{\varrho}}{1+a_2} - \varepsilon h_{1m} \left(\frac{b_1}{k_{1M}}\right) \left[\frac{\kappa_1 k_{1m}^{a_1}}{4} \left(\frac{b_1}{k_{1M}}\right)^{a_1} - \bar{\varrho}\right] \\ &= \left[\frac{\varsigma_{c2,\,\bar{\varrho}_0} a_2}{1+a_2} + \frac{\varepsilon h_{1m} b_1}{k_{1M}}\right] \bar{\varrho} - \varepsilon h_{1m} \left(\frac{\kappa_1 k_{1m}^{a_1}}{4}\right) \left(\frac{b_1}{k_{1M}}\right)^{1+a_1} \end{aligned}$$

Hence, for a sufficiently small value of $\bar{\varrho}$, such that

$$\bar{\varrho} < \bar{\varrho}_2 \triangleq \frac{\varepsilon h_{1m}(\kappa_1 k_{1m}^{a_1}/4)(b_1/k_{1M})^{1+a_1}}{\varsigma_{c2,\,\bar{\varrho}_0} a_2/(1+a_2) + \varepsilon h_{1m} b_1/k_{1M}} < \bar{\varrho}_0$$

we have that $W_{41}(\bar{q}) + W_{42}(\dot{\bar{q}}) < 0$ and consequently

$$\dot{V}(t,x) \leq -W_3(x), \quad \forall (t,x) \in \mathbb{R}_{\geq 0} \times \{\|(\bar{q}^T\, \dot{\bar{q}}^T)^T\| \geq r :\ \|\bar{q}\| \geq r\}$$

5th stage: uniform ultimate boundedness and ultimate bound.
From the results obtained in the 3rd and 4th stages above, we have that, for a sufficiently small value of $\bar{\varrho}$, such that $\bar{\varrho} < \min\{\bar{\varrho}_1,\ \bar{\varrho}_2\}$:

$$\dot{V}(t,x) \leq -W_3(x)$$

$\forall (t,x) \in \mathbb{R}_{\geq 0} \times \{\|x\| \geq \mu\}$, with $V(t,x)$ being a continuously differentiable, (globally) positive definite, decrescent and radially unbounded function such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

$\forall (t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{B}_r^{2n}$, with $\alpha_\ell \in \mathcal{K}$, $\ell = 1,2$, defined through expressions (3.110) and $r > \alpha_1^{-1}(\alpha_2(\mu))$, and consequently, by Corollary 2.1, we conclude that, for any $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{2n}$, the closed-loop solutions $x(t; t_0, x_0)$ are uniformly ultimately bounded, with ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$, *i.e.*, such that

$$\|x(t; t_0, x_0)\| \leq \varpi_0 \left(\frac{\bar{\varrho}}{\varpi_1}\right)^{\bar{\gamma}(a_1)}$$

$\forall t \geq t_0 + T$ for some $T \in [0, \infty)$, with $\varpi_0 = (\bar{\alpha}_2/\bar{\alpha}_1)^{1/2}$, ϖ_1 as defined in Eq. (3.113), and

$$\bar{\gamma}(a_1) = \frac{1+a_1}{2a_2} = \frac{(1+a_1)^2}{4a_1}$$

whence one gets that

$$\frac{d\bar{\gamma}}{da_1}(a_1) = \frac{a_1^2 - 1}{4a_1^2} \leq 0$$

$\forall a_1 \in (0,1]$, with $\frac{d\bar{\gamma}}{da_1}(a_1) = 0 \iff a_1 = 1$, and consequently $\bar{\gamma}(a_1) \geq \bar{\gamma}(1) = 1$, $\forall a_1 \in (0,1]$, with $\bar{\gamma}(a_1) = 1 \iff a_1 = 1$. ■

Remark 3.19. Observe that since $\bar{\gamma}(a_1) > 1$, $\forall a_1 \in (0,1)$, and $\bar{\gamma}(1) = 1$, and (in accordance to (3.116)) $\bar{\varrho}/\varpi_1 < 1$, the ultimate bound of the closed-loop responses obtained through finite-time controllers from the considered control scheme is lower than that gotten with their analog exponential tracking algorithms. Thus, for sufficiently small perturbation terms acting on the system, the finite-time controllers give rise to lower post-transient errors, confirming a robustness type aspect where they show superiority over their analog exponential tracking algorithms.

Remark 3.20. It is further worth to note that, in view of its sufficient character, the developed result actually shows that there is $\bar{\varrho}^* \geq \min\{\bar{\varrho}_1,\ \bar{\varrho}_2\}$ such that, for $\bar{\varrho} \leq \bar{\varrho}^*$, Proposition 3.10 holds, but for values of $\bar{\varrho}$ higher than $\bar{\varrho}^*$, ultimate boundedness could either cease to hold or keep holding but with lower ultimate bounds in the exponential tracking case.

CHAPTER 4

Simulation results

Throughout this chapter, numerical implementations are presented in order to show the performance achieved by the proposed controllers. With this goal in mind, the dynamical models of different robot manipulators, described in section 1.5, are taken into account. The simulation results are shown in accordance to the sections described in Chapter 3, and are mainly focused on showing the finite-time control implementations and to compare them with the analog exponential controller tests. Among the comparison purposes are the observation of the convergence differences, and the corroboration of the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones. Let us note that through the incorporation of exponential controller implementations in the comparison, the fastest and more desirable type of asymptotic stabilization is being considered. For the application of the controllers proposed in Chapter 3 the following functions are defined:

$$\sigma_u(\varsigma; \beta, a) = \text{sign}(\varsigma) \max\{|\varsigma|^\beta,\ a|\varsigma|\} \tag{4.1a}$$

$$\sigma_{bh}(\varsigma; \beta, a, M) = \text{sign}(\varsigma) \min\{|\sigma_u(\varsigma; \beta, a)|,\ M\} \tag{4.1b}$$

$$\sigma_{bs}(\varsigma; \beta, a, M, L) = \begin{cases} \sigma_u(\varsigma; \beta, a) & \text{if } |\varsigma| \leq L \\ \text{sign}(\varsigma)\sigma_{bs}^+(|\varsigma|; \beta, a, M, L) & \text{if } |\varsigma| > L \end{cases} \tag{4.1c}$$

where

$$\sigma_{bs}^+(\varsigma; \beta, a, M, L) = \sigma_u(L; \beta, a) + (M - \sigma_u(L; \beta, a)) \tanh\left(\frac{\sigma_u(\varsigma; \beta, a) - \sigma_u(L; \beta, a)}{M - \sigma_u(L; \beta, a)}\right)$$

for constants $\beta > 0$, $a \in \{0, 1\}$, $M > 0$, and $L > 0$ such that $\sigma_u(L; \beta, a) < M$. Examples of these functions are shown in Figure 4.1.

All the simulation results were obtained through the block-based environment Simulink of Matlab considering the Runge-Kutta integration method with a fixed-step size of 1×10^{-3}.

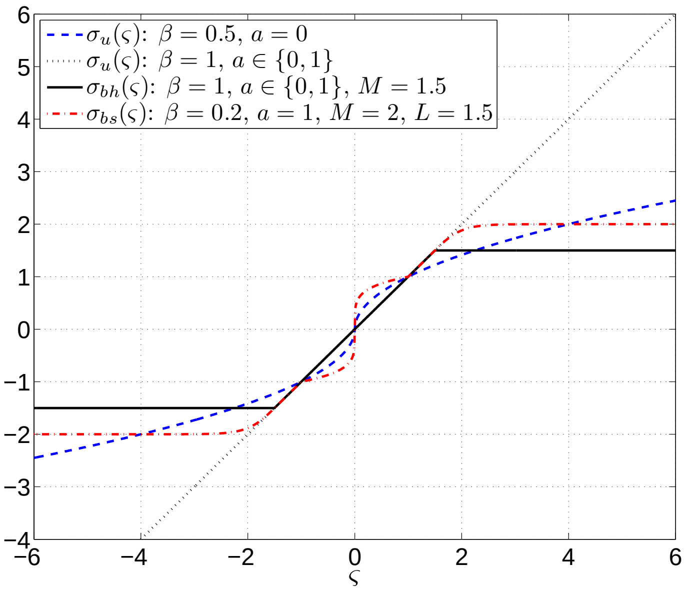

Figure 4.1: Examples of $\sigma_u(\varsigma;\beta,a)$, $\sigma_{bh}(\varsigma;\beta,a,M)$ and $\sigma_{bs}(\varsigma;\beta,a,M,L)$.

4.1 Regulation with on-line conservative force compensation

4.1.1 State-feedback control scheme

The tests of the control scheme proposed in (3.3) were run under the consideration of the model of the 2-DOF robot manipulator described in Subsection 1.5.1 and the functions defined through Eqs. (4.1), particularly —for every $j = 1, 2$— those involved in the implementations were taken as

$$\sigma_{0j}(\varsigma) = \sigma_{bs}(\varsigma; \beta_0, a_{0j}, M_{0j}, L_{0j}) \tag{4.2a}$$

$$\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, a_{ij}) \qquad i = 1, 2 \tag{4.2b}$$

Following the statement in Corollary 3.1, we fixed $\gamma = 3/2$, $\beta_1 = 1/3$, $\beta_2 = 1/2$ and $\beta_0 = 1$ for the finite-time control implementations, and $\gamma = \beta_1 = \beta_2 = \beta_0 = 1$ for the exponential stabilization tests. Observe, from the definition of σ_{0j} through (4.2a), that $B_j = M_{0j}$, $j = 1, 2$ (see (3.4) in Section 3.1). Thus, by fixing $M_{01} = 100$ and $M_{02} = 13$ [N m], the inequalities from expression (3.4) are satisfied, additionally, we fixed $L_{0j} = 0.9\, M_{0j}$, $j = 1, 2$. All the implementations of the considered control scheme were coded taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \end{pmatrix} \quad \text{[rad]}$$

and initial conditions as $q(0) = \dot{q}(0) = 0_2$. In order to compare the tests, for every

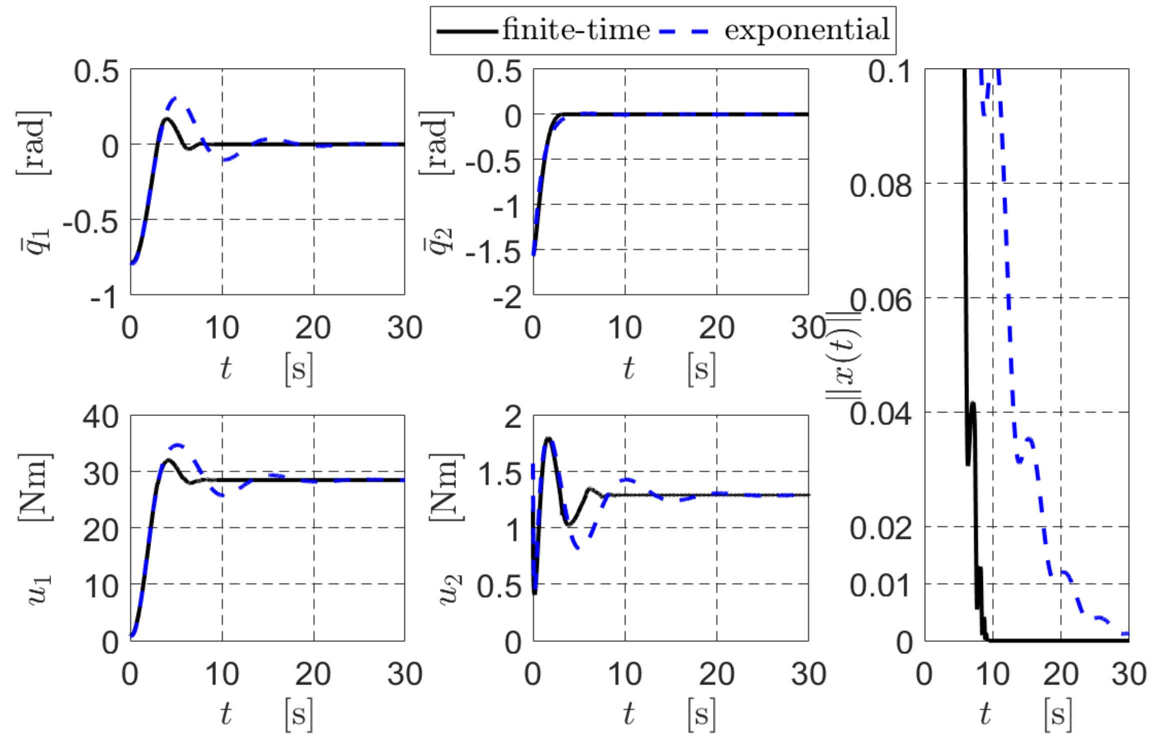

Figure 4.2: Results with $a_{ij} = 0$, $\forall i \in \{0,1,2\}$, $\forall j \in \{1,2\}$, $K_1 = K_2 = \text{diag}[1,\ 1]$ N m/rad: position errors (↑), control signals (↓), and $\|x(t)\|$ (→). Comparison between finite-time and exponential convergence, state-feedback scheme.

closed-loop response, we got (from the simulation data) the $\bar{\rho}$-stabilization time $t^s_{\bar{\rho}}$, defined as $t^s_{\bar{\rho}} \triangleq \inf\{t_s \geq 0 : \|x(t)\| \leq \bar{\rho} \quad \forall t \geq t_s\}$, where $x \triangleq (\bar{q}^T, \dot{q}^T)^T$.

Figure 4.2 shows the results obtained taking, for every $j = 1,2 :\ a_{ij} = 0$, $\forall i \in \{0,1,2\}$, $K_1 = K_2 = \text{diag}[1,\ 1]$ N m/rad. Observe that there exists a considerable difference among the concerned types of convergence; on the one hand, the responses (in the error variables) obtained through the finite-time stabilizer reach the control objective in less than 10 seconds without any change thereafter; on the other hand, the responses obtained from the exponential controller show important oscillations during the transient, for about 20 seconds. Thus, faster responses are obtained from the finite-time controller, which is clearly observed through the graph of $\|x(t)\|$ and corroborated from $\bar{\rho}$-stabilization times, determined for $\bar{\rho} = 0.01$ as $t^s_{0.01} = 8.31$ s for the finite-time control implementation and $t^s_{0.01} = 21.61$ s for the exponential stabilization test. Further simulations were obtained by adjusting the control gain values to $K_1 = K_2 = \text{diag}[10,\ 10]$ N m/rad, and keeping $a_{ij} = 0$, $\forall i \in \{0,1,2\}$, $\forall j \in \{1,2\}$. The results in Figure 4.3 show that, contrarily to the previous test, the exponential stabilizer gives rise to faster closed-loop reactions (transient responses with shorter rise times). This is confirmed through the $\bar{\rho}$-stabilization time estimations, obtained for

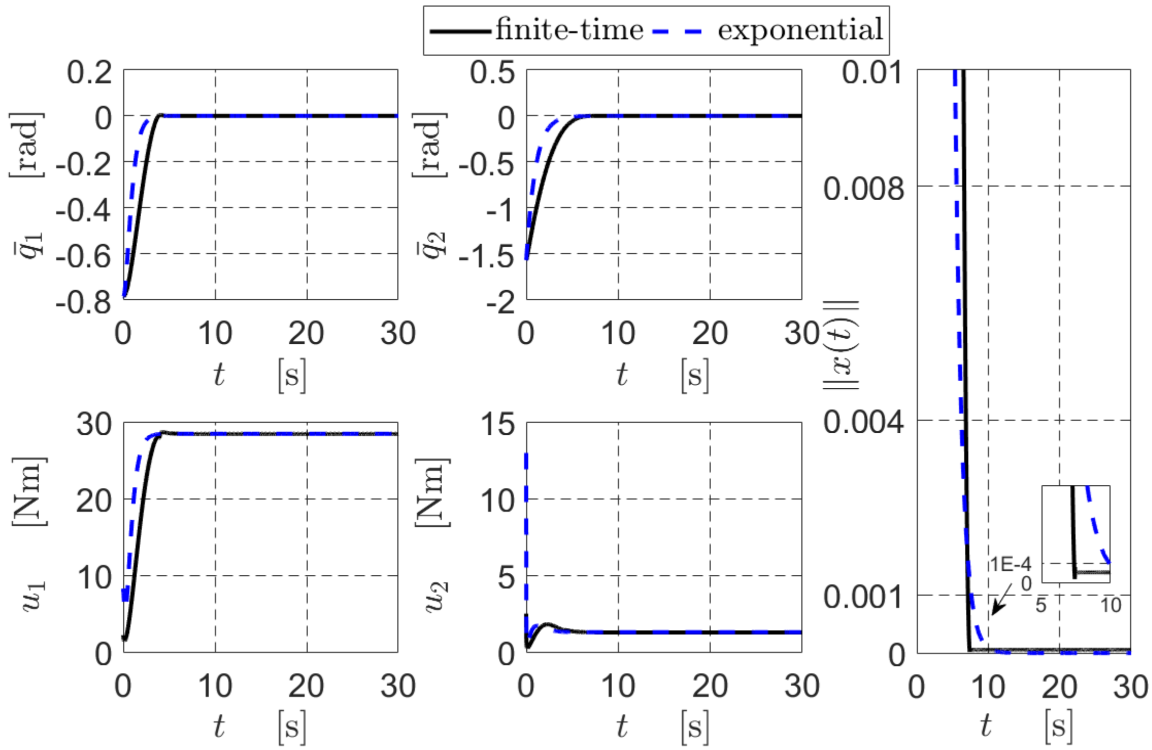

Figure 4.3: Results with $a_{ij} = 0$, $\forall\, i \in \{0, 1, 2\}$, $\forall\, j \in \{1, 2\}$, $K_1 = K_2 =$ diag[10, 10] N m/rad: position errors (↑), control signals (↓), and $\|x(t)\|$ (→). Comparison between finite-time and exponential convergence, state-feedback scheme.

$\bar{\rho} = 0.01$ as $t^s_{0.01} = 6.55$ s for the finite-time control implementation and $t^s_{0.01} = 5.35$ s for the exponential stabilization test. The differences among the results depicted in Figure 4.2 in contrast to those in Figure 4.3 arise from the characteristics of each controller. While the exponential stabilizers remain Lipschitz-continuous, the finite-time controllers loose Lipschitz-continuity at the origin. For instance, the σ_{ij}, $i, j = 1, 2$, functions of the implemented algorithm keep a unitary slope around zero in the exponential stabilization case while they adopt a vertical slope at zero for the finite-time controller. Consequently, in the finite-time control case, there is a region around zero where each one of the corresponding control force components is magnified by an additional (nonlinear) gain induced by the involved functions (see, for instance, Figure 4.1). In the case of the implemented finite-time stabilizer, by denoting ς_{ij}, i, $j = 1, 2$, the corresponding arguments of σ_{ij}, such a region is characterized as $\{|\varsigma_{ij}| \leq 1,\, i, j = 1, 2\} \triangleq D_M$. Outside this region, the involved functions have a reductive effect on their arguments in the finite-time control case, as may be corroborated, for instance, through Figure 4.1. Thus, when the closed-loop trajectories are such that the arguments of the involved functions, ς_{ij}, remain most of the time within the referred region, D_M, the corresponding (P and D type) control force components act with higher intensity in the finite-time case, forcing the resulting trajectories to attain any neighborhood of the origin faster. On the contrary, when the closed-loop trajectories spend most of the transient time outside such a region, D_M, slower (transient) reactions take place through the finite-time controller. Figures 4.4 and 4.5 show the variation of the arguments of the functions σ_{ij}, $i, j = 1, 2$,

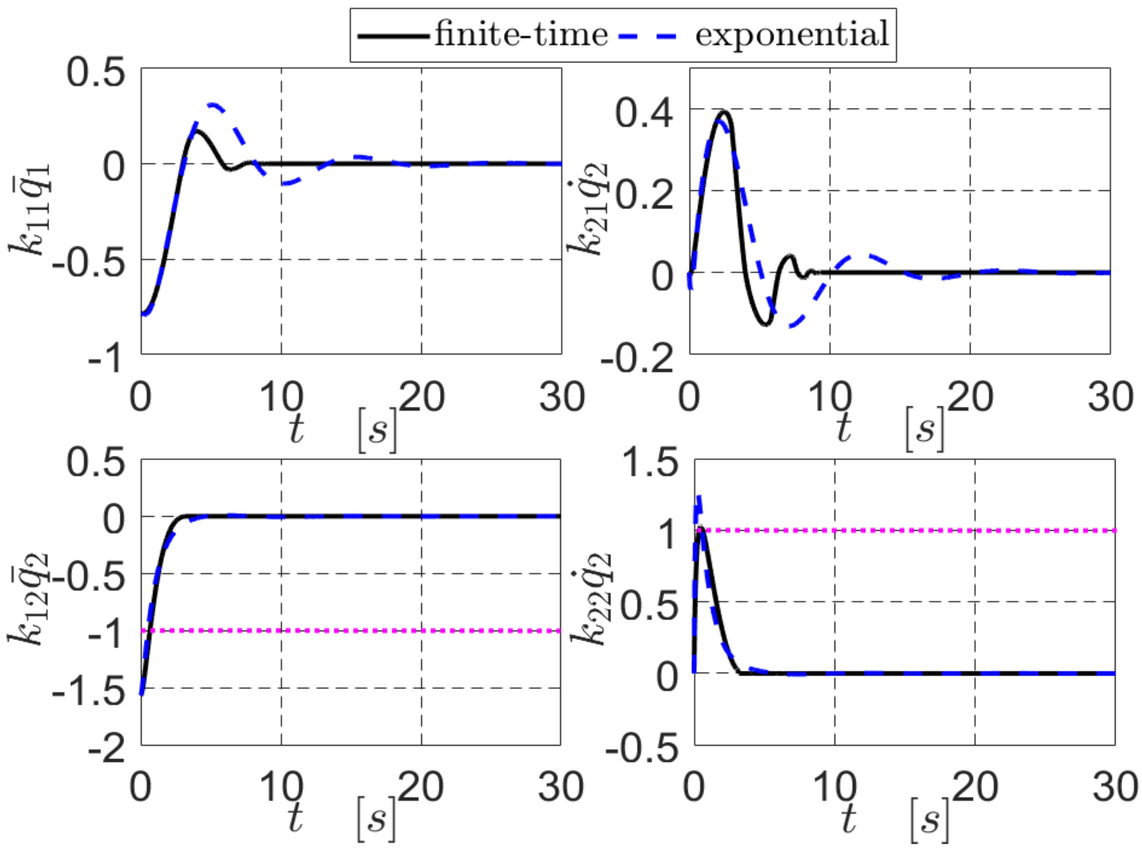

Figure 4.4: Variation of the arguments of σ_{ij}, $i, j = 1, 2$ within the region D_M, obtained from the implementations with $K_1 = K_2 = \text{diag}[1,\ 1]$ N m/rad. This explains the quicker convergence of the finite-time controller.

obtained from both implementations. One sees that with unitary gains (Figure 4.4), such arguments remained most of the time within the referred region, D_M, explaining the quicker convergence of the trajectories obtained with the finite-time controller (error-variable responses of Figure 4.2). On the contrary, with the higher gains (Figure 4.5), the referred arguments remained most of the transient time outside D_M, which explains the faster (transient) reaction of the trajectories obtained with the exponential controller. This explains the contrasting differences among the (error-variable) responses shown in Figure 4.2 and those shown in 4.3.

Nevertheless, further $\bar{\rho}$-stabilization times obtained in this latter case for $\bar{\rho} = 0.001$ (resp. $\bar{\rho} = 0.0001$) gave $t^s_{0.001} = 7.17\,\text{s}$ (resp. $t^s_{0.001} = 7.37\,\text{s}$) for the finite-time controller and $t^s_{0.001} = 7.63\,\text{s}$ (resp. $t^s_{0.001} = 9.91\,\text{s}$) for the exponential stabilizer, which seems to show that closed-loop trajectories finish up by converging quicker to zero as a result

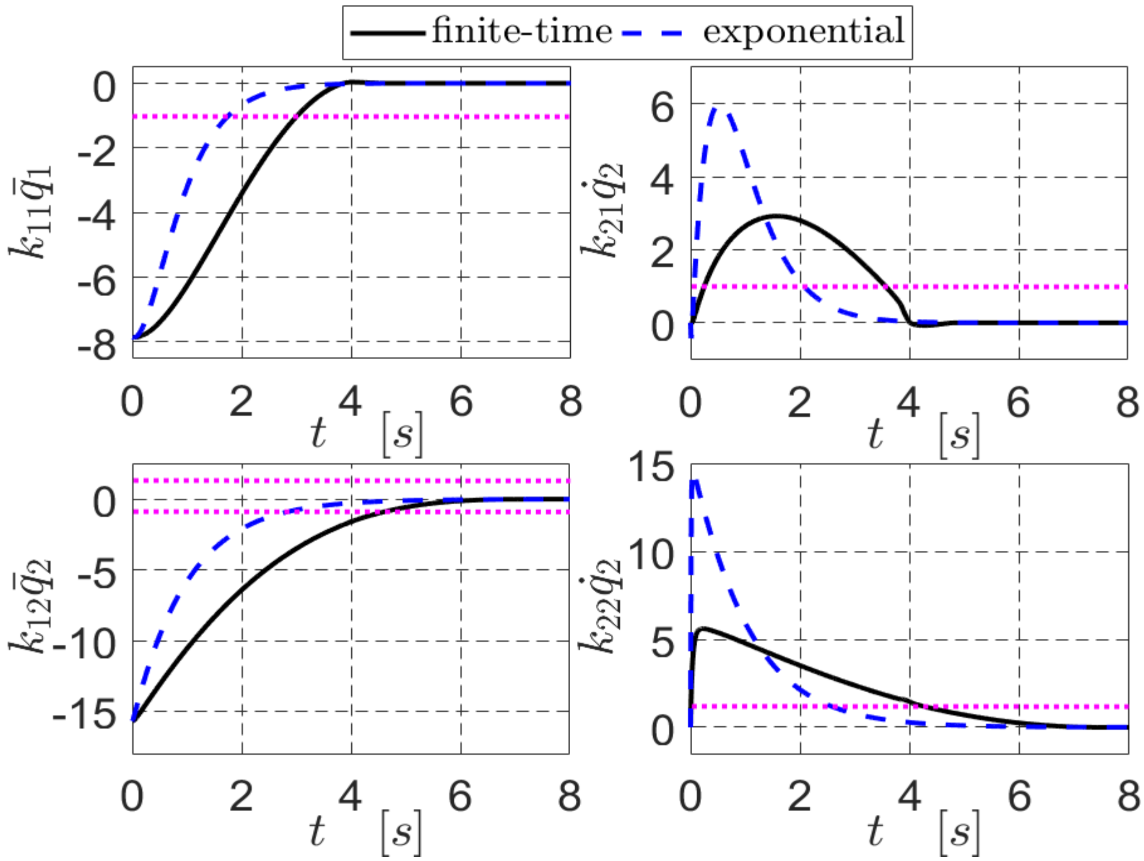

Figure 4.5: Variation of the arguments of σ_{ij}, $i, j = 1, 2$, about region D_M, obtained from the implementations with $K_1 = K_2 = \text{diag}[10,\ 10]$ N m/rad. This explains the quicker convergence of the exponential controller.

of finite-time control, (see the graph of $\|x(t)\|$ in Figure 4.3). This results from the finite-time convergence, which forces the trajectories to exactly reach the equilibrium at the settling (finite) time, contrarily to the asymptotic infinite-time attraction, which implies (divergently) longer time intervals to get to smaller neighborhoods of the origin. However, from a practical viewpoint, $\bar{\rho}$-stabilization times estimated for $\bar{\rho} = 0.01$ could be enough to determine that closed-loop trajectories practically reached the equilibrium, giving rise to the possibility to have closed-loop implementations where exponential stabilizers be considered to (practically) achieve stabilization faster than finite-time controllers.

Further simulations with alternative selections on the involved functions (the P- and D-type actions) were performed. For instance, by keeping the control gain combinations already tested ($K_1 = K_2 = \text{diag}[10,\ 10]$), but this time taking $a_{ij} = 1, \forall\, i \in \{0, 1, 2\}, \forall\, j \in \{1, 2\}$, the results shown in Figure 4.6 are obtained. Observe that, the trajectories obtained with the finite-time controller are close to those obtained with the exponential stabilizer (which remained identical). This is not surprising since, the arguments of the

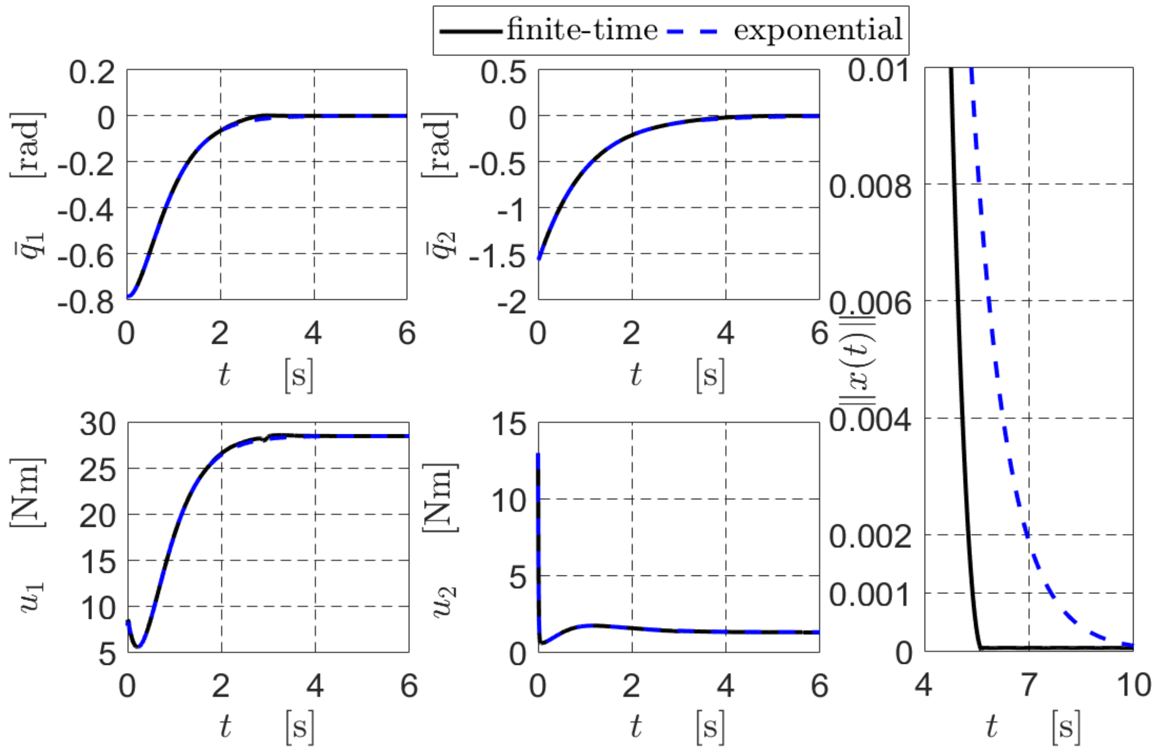

Figure 4.6: Results with $a_{ij} = 1$, $\forall\, i \in \{0, 1, 2\}$, $\forall\, j \in \{1, 2\}$, $K_1 = K_2 = 10$ N m/rad: position errors (↑),control signals (↓), and $\|x(t)\|$ (→). Comparison between finite-time and exponential convergence, state-feedback scheme.

involved functions remained, most of the transient time, outside D_M (where the P and D action-related functions keep the same form for both stabilizers). The $\bar{\rho}$-stabilization time for $\bar{\rho} = 0.01$ gave $t^s_{0.01} = 4.77$ s for the finite-time controller —against $t^s_{0.01} = 5.35$ s for the exponential stabilizer— showing that, contrarily to the corresponding precedent case, this time, the finite-time controller may indeed be concluded to achieve faster convergence through practical criteria (and not just in view of its finite-time nature). It is worth emphasizing that such a way to ensure faster stabilization —as well as input saturation avoidance— through finite-time control was achieved, thanks to the design flexibility permitted within the framework of local homogeneity, which allows to involve functions that are not forced to keep the homogeneity property globally but may rather adopt suitable changes.

4.1.2 Output-feedback control scheme

The tests in this subsection involve the proposed control scheme in (3.13)-(3.14). For implementation purposes the following functions are considered (based on Eqs. (4.1)), for every $j = 1, 2$,

$$\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij}) \qquad i = 1, 2 \tag{4.3a}$$

$$\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_{3j}, a_{3j}) \tag{4.3b}$$

Let us note that through these definitions we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (3.15)). Thus, by fixing $M_{11} = M_{21} = 50$ and $M_{12} = M_{22} = 6.4$, the inequalities from expression (3.15) are satisfied. The implementations were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \end{pmatrix} \quad \text{[rad]}$$

and initial conditions as $q(0) = \dot{q}(0) = \vartheta_c(0) = 0_2$.

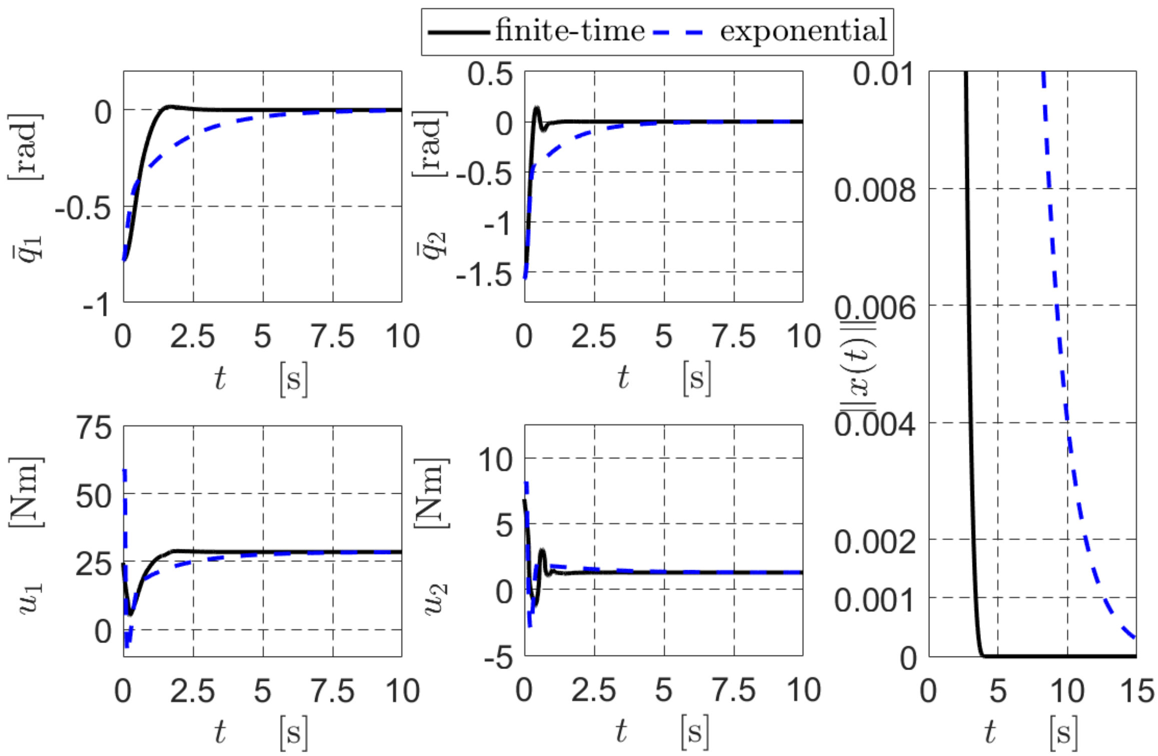

Figure 4.7: Finite-time *vs* exponential stabilization, output-feedback scheme: position errors (↑), control signals (↓), and $\|x(t)\|$ (→).

Following the design procedure in accordance to Corollary 3.2, we present a test where the aim is to corroborate the convergence difference among the closed-loop trajectories obtained with the proposed finite-time controller, taking $\beta_1 = \beta_2 = 1/2$ and $\beta_3 = 3/4$, and the analog exponential stabilizer, *i.e.*, with $\beta_1 = \beta_2 = \beta_3 = 1$. For this test we took $a_{ij} = 0,\ i = 1, 2, 3,\ j = 1, 2$. As a performance index recall the ρ-stabilization time, as defined in the previous subsection, but this time with $x \triangleq (\bar{q}^T\ \dot{q}^T\ \vartheta^T)^T$.

Figure 4.7 shows the results obtained taking $K_1 = K_2 = \mathrm{diag}[10, 1]$, $A = \mathrm{diag}[30, 15]$ and $B = \mathrm{diag}[60, 20]$. One sees that the stabilization objective was achieved by both controllers avoiding input saturation. Moreover, the contrast among the different types of trajectory convergence in accordance to the corresponding controller nature, is clear from the graphs. In particular, one sees that, with the finite-time controller, the position errors, and actually the (norm of the) whole state vector in the extended state space, converge to zero in less than 5 s, remaining invariant thereafter. The exponential controller, instead, generated asymptotically convergent closed-loop trajectories with longer stabilization time. In terms of the $\bar{\rho}$-stabilization time for $\bar{\rho} = 0.01$, we obtained $t^s_{0.01} = 8.27$ s for the exponential controller *vs* $t^s_{0.01} = 2.7$ s for the finite-time stabilizer. Let us note that, in view of the different types of trajectory convergence, whatever the control parameter tuning be, there will always be a sufficiently small value ρ^* such that $t^s_{\bar{\rho}}$ is smaller in the finite-time controller case for all $\bar{\rho} < \rho^*$. The control gain tuning was fixed so as to render such a convergence difference visibly clear from the graphs.

4.2 Regulation with desired conservative force compensation

4.2.1 State-feedback control scheme

The proposed control scheme in (3.22) was implemented taking into account the 2-DOF robot manipulator described in Section 1.5.1 and the functions defined through Eqs. (4.1). Particularly —for every $j = 1, 2$— the involved functions were taken as

$$\sigma_{0j}(\varsigma) = \varsigma \tag{4.4a}$$

$$\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij}) \qquad i = 1, 2 \tag{4.4b}$$

with $a_{ij} = 0$, $i = 1, 2$, $j = 1, 2$. Conditions on their parameters under which (3.24) is fulfilled are

$$k_{1j} > k_g (2B_{gj})^{(1-\beta_1)/\beta_1} \tag{4.5a}$$

$$M_{1j} > 2B_{gj} \tag{4.5b}$$

(this is shown in Appendix B). The test was run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \end{pmatrix} \quad [\mathrm{rad}]$$

and initial conditions as $q(0) = \dot{q}(0) = 0_2$. Note, from the definition of σ_{0j} and Corollary 3.3, that $\beta_{0j} = 1$ and, consequently, $\beta_{2j} = 2\beta_{1j}/(1 + \beta_{1j})$, $j = 1, 2$. Thus, we fixed $\beta_1 = 3/5$ and $\beta_2 = 3/4$ for the finite-time control implementation, and $\beta_1 = \beta_2 = 1$

for the exponential stabilization test. Note, from the definition of σ_{0j} and σ_{ij} through Eqs. (4.4), that $B_j = M_{1j} + M_{2j}$, $j = 1, 2$. Thus, by fixing $M_{11} = 82$, $M_{12} = 18$, and $M_{21} = M_{22} = 6$, the inequalities from expression (3.23) are satisfied.

Figure 4.8 shows the results obtained taking, for both controllers, $K_1 = \text{diag}[754,\ 96]$

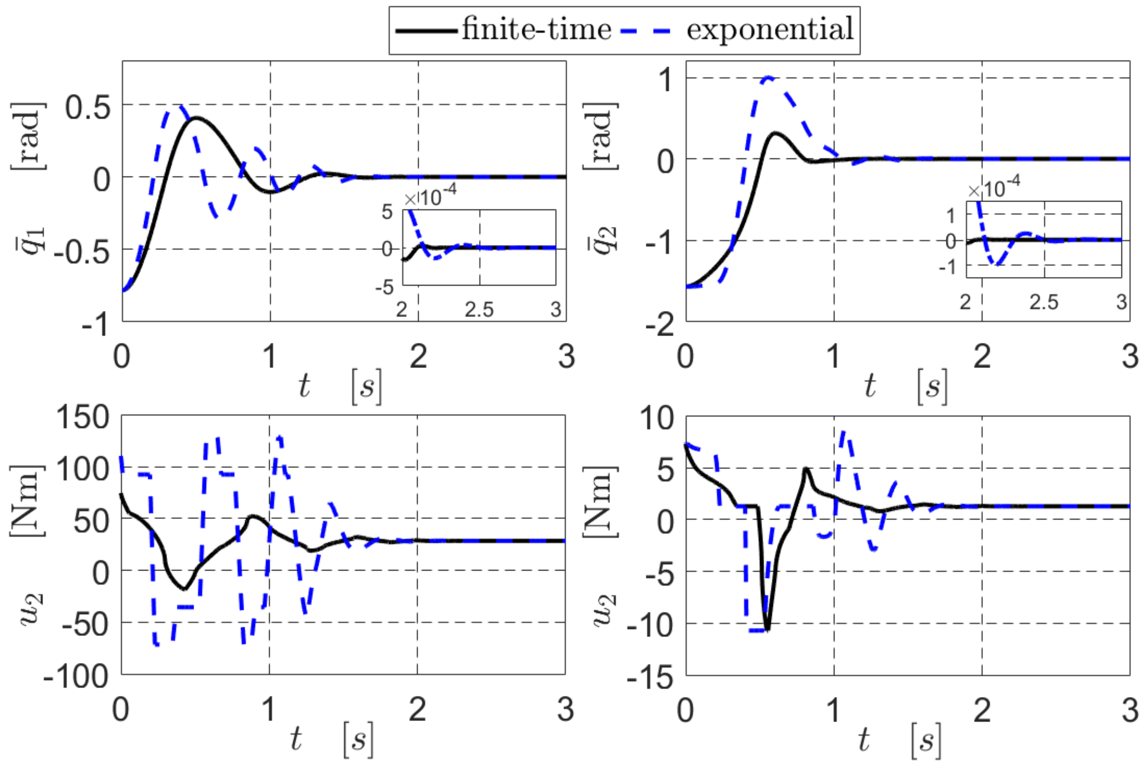

Figure 4.8: Finite-time *vs* exponential stabilizer, state-feedback control scheme with desired conservative-force compensation: position errors (↑), control signals (↓).

and $K_2 = \text{diag}[35,\ 3]$. One sees that the proposed scheme achieves both types of convergence avoiding input saturation, with the closed-loop trajectory arising through the exponential controller presenting a longer and more important transient. On the other hand, the finite-time stabilizer shows a more efficient ability to counteract the inertial effects through control signals with considerably less and lower variations during the transient.

4.2.2 Output-feedback control scheme

The application of the control scheme stated in (3.44)–(3.45) involved the Phantom haptic device shown in Subsection 1.5.2 and the functions, for every $j = 1, 2, 3$, were defined as $\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij})$, $i = 1, 2$, and $\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_{3j}, a_{3j})$, with $a_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2, 3$. In this case, the conditions on their parameters under which (3.47) is fulfilled are the same as those shown in (4.5).

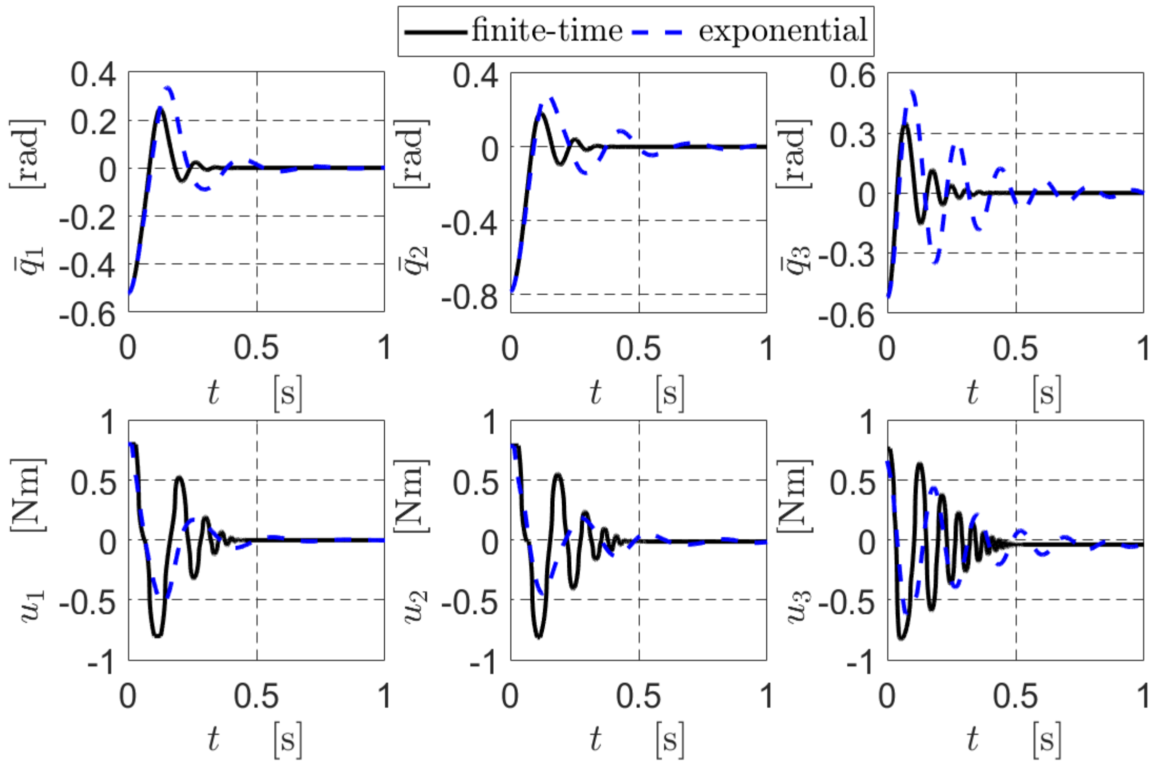

Figure 4.9: Finite-time *vs* exponential stabilization, output-feedback control scheme with desired conservative-force compensation: positions errors (↑) and control signals (↓)

Observe, from the involved functions, that $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$. Thus, by fixing $M_{ij} = 0.4$, $i = 1, 2$, $j = 1, 2, 3$, the inequalities from expressions (3.46) and (4.5b) have been simultaneously satisfied. The rest of the control gain/parameter values were chosen taking care that the design requirements were always satisfied. All the implementations were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{4} \\ \frac{\pi}{6} \end{pmatrix} \quad \text{[rad]}$$

and initial conditions: $q(0) = \dot{q}(0) = (0 \quad 0 \quad 0)^T$.

The test aim is to focus on the performance of the finite-time stabilization in contrast to analog exponential regulation implementations. Figure 4.9 shows results obtained taking $\beta_1 = \beta_2 = 1/2$ and $\beta_3 = 3/4$, for the finite-time controller, while $\beta_1 = \beta_2 = \beta_3 = 1$ for the exponential stabilizer, and the remaining control gain/parameters were taken, for both (finite-time and exponential) controllers, as: $K_1 = \text{diag}[1,\ 1,\ 1]$ (satisfying (4.5a)), $K_2 = \text{diag}[0.3,\ 0.2,\ 0.1]$, $A = \text{diag}[40,\ 40,\ 40]$, and $B = \text{diag}[5,\ 5,\ 5]$. One sees that both tested controllers achieved the regulation objective avoiding input saturation and with the corresponding types of trajectory convergence. In particular, while the finite-time convergence trajectories attain the desired position fast enough and remain thereafter, the exponentially convergent responses keep on oscillating entailing a longer practical stabilization.

4.2.3 Desired *versus* on-line conservative-force compensation

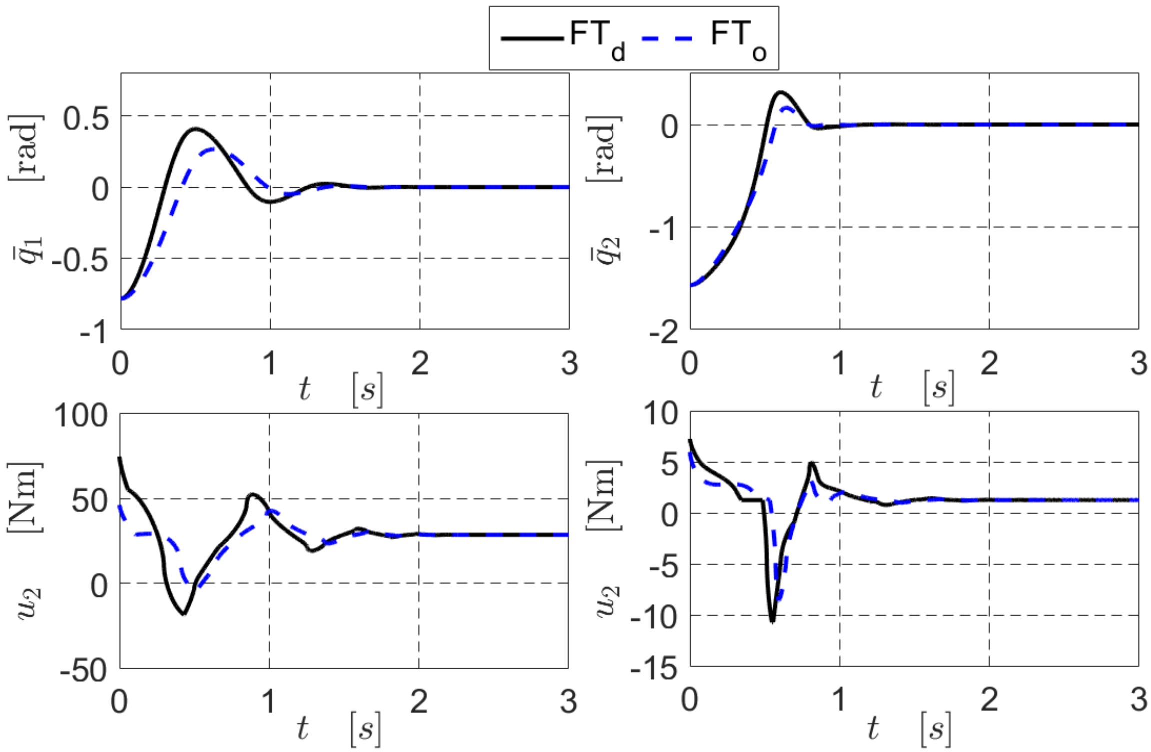

Figure 4.10: State-feedback controller tests, desired ($\mathrm{FT_d}$) *vs* online ($\mathrm{FT_o}$) conservative-force compensation: position errors (↑), control signals (↓).

The following tests are focused on the comparison among finite-time control implementations involving the on-line and desired conservative-force compensation versions of the control schemes proposed in Sections 3.1 and 3.2, respectively. The closed-loop responses were compared taking the same control gain/parameter values and saturating structures but differ only on the type of conservative-force compensation. In this direction, the finite-time control tests shown in the state-feedback case of Subsection 4.2.1 and those shown in the output-feedback case of Subsection 4.2.2 are repeated here, by just alternating the referred compensation term. First, we present the state-feedback control implentations with the 2-DOF robot manipulator considered in Subsection 4.2.1 and the parameter/gain values fixed in such subsection, this in order to compare the desired conservative force compensation tests already obtained in Subsection 4.2.1 with those that will be obtained by changing the compensation term for the on-line conservative force. Figure 4.10 shows the comparison among the tested state-feedback controllers with $\mathrm{FT_d}$ and $\mathrm{FT_o}$ denoting the finite-time controllers with the desired and online compensation term, respectively. Observe that there are not meaningful differences among the closed-loop responses during the transient, keeping a similar behavior. Further, analog tests were run for the output-feedback control schemes. In this case, the 3-DOF haptic device and the parameter/gain values used in Subsection 4.2.2 are considered. Figure 4.11 shows the results by comparing the ($\mathrm{FT_d}$ versus $\mathrm{FT_o}$) responses obtained

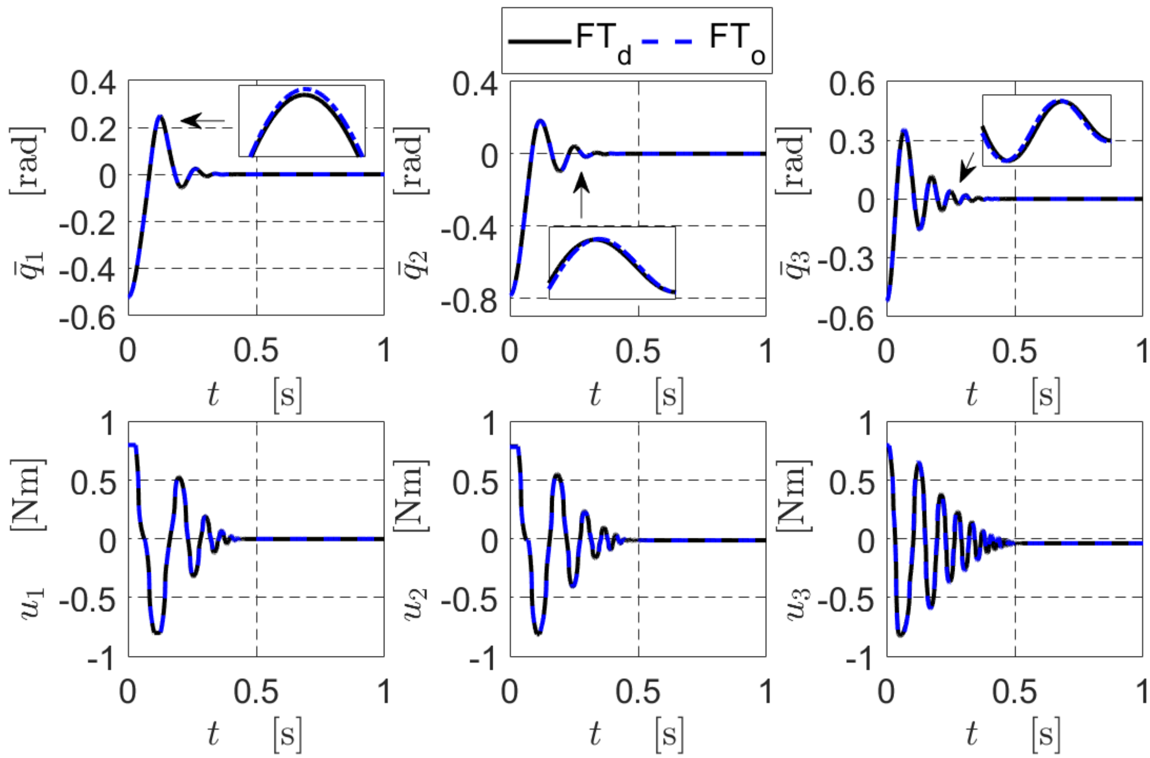

Figure 4.11: Output-feedback controller tests, desired ($\mathrm{FT_d}$) *vs* online ($\mathrm{FT_o}$) conservative-force compensation: position errors (↑), control signals (↓).

from the output-feedback controllers. Observe that there is a negligible difference among the closed-loop performances.

From the previous tests and analog results obtained from other implementationts (not shown here), we observe that the effect on the performance for the implementation simplification earned by the desired compensation version of the controllers is negligible, in spite of the open-loop conservative-force term that is left acting on the system.

4.3 Tracking problem implementations

The proposed scheme (3.70) in Section 3.3 was implemented through the functions defined in accordance to Eq. (4.1b) with $a_{ij} = 0$, *i.e.*, $\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)\min\{|\varsigma|^{\beta_i}, M_{ij}\}$, $i, j = 1, 2$, for constants $\beta_i > 0$ and $M_{ij} > 0$. For each $i = 1, 2$, such functions prove to be bounded strongly passive functions for $(\kappa_i, \beta_i, b_i, \bar{\kappa}_i, \beta_i, b_i)$, where $b_i = \min\{b_{i1}, b_{i2}\}$, $\kappa_i \leq 1$ and $\bar{\kappa}_i \geq b_i^{-a_i} \max\{M_{i1}, M_{i2}\}$, with —for every $j = 1, 2$— $b_{ij} = M_{ij}^{1/\beta_i}$. Following the proposed design procedure, we fixed $\beta_1 = 1/2$ and $\beta_2 = 2/3$ for the finite-time control implementation, and $\beta_1 = \beta_2 = 1$ for the exponential tracking test. Let us note that by the defined functions σ_{ij}, we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (3.72)). On the other hand, the simulations were run taking initial conditions at $q(0) = \dot{q}(0) = 0_2$, and the desired trajectory as

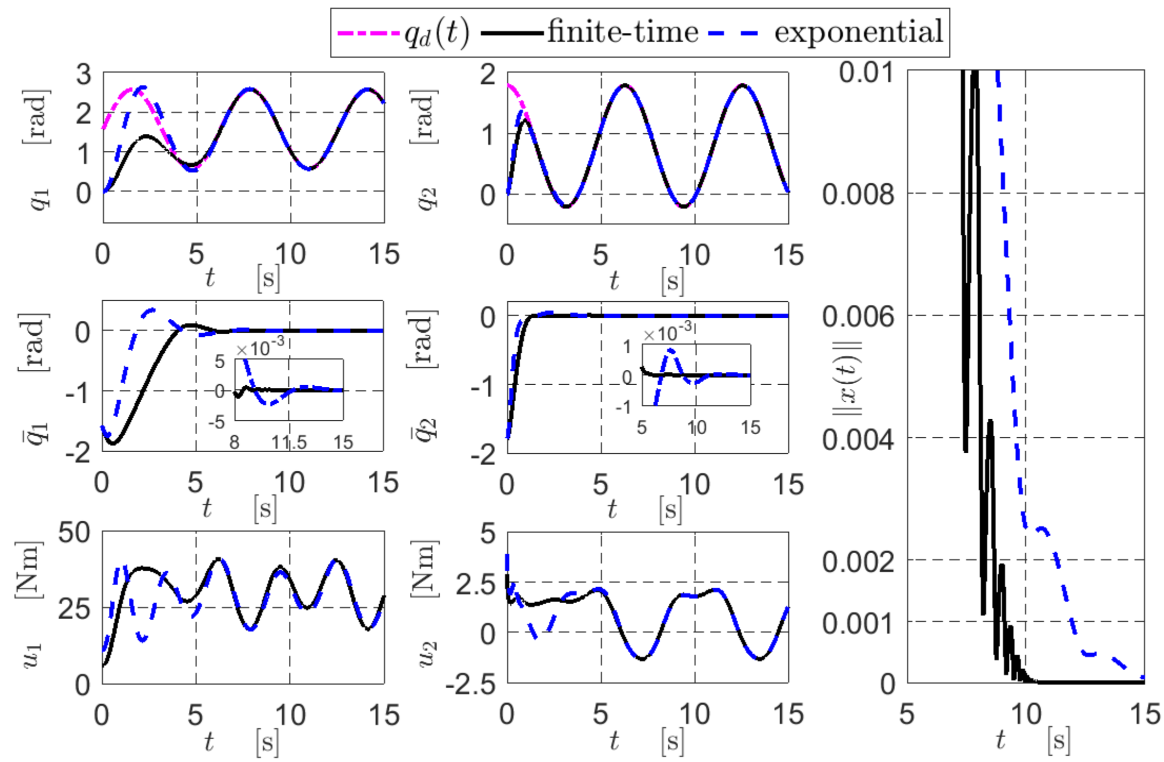

Figure 4.12: Results from the tracking implementation, finite-time *vs* exponential convergence: position responses (top), position errors (middle), control signals (bottom), and $\|x(t)\|$ (right).

$$q_d(t) = \begin{pmatrix} \frac{\pi}{2} + \sin(t) \\ \frac{\pi}{4} + \cos(t) \end{pmatrix}$$

for which $B_{dv} = 1$ and $B_{da} = 1$. From this and the values of the parameters characterizing Property 1.1, Assumptions 1.1–1.3 and 3.2 for the considered robot manipulator, one can corroborate that Assumption 3.3 is satisfied too. Moreover, from the considered desired trajectory, one sees that (3.72) is satisfied provided that $M_{11} + M_{21} < 104.76$ and $M_{12} + M_{22} < 12.7$. Hence, for the simulation, we took $M_{11} = M_{21} = 50$ and $M_{12} = M_{22} = 6$. We also took the ρ-stabilization time as a performance index, but now by taking $x \triangleq (\bar{q}^T \ \dot{\bar{q}}^T)^T$.

Figure 4.12 shows results obtained taking control gains $K_1 = \text{diag}[5, 5]$ and $K_2 = \text{diag}[1, 1]$ for both implemented controllers. The tracking objective is observed to be achieved avoiding input saturation through the proposed scheme for both the finite-time and exponential versions. Particularly, one sees that, with the finite-time controller, the position errors, and actually the (norm of the) whole state vector, converge to zero in almost 10 s, remaining invariant thereafter. The exponential controller, instead, generated asymptotically convergent closed-loop trajectories with longer stabilization time. In terms of the $\bar{\rho}$-stabilization time for $\bar{\rho} = 0.01$, we obtained $t^s_{0.01} = 6.62$ s for the finite-time controller *vs* $t^s_{0.01} = 7.64$ s for the exponential stabilizer. Further, for $\bar{\rho} = 0.001$, we obtained $t^s_{0.01} = 8.32$ s for the finite-time controller *vs* $t^s_{0.01} = 11.60$ s for the exponential stabilizer, which is clearly observed through the graph of $\|x(t)\|$.

4.4 Robustness problem implementations

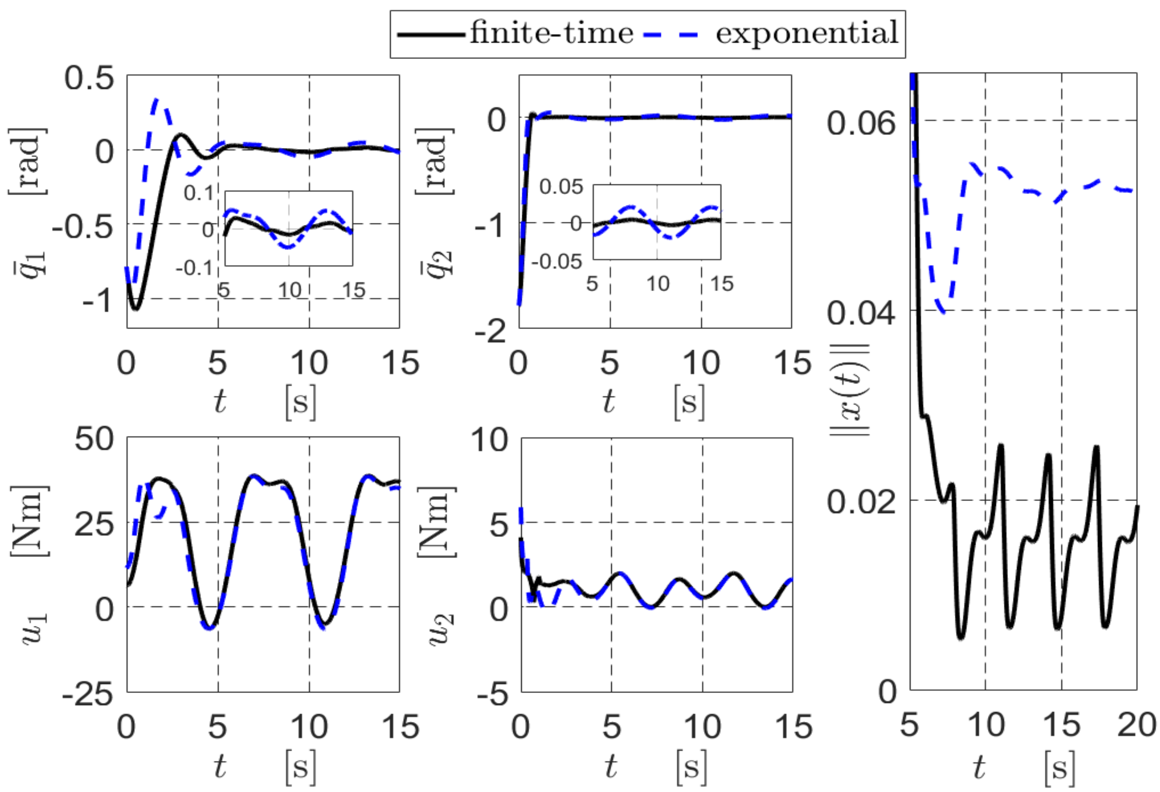

Figure 4.13: Test of the tracking control scheme under perturbation, with $K_1 = \text{diag}[10, 10]$ and $K_2 = \text{diag}[1, 1]$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

Following what is exposed in Section 3.4, the control scheme defined in (3.70) is implemented in presence of perturbation. All the simulations were run considering the 2-DOF robot manipulator and the $\sigma_{ij}(\cdot)$ —for every $i = 1, 2,\ \ j = 1, 2$—

functions used in the previous section. Particularly, the functions were defined as $\sigma_{ij}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|^{\beta_i}, M_{ij}\}$, and initial conditions at $q(0) = \dot{q}(0) = 0_2$, including a perturbation of the form

$$\varrho(t) = \begin{pmatrix} 0.4\cos(t) \\ 0.2\sin(t) \end{pmatrix} \tag{4.6}$$

that satisfies Assumptions 3.4 and 3.5 (with $\bar{\varrho}_1 = 0.4$ and $\bar{\varrho}_2 = 0.2$), and the desired

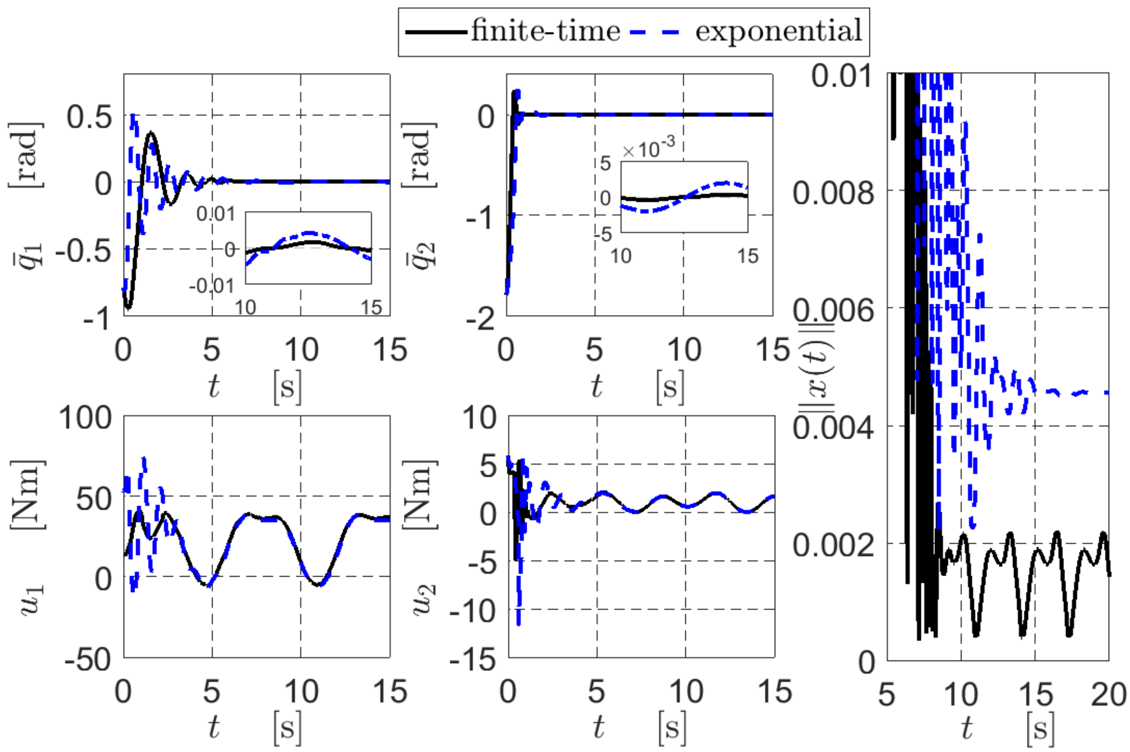

Figure 4.14: Test of the tracking control scheme under perturbation, with K_1 = diag[100, 100] and K_2 = diag[1, 1], finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

trajectory as

$$q_d(t) = \begin{pmatrix} \frac{\pi}{4} + \sin(t) \\ \frac{\pi}{4} + \cos(t) \end{pmatrix}$$

(for which $B_{dv} = 1$ and $B_{da} = 1$), with which Assumption 3.3 is fulfilled too. The controller was implemented taking $\beta_1 = 1/2$ and $\beta_2 = 2/3$ for the finite-time controller, while $\beta_1 = \beta_2 = 1$ for the exponential control algorithm, and $M_{i1} = 50$, $M_{i2} = 6$, $i = 1, 2$, in accordance to the inequality in (3.101) for both (finite-time and exponential) controllers. The simulations are presented through two tests with different control gain values. Figure 4.13 shows the results obtained with $K_1 = \text{diag}[10,\ 10]$ and $K_2 = \text{diag}[1,\ 1]$, where, in both cases (finite-time and exponential controllers), the control signals are observed to avoid input saturation. But in view of the input-matching perturbation term, post-transient variations are noticed to take place in the position-error closed-loop responses. Observe that, smaller post-transient variations are observed to arise in the finite-time controller case [which is more visible through the graph of $\|x(t)\|$: variations of $\|x(t)\|$ take place within a smaller range of values in the finite-time controller case]. Similar observations arose through the test showed in Figure 4.14 where the results were obtained with $K_1 = \text{diag}[100,\ 100]$ and $K_2 = \text{diag}[1,\ 1]$. The control signals obtained from both controllers succeed on the input avoidance. Although, post-transient variations are reduced in both controller responses, those gotten from the finite-time controller are again observed to be smaller than the post-transient variations obtained through the exponential case.

Further discussion

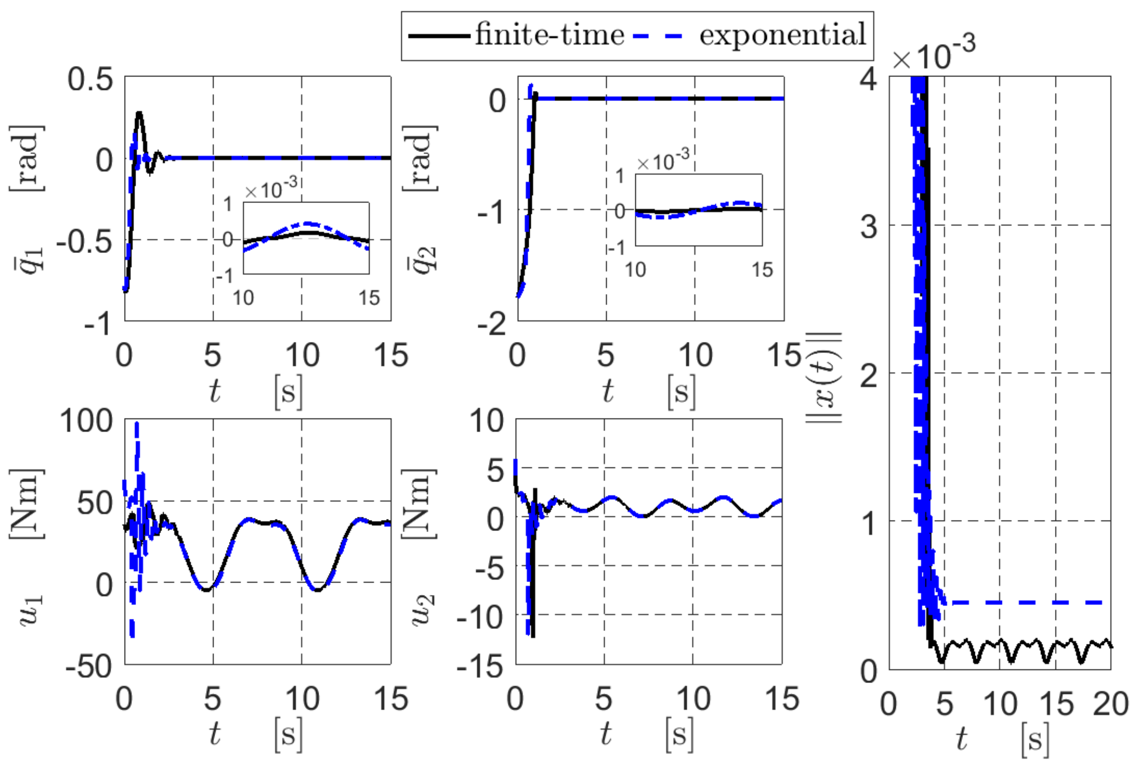

Figure 4.15: Simulation of the tracking control scheme under perturbation, with $K_1 = \text{diag}[1000,\ 1000]$ and $K_2 = \text{diag}[10,\ 10]$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

The results presented above show evidence on the ultimate boundedness of the closed-loop trajectories and the smaller post-transient variation obtained in the finite-time controller case (in contrast to the exponential control algorithm case) when the perturbation bound is sufficiently small under constant (unchanged) control parameters. Could such a panorama differ under an arbitrary change on the control gain values? For instance, could a change on the control gain values inverse the smaller post-transient variation relation among the finite-time and exponential controllers? This could hardly be analytically investigated through the expressions obtained in the proof of Proposition 3.10 since they involve quantities whose exact dependence on the control gains is unknown, and (more precisely) an exact expression of the *critical value* $\bar{\varrho}^*$ —see Remark 3.20— of the perturbation bound $\bar{\varrho}$ is unavailable. Thus, we have explored this point through further simulation tests, that were run reproducing the above considered case, but taking different control gain values reaching very high orders. The observed results are shown in Figures 4.15–4.17. Figure 4.15 presents the results obtained with $K_1 = \text{diag}[1000,\ 1000]$ and $K_2 = \text{diag}[10,\ 10]$, Figure 4.16 shows the results obtained with $K_1 = 10^4\, I_2$ and

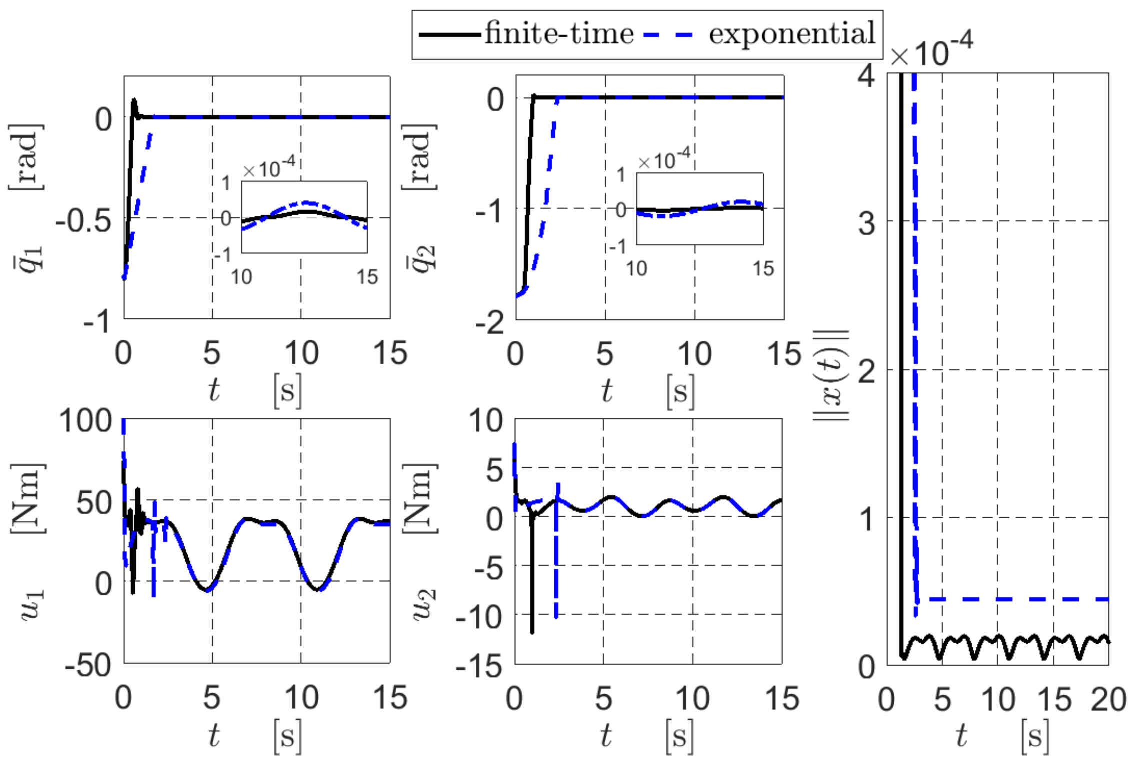

Figure 4.16: Simulation of the tracking control scheme under perturbation, with $K_1 = 10^4\, I_2$ and $K_2 = 10^2\, I_2$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

$K_2 = 10^2\, I_2$, and Figure 4.17 displays those obtained with $K_1 = 10^5\, I_2$ and $K_2 = 10^3\, I_2$. Observe that, all the cases present the same results: post-transient variations were smaller in the finite-time control case. We conclude from these tests that (arbitrary) changes in the control gains must entail changes in the critical value $\bar{\varrho}^*$ within a restricted range, so that for sufficiently small values of $\bar{\varrho}$, the result and conclusions obtained in

Section 3.4 hold, with (certain degree of) immunity to changes on the control gains.

It is worth adding that further tests —in the same simulation context considered in this subsection— were performed with a perturbation term having a considerably higher bound $\bar{\varrho}$, where, by taking small control gain values, the responses obtained through the finite-time controller presented smaller post-transient variations than those obtained from the exponential stabilizer, similarly to the results so far obtained. However, with higher control gain values such a relation among the responses was inverted. This was corroborated, for instance, by replacing the previous perturbation term in (4.6), by

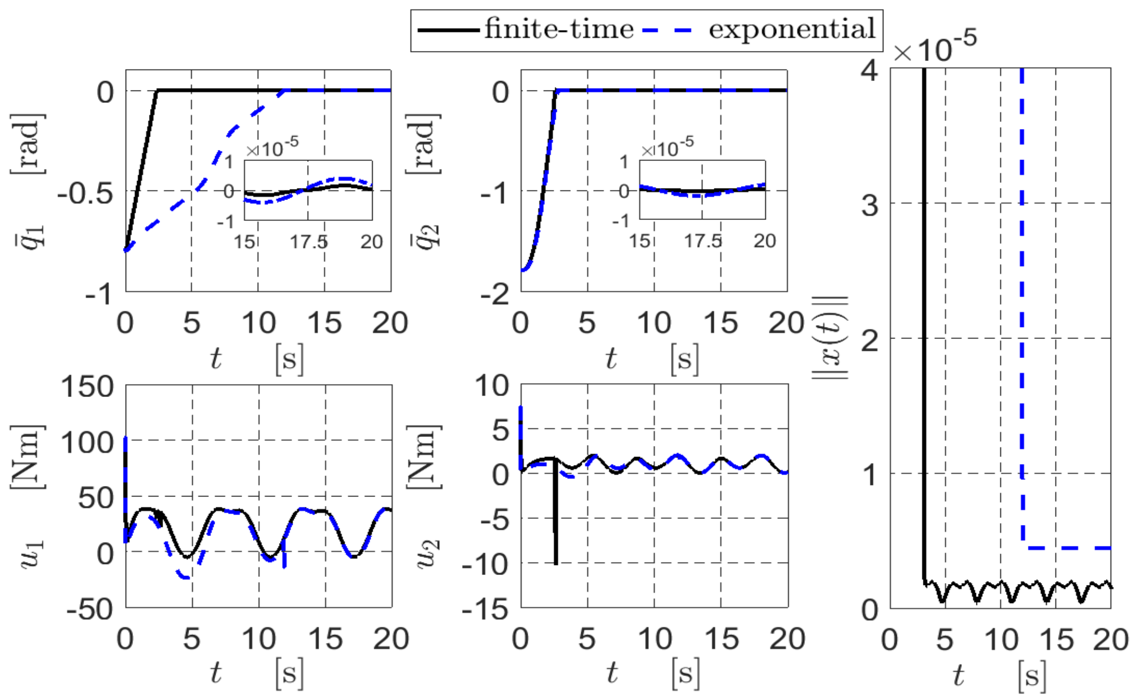

Figure 4.17: Simulation of the tracking control scheme under perturbation, with $K_1 = 10^5\, I_2$ and $K_2 = 10^3\, I_2$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

$$\varrho(t) = \begin{pmatrix} 1.4\cos(t) \\ \sin(t) \end{pmatrix}$$

In particular Figures 4.18 and 4.19 show the results taking $K_1 = K_2 = \text{diag}[1, 1]$ and $K_1 = \text{diag}[10, 10]$, $K_2 = \text{diag}[1, 1]$, respectively, where what we just described is corroborated. We thus conclude that with high enough perturbation bounds, happening to be close to their critical value $\bar{\varrho}^*$, the previously referred immunity to changes on the control gains could (or would in general) be lost.

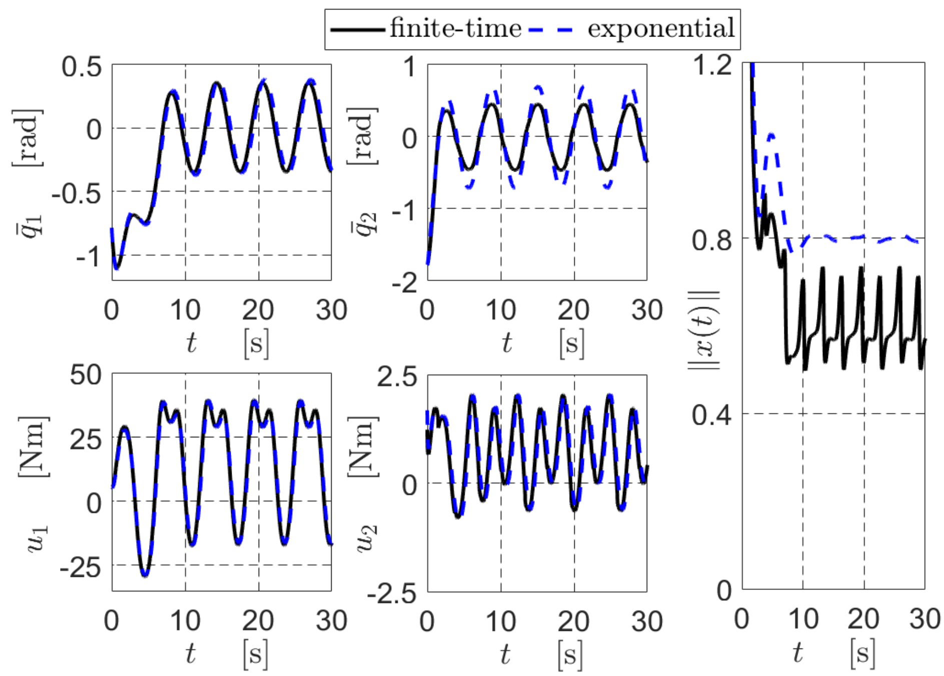

Figure 4.18: Simulation of the tracking control scheme under perturbation, with $K_1 = K_2 = I_2$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

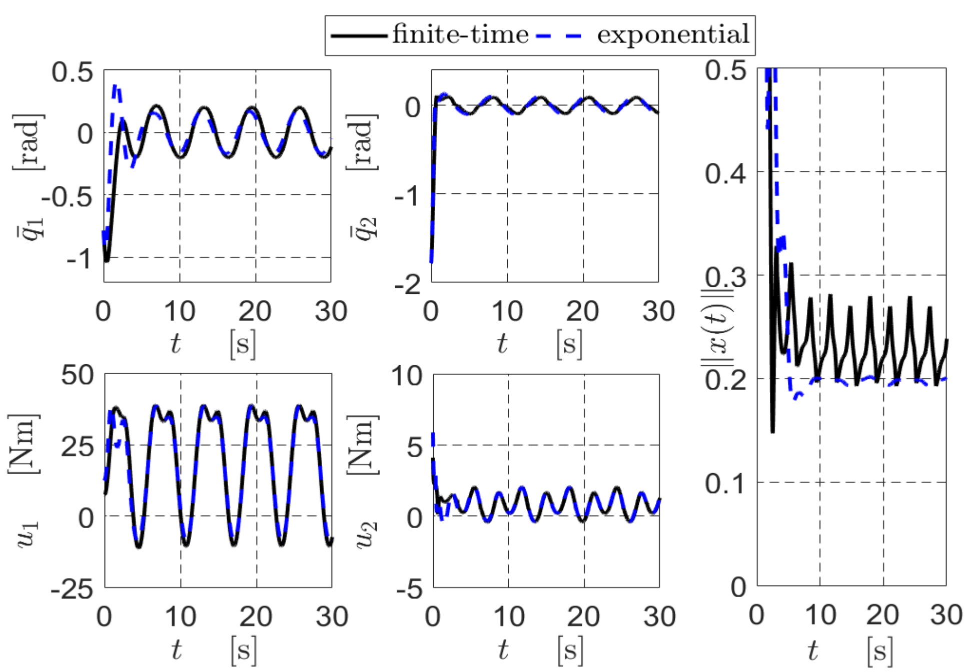

Figure 4.19: Simulation of the tracking control scheme under perturbation, with $K_1 = 10\, I_2$ and $K_2 = I_2$, finite-time *vs* exponential convergence: positions (↑), control signals (↓), and $\|x(t)\|$ (→).

CHAPTER 5

Experimental results

The control schemes presented in Chapter 3 were implemented through experimental tests, whose results are shown throughout the following sections. The mechanical systems shown in Section 1.5 are considered in order to carry out the tests. The experiments do not only show finite-time control implementations but also include a comparison among these and analog exponential stabilization tests. All the tests involve the functions defined in Chapter 4, through Eqs. (4.1). The results are shown in accordance to the sections presented in Chapter 3.

5.1 Regulation with on-line conservative force compensation

5.1.1 State-feedback control scheme

The control scheme proposed in (3.3) was applied on the 2-DOF robot manipulator described in Subsection 1.5.1, where is it indicated that joint positions are obtained from incremental encoders located on the motors, which have a resolution of 1,024,000 pulses/rev for the first motor and 655,300 for the second one (accuracy of 0.0069° for both motors), and the standard backwards difference algorithm is used to obtain the velocity signals. The setup includes a PC-host computer with an acquisition board (the *Multi-Q* I/O card from *Quanser*) to get the encoder data and generate reference voltages. The robot is programmed through *WinMechLab* [45], which is a general-purpose computer system for real-time control of mechanisms that runs on a Windows platform based on C language. The control algorithm is executed at a 2.5 ms sampling period (holding constant the control signals among the samples). This has proven to be fast enough to suitably approximate the continuous control signals generated by the implemented continuous-time scheme. Property 1.1, Assumptions 1.1–1.4 and 3.2 are thus satisfied. The experiments were run keeping the functions defined in Eqs. (4.2), *i.e.* —for every $j = 1, 2$—, $\sigma_{0j}(\varsigma) = \sigma_{bs}(\varsigma; \beta_0, a_{0j}, M_{0j}, L_{0j})$ and $\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, a_{ij})$, $i = 1, 2$. Observe, from the definition of σ_{0j}, that $B_j = M_{0j}$, $j = 1, 2$, (analog to the case mentioned in Subsection 4.1). Thus, by fixing $M_{01} = 100$ and $M_{02} = 12$ [N m], the inequalities from expression (3.4) are satisfied, while the constants L_{0j} were taken as $L_{0j} = 0.9\, M_{0j}$, $j = 1, 2$.

All the tests of the considered control scheme were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{3} \end{pmatrix} \quad [\text{rad}]$$

and initial conditions at $q(0) = \dot{q}(0) = 0_2$.

Figure 5.1 shows the results obtained (in accordance to Corollary 3.1) with $\gamma = 6/5$,

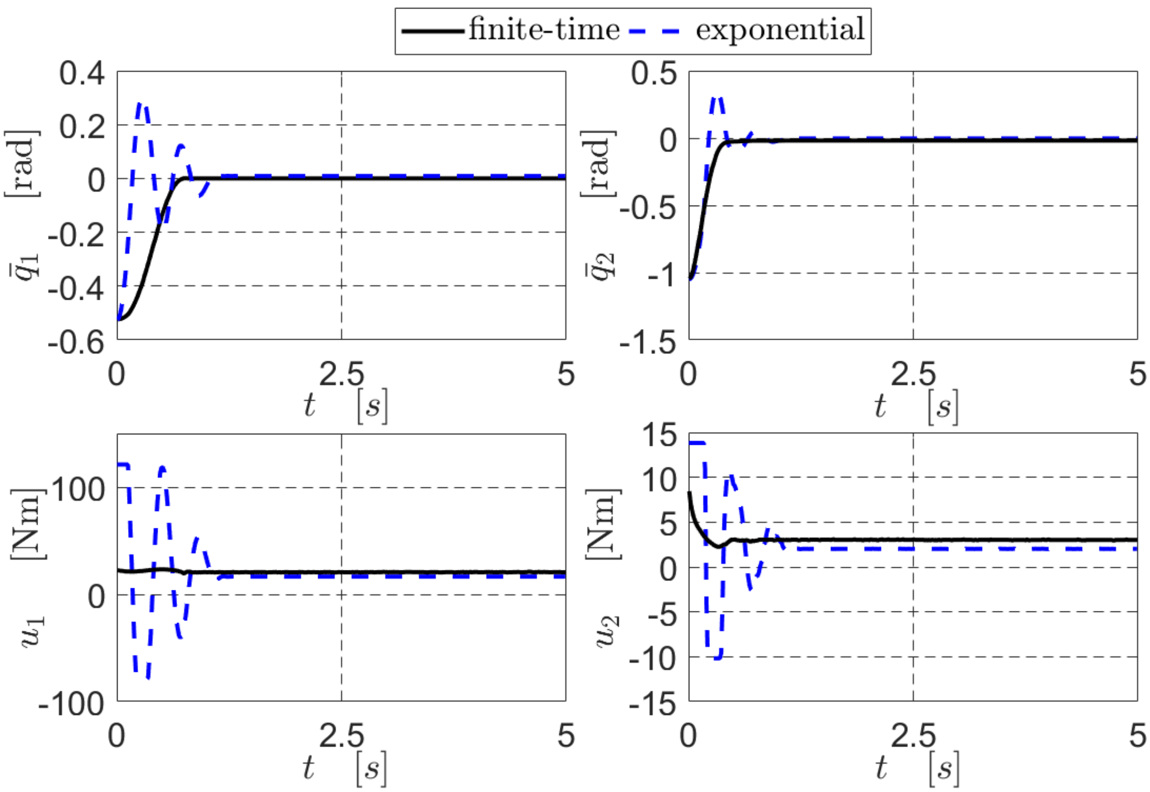

Figure 5.1: State-feedback experimental results (on-line approach), finite-time *vs* exponential convergence: position errors (↑) and control signals (↓).

$\beta_1 = 1/2$, $\beta_2 = 3/5$ and $\beta_0 = 4/3$ for the finite-time control implementation, while $\gamma = \beta_1 = \beta_2 = \beta_0 = 1$ for the exponential stabilization test, and the remaining control gain/parameter values were taken, for both controllers, as $K_1 = \text{diag}[500, 78]$ and $K_2 = \text{diag}[2.5, 2.5]$. Further parameters were fixed as $L_{01} = 59.78$ and $L_{02} = 7.58$ for the finite-time controller, and $L_{01} = 94.17$ and $L_{02} = 9.49$ for the exponential stabilizer. One sees that both controllers present fast responses and their control signals avoid input saturation. However, there exists a contrast among the two types of trajectory convergence (finite-time *vs* exponential): the closed-loop trajectories arising through the exponential stabilizer were observed to present important oscillations during the transient, contrarily to those gotten from the finite-time controller. This is also observed in the control signals, whence the finite-time controller is concluded to show a more efficient ability to counteract the inertial effects.

5.1.2 Output-feedback control scheme

The experimental results in this case were obtained from the application of the proposed control scheme (3.13)–(3.14) on the 3-DOF haptic device described in Subsection 1.5.2, for the implementation of the controller and the communication software was used the environment Simulink of Matlab and the PhanTorque libraries, more details can be found in [33].For the considered robot, Property 1.3 and Assumption 1.4 are satisfied. The considered functions correspond to those defined in Eqs. (4.3), *i.e.*, for every $j = 1, 2, 3$, $\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij})$, $i = 1, 2$, and $\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_{3j}, a_{3j})$, with $a_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2, 3$. Let us note that through these definitions we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$ (see (3.15)). Thus, by fixing $M_{ij} = 0.4$, $i = 1, 2$, $j = 1, 2, 3$, the inequalities from expression (3.15) are satisfied. The implementations were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{4} \\ \frac{\pi}{6} \end{pmatrix} \quad \text{[rad]}$$

and initial conditions: $q(0) = (-0.00863 \quad 0.02467 \quad -0.01834)^T$ [rad], $\dot{q}(0) = 0_3$.

Figure 5.2 shows results obtained taking $\beta_1 = \beta_2 = 3/5$ and $\beta_3 = 4/5$, for the finite-time controller, while $\beta_1 = \beta_2 = \beta_3 = 1$ for the exponential case. The control gains were taken for both (finite-time and exponential) controllers as $K_1 = K_2 = \text{diag}[0.8,\ 0.8,\ 0.8]$ and the parameters involved in the auxiliary subsystem as $A = B = \text{diag}[1,\ 1,\ 1]$. Notice, from these results, that the responses corresponding to the finite-time controller present less oscillations during the transient than those obtained from the exponential stabilizer. Moreover, despite the presence of overshoots in the transient of both controllers, those shown by the exponential stabilizer are more important (specially in the first and second link responses). Observe that, regardless of the modelling imprecisions, a considerably smaller (almost imperceptible) steady-state error is noticed to take place with the finite-time stabilizer, while a notorious steady-state error is observed to be obtained with the exponential controller. Although the closed-loop trajectory arising through the exponential stabilizer was observed to present a more important transient, one sees that the control signals avoid input saturation in both implementations.

5.2 Regulation with desired conservative force compensation

5.2.1 State-feedback control scheme

The following experimental tests, in addition to the comparison among the finite-time controller and the analog exponential scheme, will show results focusing on the ability

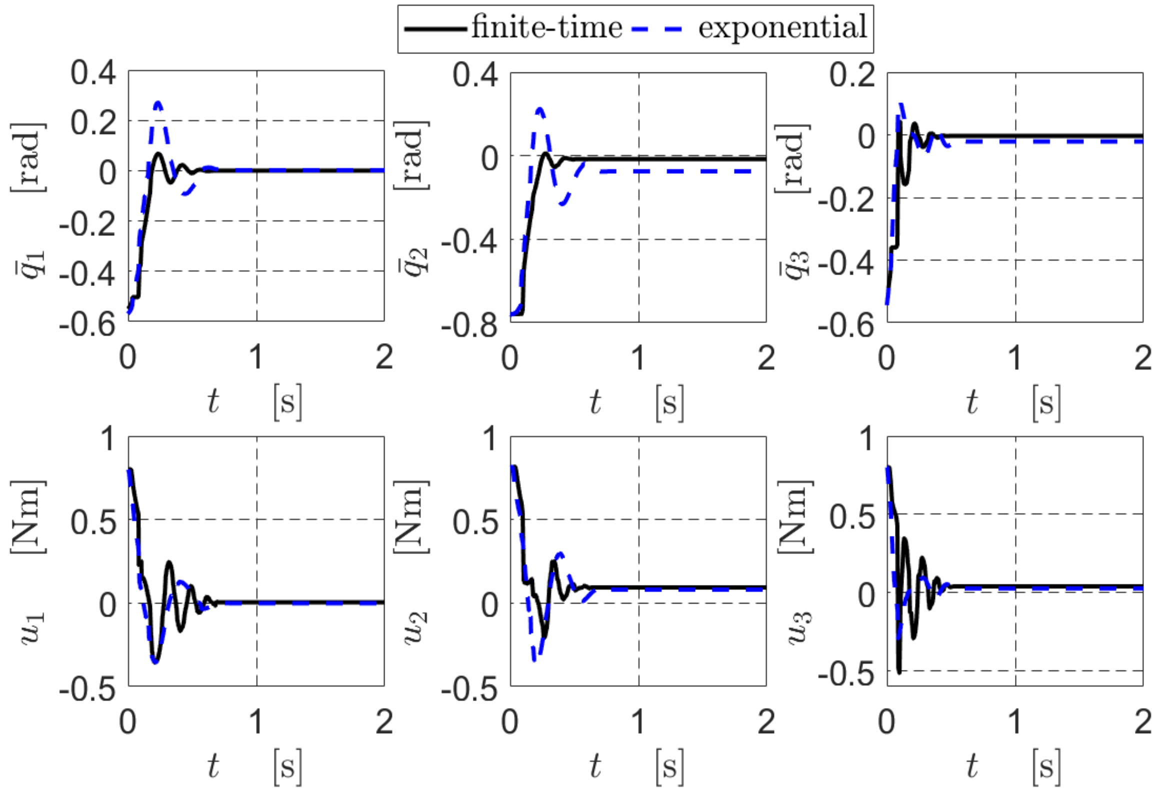

Figure 5.2: Output-feedback experimental results (on-line approach), finite-time *vs* exponential convergence: position errors (↑) and control signals (↓).

of the proposed controller in (3.22) to adopt different saturating structures. This is developed under the consideration of the 2-DOF robot manipulator used in the state-feedback experiments of Subsection 5.1.1. Thus, Assumptions 1.1–1.3 and 3.1, with $\eta = 3$, are fulfilled too. All the implementations were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{3} \end{pmatrix} \quad [\text{rad}]$$

and initial conditions as $q(0) = \dot{q}(0) = 0_2$.

The involved functions were defined in accordance to Eqs. (4.2) (keeping the same structure shown in the *on-line* case), such that —for every $j = 1, 2$— $\sigma_{0j}(\varsigma) = \sigma_{bs}(\varsigma; \beta_0, a_{0j}, M_{0j}, L_{0j})$ and $\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, a_{ij})$, $i = 1, 2$, with $a_{ij} = 0$, $i = 0, 1, 2$, $j = 1, 2$. Conditions on their parameters under which (3.24) is fulfilled are

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \tag{5.1a}$$

$$2B_{gj} \leq L_{0j}^{\beta_0} < M_{0j} \tag{5.1b}$$

(see Appendix B); the right-most inequality in (5.1b) actually comes from the specifications of σ_{bs} in (4.1c). Observe, from the involved functions, that $B_j = M_{0j}$, $j =$

1, 2 (as stated in previous sections). Hence, (3.23) and (5.1b) simultaneously require that $2B_{gj} < M_{0j} < T_j - B_{gj}$, $j = 1, 2$, which has been fulfilled by fixing $M_{01} = 100$ and $M_{02} = 12$. The rest of the control gain/parameter values were chosen under the consideration of inequalities (5.1).

Figure 5.3 shows the results obtained taking $\beta_1 = 3/5$, $\beta_2 = 18/25$ and $\beta_0 = 10/9$

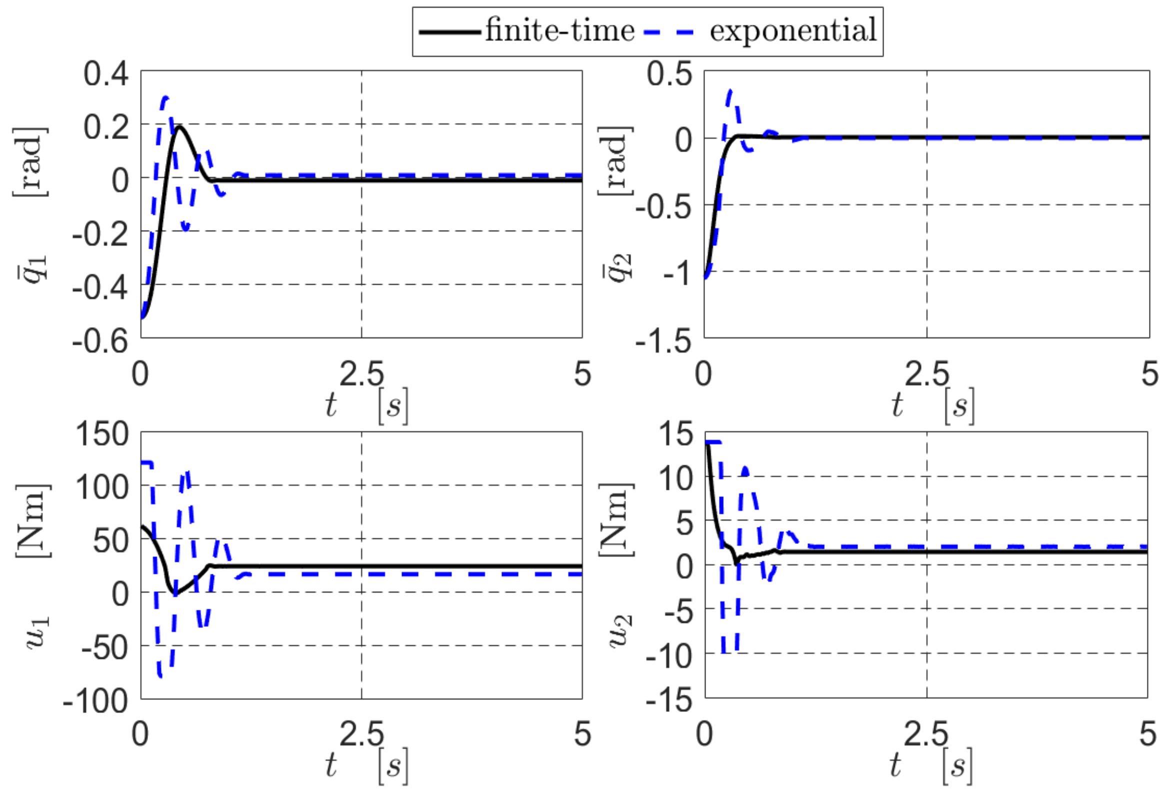

Figure 5.3: Finite-time *vs* exponential stabilization, state-feedback controller with desired conservative force compensation: position errors (↑) and control signals (↓).

for the finite-time controller with $\gamma = 6/5$, while $\beta_1 = \beta_2 = \beta_0 = 1$ for the exponential case, and the control gains were taken, for both controllers, as $K_1 = \text{diag}[500, 78]$ and $K_2 = \text{diag}[2.5, 2.5]$. Further parameters were fixed as $L_{01} = 59.78$ and $L_{02} = 7.58$ for the finite-time controller, as long as $L_{01} = 94.17$ and $L_{02} = 9.49$ for the exponential stabilizer. Observe that both stabilizers show to have fast responses, nevertheless differences among the two types of trajectory convergence (finite-time *vs* exponential) are perceptible. The responses obtained with the finite-time controller achieved the steady-state in less than one second and remained invariant thereafter, while, those gotten with the exponential stabilizer converge towards zero after a longer transient variation. Additionally, an important overshoot is perceptible in the transient response of the exponential controller. Notice that input saturation is avoided in both cases, however the finite-time controller shows a more efficient ability to counteract the inertial effects through control signals with considerably less and lower variations during the transient, which constitute an advantage for the application of the finite-time controller. Although there exist model inaccuracies, the closed-loop trajectory arising through the finite-time controller was observed to present a smaller steady-state error.

An alternative test is presented, where the proposed desired-compensation scheme adopts two different saturating structures. It is worth pointing out that the proposed design methodology does not force to keep the same saturating structure at every one of the controlled DOF but rather permits different choices among them. However, for our comparison purposes, the saturating structures are chosen different among the controllers but are kept the same among the controlled DOFs for each one of the implementations.

One of the implemented finite-time controllers adopts the same saturating structure of the previous test. Since this stabilizer uses, at every controlled DOF, a single saturation function that includes both the P and D actions, it will be referred to as the SPD controller. The alternative finite-time controller is structured taking, for every $j = 1, 2$,

$$\sigma_{0j}(\varsigma) = \sigma_u(\varsigma; \beta_0, a_{0j}) \tag{5.2a}$$

$$\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij}) \quad i = 1, 2 \tag{5.2b}$$

with $a_{ij} = 0$, $i = 0, 1, 2$, and $j = 1, 2$. Owing to the use of saturation functions for each one of the P and D actions (separately), this finite-time stabilizer will be referred to as the SP-SD controller. Conditions on the parameters of the functions involved in this case —as defined through (5.2)— under which (3.24) is fulfilled are

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \tag{5.3a}$$

$$M_{1j}^{\beta_0} > 2B_{gj} \tag{5.3b}$$

(see Appendix B). For both —the SPD and SP-SD— finite-time controllers with $\gamma = 5/4$, we took $\beta_1 = 3/5$, $\beta_2 = 3/4$ and $\beta_0 = 1$. Notice that with such a unitary value of β_0, for the SP-SD algorithm, we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (in accordance to (3.23)). Hence, while M_{01} and M_{02} were kept the same for the SPD controller (as in the precedent case), by taking $M_{11} = 82$, $M_{21} = 18$, and $M_{12} = M_{22} = 6$, the inequalities from expressions (3.23) and (5.3b) have been simultaneously satisfied. By further fixing $K_1 = \text{diag}[3260, 400]$ and $K_2 = \text{diag}[250, 25]$, the common inequalities (5.1a) and (5.3a) have been fulfilled (for both controllers). We further fixed $L_{01} = 94.17$ and $L_{02} = 9.49$ for the SPD algorithm, under the consideration of (5.1b).

Figure 5.4 shows the results obtained from the implementations. One sees that while both controllers achieve the finite-time stabilization objective avoiding input saturation, the closed-loop responses show different performances, with the SP-SD stabilizer giving rise to overshoots. Such a result corroborates the usefulness of the structural variety offered by the proposed approach in searching for performance improvement. It is worth further noticing that the SPD finite-time controller shows again —as in previous test but this time compared to the SP-SD finite-time stabilizer— a more efficient ability to counteract the inertial effects through signals with considerably less and lower variations during the transient, concluding that such a nice feature is related not only to the finite-time nature of the controller but also to its (combined) SPD-type structure.

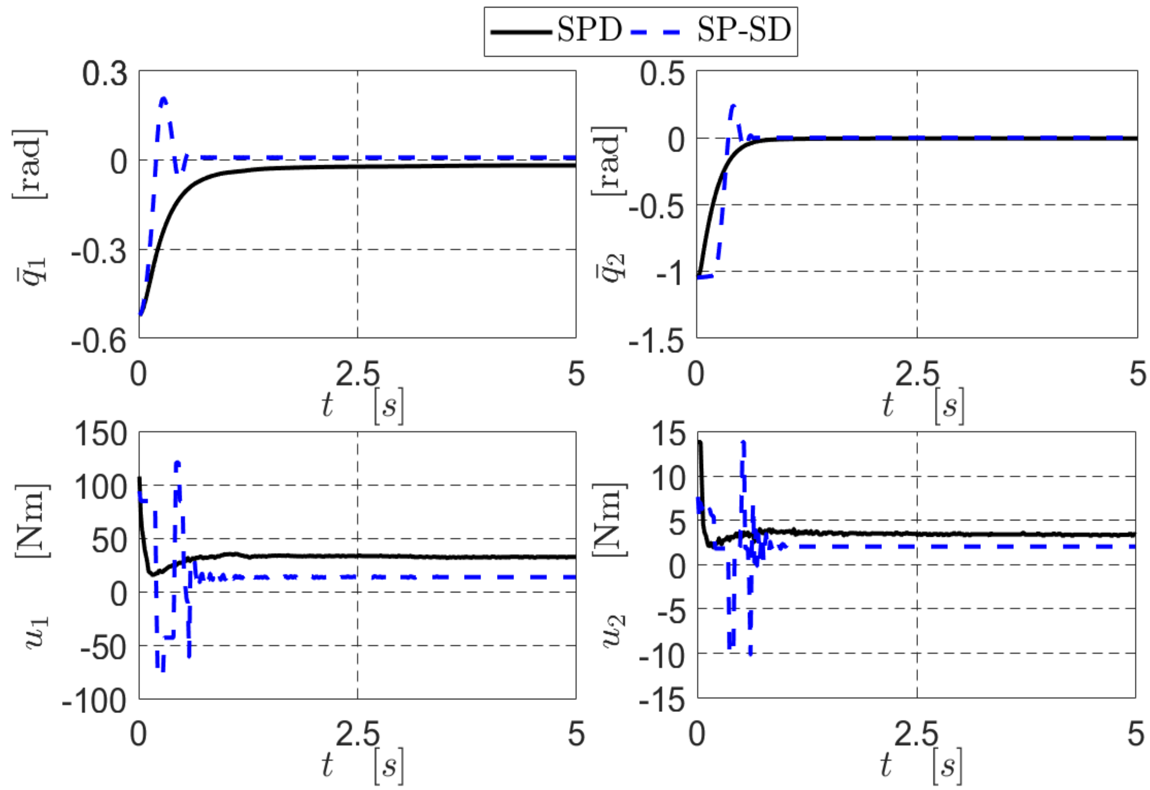

Figure 5.4: SPD *versus* SP-SD finite-time controllers, state-feedback scheme with desired conservative force compensation.

5.2.2 Output-feedback control scheme

The experimental results in this case were run on the 3-DOF haptic device used in the on-line case. Thus, as previously mentioned (in Section 5.2.1), Property 1.3 and Assumption 3.1 (with $\eta = 3$) are satisfied. Furthermore, the application of the control scheme stated in (3.44)–(3.45) also involves the functions defined in such a case, *i.e.*, for every $j = 1,2,3$, $\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij})$, $i = 1,2$, and $\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_{3j}, a_{3j})$, with $a_{ij} = 0$, $i = 1,2,3$, $j = 1,2,3$. Conditions on their parameters under which (3.47) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{1-\beta_1/\beta_1} \tag{5.4a}$$

$$M_{1j} > 2B_{gj} \tag{5.4b}$$

(see Appendix B). Let us note, from the involved functions, that $B_j = M_{1j} + M_{2j}$, $j = 1,2,3$. Thus, by fixing $M_{ij} = 0.4$, $i = 1,2$, $j = 1,2,3$, the inequalities from expressions (3.47) and (5.4) have been simultaneously satisfied. The rest of the control gain/parameter values were chosen taking care that the design requirements were always satisfied. All the implementations were run taking the desired configuration at

$$q_d = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{4} \\ \frac{\pi}{6} \end{pmatrix} \quad [\text{rad}]$$

and initial conditions: $q(0) = (-0.00863 \quad 0.02467 \quad -0.01834)^T$ [rad], $\dot{q}(0) = 0_3$.

Figure 5.5 shows results obtained taking $\beta_1 = \beta_2 = 3/5$ and $\beta_3 = 4/5$, for the

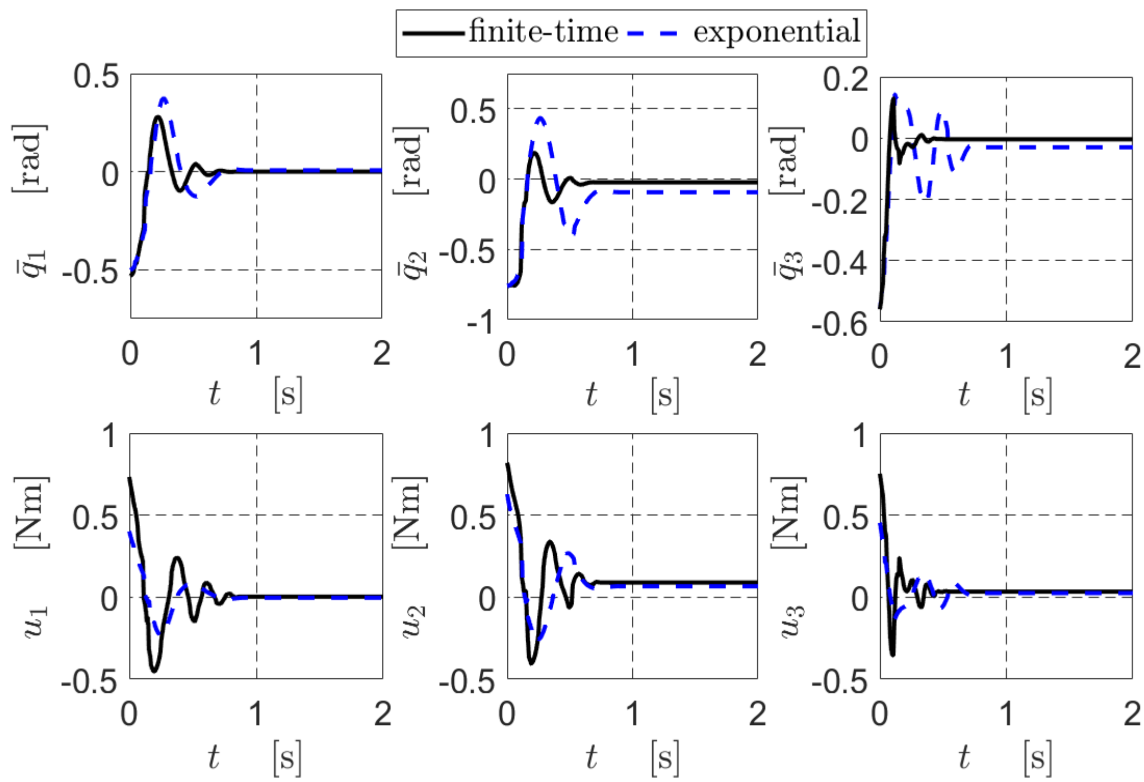

Figure 5.5: Finite-time *vs* exponential stabilization, output-feedback scheme with desired conservative force compensation: positions errors (↑) and control signals (↓).

finite-time controller, while $\beta_1 = \beta_2 = \beta_3 = 1$ for the exponential stabilizer, and the remaining control gain/parameters were taken, for both (finite-time and exponential) controllers, as: $K_1 = \text{diag}[0.5,\ 0.5,\ 0.5]$ (satisfying (5.4a)), $K_2 = \text{diag}[0.3,\ 0.3,\ 0.3]$ and $A = B = \text{diag}[1,\ 1,\ 1]$. Observe, from these results, that control signals are within the saturation bounds in both implementations. Moreover, the closed-loop trajectory arising through the exponential stabilizer was observed to present a longer and more important transient in the three link responses. On the other hand, while a notorious steady-state error —due to modelling imprecisions such as static friction and biased parameters involved in the gravity vector model— is observed to be obtained with the exponential controller, a considerably smaller one (almost imperceptible) is noticed to take place with the finite-time stabilizer. Further tests repeatedly showed the same result: considerably smaller (always almost imperceptible) steady-state errors arisen with the finite-time controller compared to those obtained with the exponential stabilizer, which were generally notorious.

5.2.3 Desired *versus* on-line conservative-force compensation

Further experimental tests are focused on the comparison among the on-line and desired conservative-force compensation versions of the finite-time controller. The closed-loop responses were compared taking the same control gain/parameter values and saturating structures but differ only on the type of conservative-force compensation. In this direction, we repeat here the finite-time control tests shown in the state-feedback case of Subsection 5.2.1 and those shown in the output-feedback case of Subsection 5.2.2, just alternating the referred compensation term. We refer to the finite-time controllers with desired conservative-force compensation as FT_d and the finite-time controllers with on-line conservative-force compensation as FT_o. First, we present the state-feedback

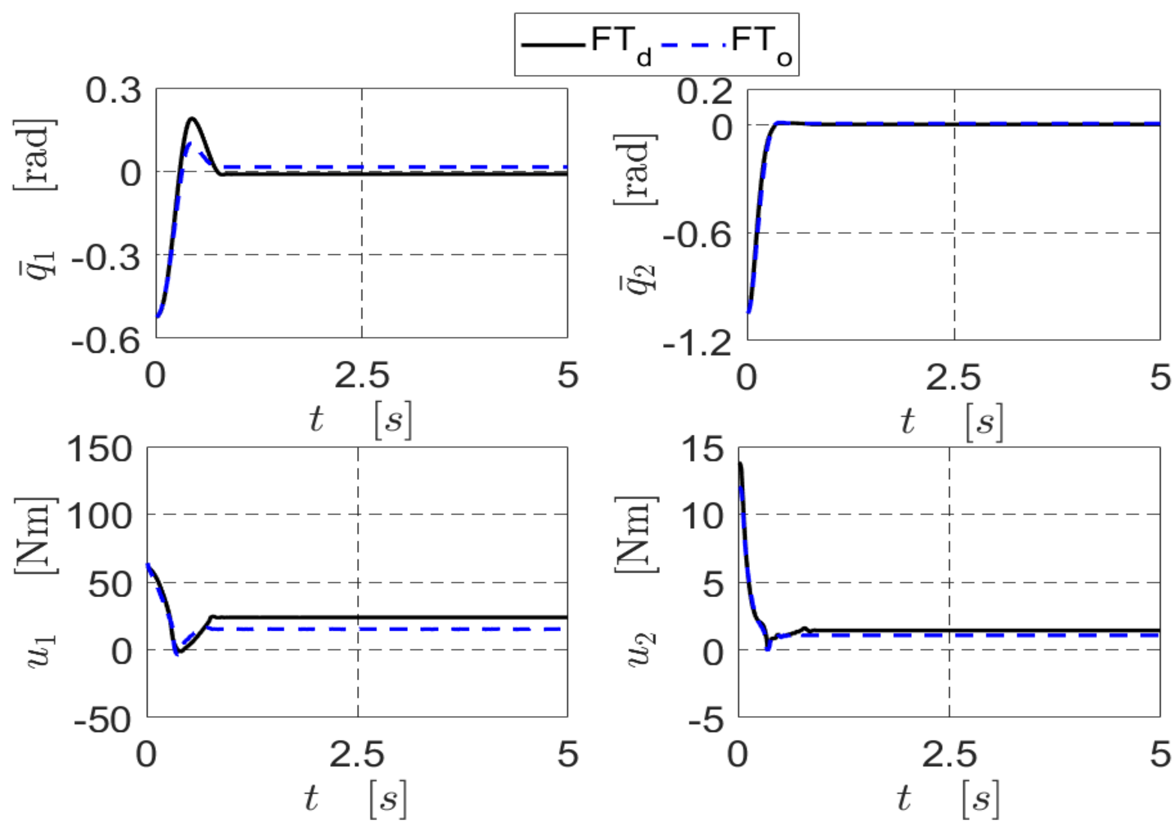

Figure 5.6: State-feedback controller experiments, desired (FT_d) *vs* online (FT_o) conservative-force compensation: position errors (↑), control signals (↓).

control experiments with the 2-DOF robot manipulator and the parameter/gain values considered in Subsection 5.2.1, this in order to compare the tests already obtained in such subsection with those that will be obtained by changing the compensation term for the on-line conservative force. Figure 5.6 shows the comparison among the tested state-feedback controllers. Observe that no significant differences among the responses are obtained.

Further, analog tests were run for the output-feedback control schemes. Such tests were run taking the 3-DOF haptic device and the parameter/gain values used in Subsection 5.2.2. Figure 5.7 shows the results by comparing the ($\mathrm{FT_d}$ versus $\mathrm{FT_o}$)

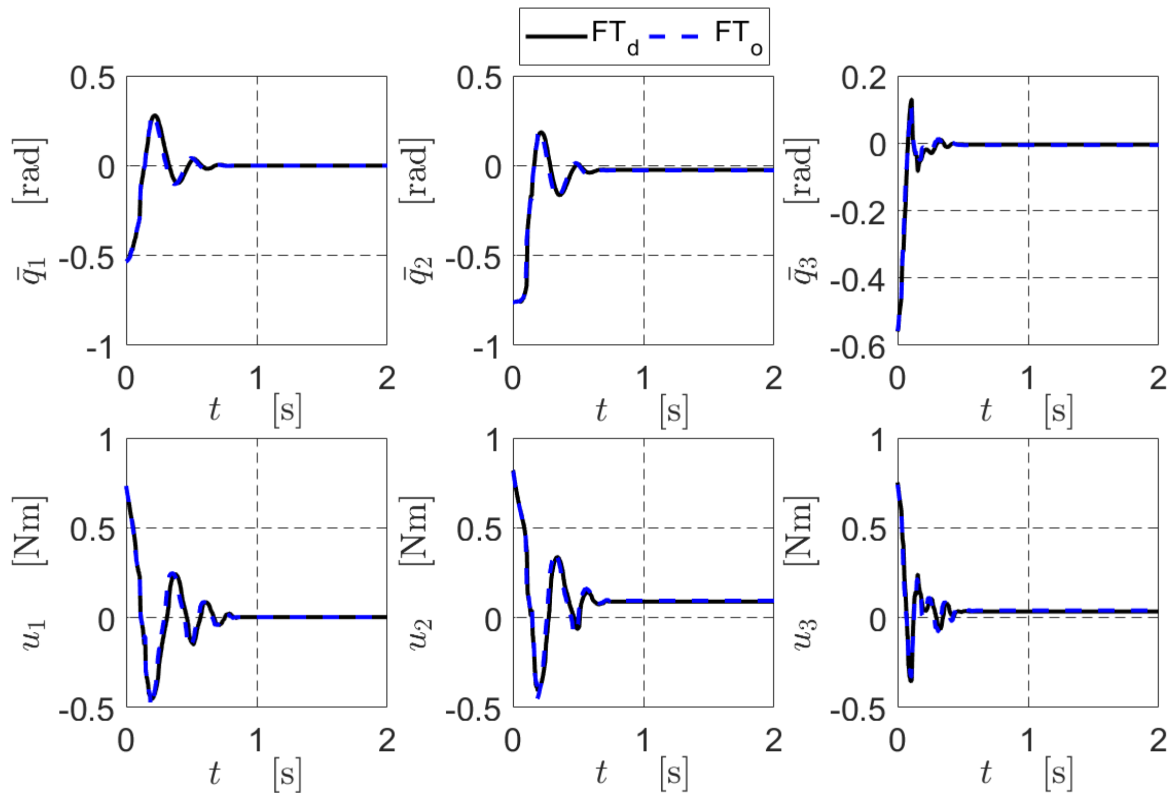

Figure 5.7: Output-feedback controller experiments, desired ($\mathrm{FT_d}$) *vs* online ($\mathrm{FT_o}$) conservative-force compensation: position errors (↑), control signals (↓).

responses obtained from the output-feedback controllers. Observe that there is a negligible difference among the closed-loop performances.

The previous tests confirm the observed in chapter 4, *i.e.*, the effect on the performance for the implementation simplification earned by the desired compensation version of the controllers is negligible, in spite of the open-loop conservative-force term that is left acting on the system.

5.2.4 Tracking problem

For the following test the 3-DOF anthropomorphic-type described in Subsection 1.5.2 was considered. As mentioned in such a description, the joint positions are measured using incremental encoders on the motors and the standard backwards difference algorithm is used to obtain the velocity signals. Measurement of encoder data and reference voltage generation are carried out through an electronic interface composed by the PMDi LC228 model from *Precision MicroDynamic Inc.* The control algorithm is written in C language and executed at a 2.5 milliseconds sampling period on a PC host computer. For such a robot, Property 1.1, Assumptions 1.1–1.3 and 3.2 are satisfied, particularly with $\mu_{M1} = 2.975$ kg m^2, $\mu_{M2} = 1.6589$ kg m^2, $\mu_{M3} = 0.1757$ kg m^2, $k_{C1} = 0.989$ kg m^2/s, $k_{C2} = 0.4681$ kg m^2/s, $k_{C3} = 0.1997$ kg m^2/s, $f_1 = 0.4$ kg m^2/s, $f_2 = 1.2806$ kg m^2/s, and $f_3 = 0.64$ kg m^2/s. Furthermore, the input saturation bounds are $T_1 = 15$ Nm, $T_2 = 50$ Nm and $T_3 = 4$ Nm for the first, second and third links respectively. For the sake of simplicity, units are subsequently omitted.

For the implementation of the proposed scheme in (3.70), we define $\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, 0, M_{ij})$, $i = 1, 2$, $j = 1, 2, 3$, for constants $\beta_i \in (0, 1]$ and $M_{ij} > 0$. For each $i = 1, 2$, such functions prove to be bounded strongly passive functions for $(\kappa_i, \beta_i, b_i, \bar{\kappa}_i, \beta_i, b_i)$, where $b_i = \min\{b_{i1}, b_{i2}, b_{i3}\}$, $\kappa_i \leq 1$ and $\bar{\kappa}_i \geq b_i^{-\beta_i} \max\{M_{i1}, M_{i2}, M_{i3}\}$, with —for every $j = 1, 2, 3$— $b_{ij} = M_{ij}^{1/\beta_i}$. Let us note that by the functions σ_{ij}, we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$ (recall (3.101)). The test was run taking initial conditions at $q(0) = \dot{q}(0) = 0_3$, and the desired trajectory as

$$q_d(t) = \begin{pmatrix} \frac{\pi}{4} + \pi/6\cos(1.2t) \\ \frac{\pi}{2} + \pi/6\sin(1.2t) \\ \frac{\pi}{2} + \pi/6\cos(1.2t) \end{pmatrix}$$

for which $B_{dv} = 0.888$ and $B_{da} = 1.066$. From this and the above-listed values of the parameters characterizing Property 1.1, Assumptions 1.1–1.3 and 3.2, one can corroborate that Assumption 3.3 is satisfied too.

Figure 5.8 shows the results obtained with $\beta_1 = 7/10$ and $\beta_2 = 14/17$ to obtain the finite-time convergence, the rest of the control parameters were taken as $K_1 = \text{diag}[70, 450, 6]$, $K_2 = \text{diag}[0.5, 4.5, 0.0015]$ and, in accordance to (3.101) (with $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$), for every $i = 1, 2$: $M_{i1} = 5$, $M_{i2} = 14.5$, $M_{13} = 1$, and $M_{23} = 0.9$. The tracking objective is observed to be achieved and input saturation is avoided. The control gains were chosen so as to considerably reduce the post-transient tracking error due to unmodelled phenomena. Such election made almost imperceptible the difference among the finite-time and exponential convergence. Further discussion about the post-transient tracking error due to modelling imprecisions will be given in the following subsection.

5.2.5 Robustness problem

The tests in this subsection were implemented so as to verify the conclusion of the robustness problem exposed in Section 3.4 of Chapter 3. Thus, with this goal in mind, the

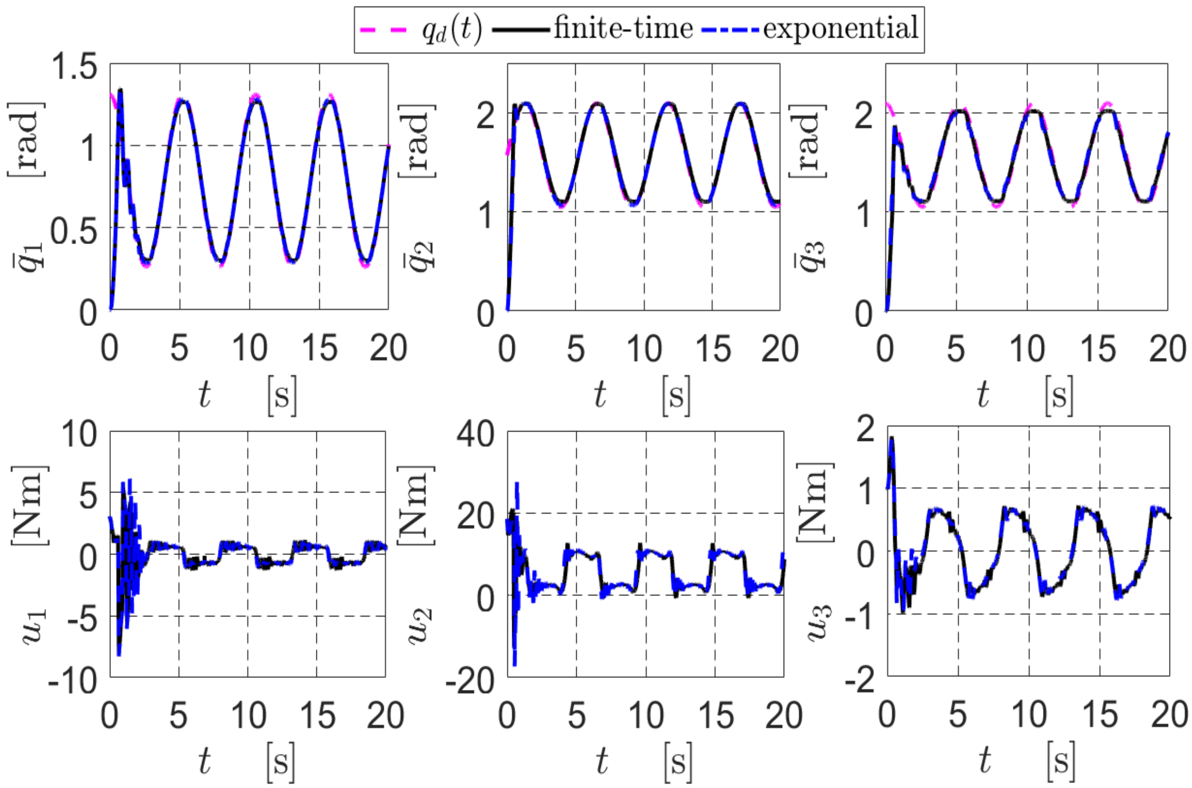

Figure 5.8: Experiments obtained with the tracking control scheme, finite-time *vs* exponential convergence: positions (↑), position errors (→) and control signals (↓).

3-DOF manipulator as well as the control scheme used in the previous subsection are also taken into account. For the implementations, the functions kept the same structure as those defined in the previous test, *i.e.*, $\sigma_{ij}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|^{\beta_i}, M_{ij}\}$, $i = 1, 2$, $j = 1, 2, 3$, for constants $\beta_i \in (0, 1]$ and $M_{ij} > 0$. For each $i = 1, 2$, such functions prove to be bounded strongly passive functions for $(\kappa_i, \beta_i, b_i, \bar{\kappa}_i, \beta_i, b_i)$, where $b_i = \min\{b_{i1}, b_{i2}, b_{i3}\}$, $\kappa_i \leq 1$ and $\bar{\kappa}_i \geq b_i^{-a_i} \max\{M_{i1}, M_{i2}, M_{i3}\}$, with —for every $j = 1, 2, 3$— $b_{ij} = M_{ij}^{1/\beta_i}$. From the definition of σ_{ij} functions, we have that $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$ (recall (3.101)). All the experiments were run taking initial conditions at $q(0) = \dot{q}(0) = 0_3$, and the desired trajectory as

$$q_d(t) = \begin{pmatrix} \frac{\pi}{4} + A_{d1}\cos\omega_{d1}t \\ \frac{\pi}{2} + A_{d2}\sin\omega_{d2}t \\ \frac{\pi}{2} + A_{d3}\cos\omega_{d3}t \end{pmatrix} \tag{5.5}$$

We show here results carried out on the experimental setup, which naturally implies the presence of an input-matching perturbation term, ϱ, arisen from unmodelled phenomena, such as Coulomnb and static friction, and system parameter estimation bias (in view of the inclusion of the system model in the control scheme), all these giving

—or considered to give— rise to bounded perturbation components, in accordance to Assumption 3.4; control parameter values that permit a clear visualization on the effect of the perturbation term on the closed-loop trajectories were thus taken at every experiment. In our implementations, we considered $\bar{\varrho}_j = 0.05\,T_j,\ j = 1, 2, 3$ (although no proof about such a consideration is available, but it seemed acceptable in our experiments), through which Assumption 3.5 is corroborated to be satisfied. The evaluation of the results will additionally involve the Integral-of-the-Square-of-the-Error-variables (ISE) performance index —*i.e.* letting $x = (\bar{q}^T\ \dot{\bar{q}}^T)^T$: $\int_{t_1}^{t_1+\Delta} \|x(t)\|^2 dt$— applied during the post-transient phase. More specifically, at every one of the performed test, such an index was evaluated with $t_1 = 15$ s and $\Delta = 10$ s. Tests were run for different combinations of A_{dj} and ω_{dj}, $j = 1, 2, 3$, in (5.5) for which Assumption 3.6 was always corroborated to be satisfied. Figure 5.9 shows the results obtained with $A_{dj} = 0.2$ and $\omega_{dj} = 0.8$, $j = 1, 2, 3$ (for which

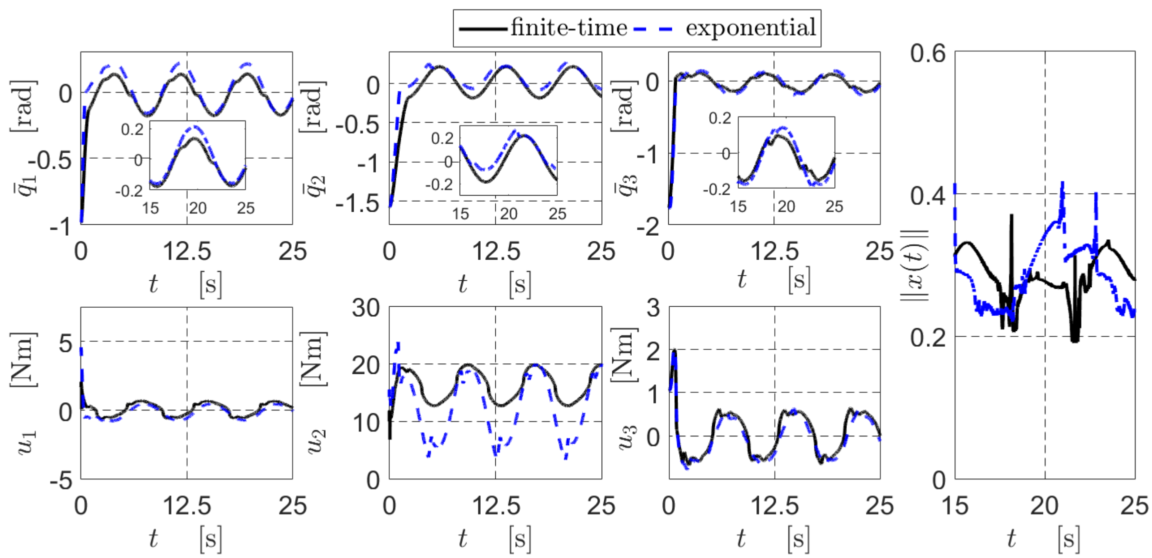

Figure 5.9: Experiments obtained with the tracking control scheme under perturbation, with $A_{dj} = 0.2$ and $\omega_{dj} = 0.8$, $j = 1, 2, 3$: position errors (↑), control signals (↓) and $\|x(t)\|$(→).

$B_{dv} = 0.2263$ and $B_{da} = 0.181$). This test was run taking $\beta_1 = 11/20$ and $\beta_2 = 22/31$ for the finite-time controller, and $\beta_1 = \beta_2 = 1$ for the exponential control algorithm, while, for both (finite-time and exponential) controllers, the rest of the control parameters were taken as $K_1 = \text{diag}[5, 48, 3]$, $K_2 = \text{diag}[0.5, 4.5, 0.0015]$ and, in accordance to (3.101) (with $B_j = M_{1j} + M_{2j}$, $j = 1, 2, 3$), for every $i = 1, 2$: $M_{i1} = 6.5$, $M_{i2} = 14.5$ and $M_{i3} = 1$. In both cases, control signals that continually aim at reducing the motion errors are observed to be generated avoiding actuator saturation. But in view of the (implicit) input-matching perturbation term, post-transient variations are noticed to take place in the position-error closed-loop responses. More, importantly, smaller post-transient variations are observed to arise in the finite-time controller case [which is more visible through the (right-most) post-transient graph of $\|x(t)\|$: for the finite-time controller $\|x(t)\| \leq 0.3717$, for all $15 \leq t \leq 25$, while $\|x(t)\| \leq 0.4179$, for all

$15 \leq t \leq 25$ in the exponential case]. This observation is further complemented through the evaluation of the ISE index, which gave values of 0.7222 *vs* 0.8357 for the finite-time and exponential controllers respectively. Similar observations arose through different tests with alternative values of A_{dj} and ω_{dj}, $j = 1, 2, 3$, in (5.5). For instance, Figure 5.10 shows additional results obtained with $A_{d1} = A_{d2} = \pi/18$, $A_{d3} = \pi/6$, $\omega_{d1} = \omega_{d2} = 0.6$

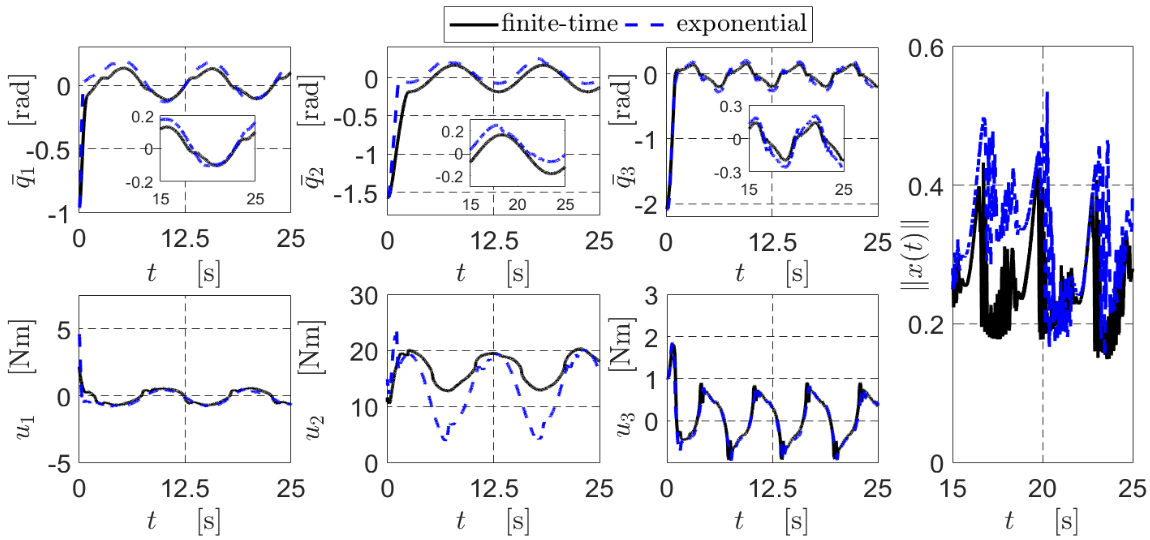

Figure 5.10: Experiments obtained with the tracking control scheme under perturbation, with $A_{d1} = A_{d2} = \pi/18$, $A_{d3} = \pi/6$, $\omega_{d1} = \omega_{d2} = 0.6$ and $\omega_{d3} = 1$: position errors (↑), control signals (↓) and $\|x(t)\|$(→).

and $\omega_{d3} = 1$ (for which $B_{dv} = 0.534$ and $B_{da} = 0.5273$), where the experiments were run with the same values for β_i and K_i, $i = 1, 2$ and, in accordance to (3.101), for every $i = 1, 2$: $M_{i1} = 6$, $M_{i2} = 14$ and $M_{i3} = 1$ were taken for both cases. Smaller post-transient variations are again corroborated to arise in the finite-time controller case with $\|x(t)\| \leq 0.4326$, for all $15 \leq t \leq 25$, while $\|x(t)\| \leq 0.5359$, for all $15 \leq t \leq 25$ in the exponential case. Furthermore, the ISE index for this test gave values of 0.6888 *vs* 1.126 for the finite-time and exponential controllers respectively.

Further tests were implemented adding an artificial bounded perturbation term $\varrho_a(t)$ through the input variable, *i.e.*, $u = u(t, q, \dot{q}) + \varrho_a(t)$, with $\|\varrho_a(t)\| \leq \bar{\varrho}_a$, $\forall t \geq 0$, or equivalently $|\varrho_{aj}(t)| \leq \bar{\varrho}_{aj}$, $j = 1, 2, 3$, $\forall t \geq 0$, for every $j = 1, 2, 3$, $\varrho_j = \varrho_{nj} + \varrho_{aj}$, where ϱ_{nj} represents the j^{th} component of the natural perturbation term (implicit in the experimental setup), and consequently $\bar{\varrho}_j = \bar{\varrho}_{nj} + \bar{\varrho}_{aj} = 0.05\, T_j + \bar{\varrho}_{aj}$. More precisely, we defined

$$\varrho_a(t) = \begin{pmatrix} A_{a1} \cos t \\ A_{a2} \sin t \\ A_{a3} \cos t \end{pmatrix} \tag{5.6}$$

for which $\bar{\varrho}_{aj} = A_{aj}$, $j = 1, 2, 3$. By gradually increasing the size of ϱ through ϱ_a, we observed that above some perturbation term size $\bar{\varrho}^*$, the (range of) ultimate variations happened to become smaller in the exponential controller case, thus confirming Remark 3.20. This is shown through Figure 5.11, where results from a test with $A_{dj} = 0.2$ and

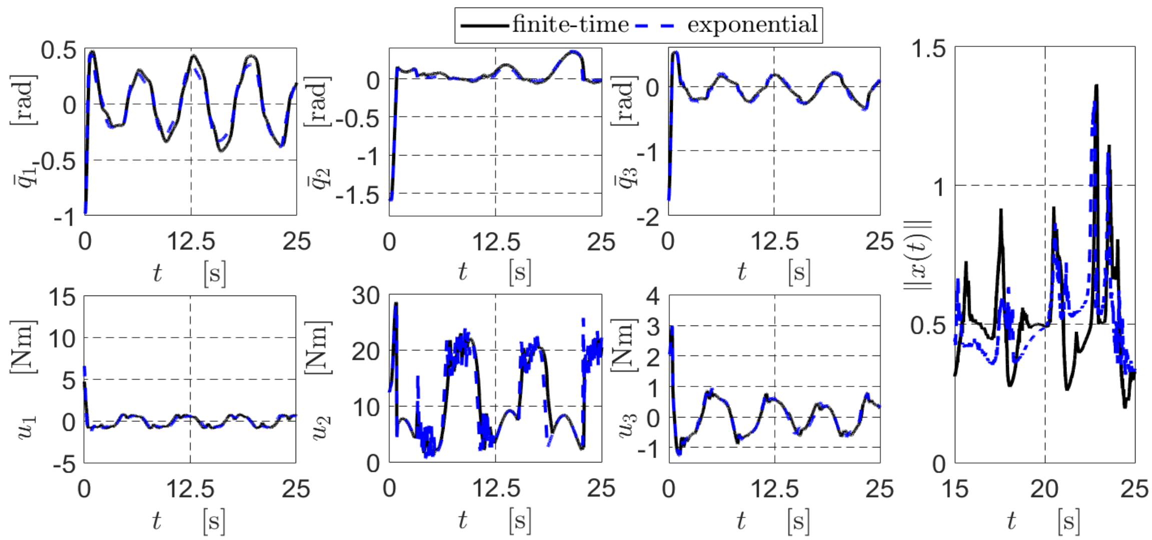

Figure 5.11: Experiments obtained with the tracking control scheme with artificial perturbation added: position errors (↑), control signals (↓) and $\|x(t)\|$(→).

$\omega_{dj} = 0.8$, $j = 1, 2, 3$, in (5.5), and $A_{a1} = 2$, $A_{a2} = 4.5$ and $A_{a3} = 1.2$ in (5.6), are shown. This test was run taking $\beta_1 = 7/10$ and $\beta_2 = 14/17$ for the finite-time controller, and $\beta_1 = \beta_2 = 1$ for the exponential control algorithm, while for both cases, the rest of the control parameters were taken as $K_1 = \text{diag}[5, 450, 3]$, $K_2 = \text{diag}[0.5, 1, 0.0015]$ and, in accordance to (3.101) (with $\bar{\varrho}_j = 0.05\, T_j + A_{aj}$, $j = 1, 2, 3$): $M_{11} = M_{21} = 5.5$, $M_{12} = M_{22} = 12$, $M_{13} = 0.8$ and $M_{23} = 0.5$. Smaller ultimate variations are corroborated to take place, in this case, through the exponential controller with $\|x(t)\| \leq 1.3075$, for all $15 \leq t \leq 25$, while $\|x(t)\| \leq 1.365$, for all $15 \leq t \leq 25$ in the finite-time case. This observation is further complemented through evaluation of the ISE index, which gave values of 3.2914 *vs* 2.9329 for the finite-time and exponential controllers respectively. The importance of this test relies on the following observation: the —so cited— robustness-related superiority of the finite-time controllers over asymptotical ones cannot be taken to be *universal*, but only holds as long as the perturbation term remains *sufficiently small*. It is however worth to note that such a *characteristic inversion* observed through this test, arises through a perturbation term being considerably higher than in the previous tests, which does not seem to be a tolerable situation in practice. For instance, it is reasonable to expect that a motion control design with a high precision exigency should involve friction compensation techniques [46], [47] to reduce the interference of friction phenomena.

CHAPTER 6

Conclusions and perspectives

Continuous finite-time/exponential control schemes for mechanical systems with input constraints have been proposed throughout this dissertation. The proposed schemes keep generalized forms permitting multiple saturating structures and involving P- and D-type actions. Such proposed schemes allow the election among finite-time or exponential convergence through a simple parameter. Moreover, the closed-loop analyses were developed within the innovating framework of *local homogeneity*; the applicability and advantages of such a framework (which allowed to analyze the considered bounded-input cases) were shown throughout this dissertation. In this direction, the novelty of the developed work is based on the solution of the finite-time control problem under the consideration of the real input-constrained case of mechanical systems within a suitable analytical framework, since for both the regulation and tracking problems addressed in this work, it was not clear how to deal with the bounded-input case, which was successfully solved. Throughout the dissertation all the analytical issues were clarified and developed; this is important in view of the lack of suitable methodologies in the literature for the formulated problems. Thus, this work provides an advancement on both analytical and design methodologies (that were unclear at the beginning) to solve the finite-time continuous control problem for constrained-input mechanical systems.

The position control problem was first considered through a state-feedback scheme with on-line conservative force compensation, that guaranteed global stabilization with either finite-time or (local) exponential convergence avoiding input saturation. This control law has been implemented through simulations and experiments, whose results have made possible to illustrate the application of the proposed method and confirm the analytical results. In particular, the simulations were further focused on studying the veracity of the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones. This was actually shown to depend on the specific locally homogeneous functions involved in the SPD term of the controller and the precision used to practically evaluate the stabilization time. Furthermore, a way to define such functions has been shown through which finite-time controllers indeed prove to be faster than asymptotical stabilizers. This was made possible, thanks to the design flexibility permitted within the framework of local homogeneity, which allows to involve functions that are not forced to keep the homogeneity property globally but may rather adopt suitable changes. Experimental implementations further confirmed that the proposed scheme achieves the control objectives.

A suitable output-feedback version of the previous mentioned scheme was proposed so as to achieve the regulation objective excluding the velocity variables in the feedback. In this case, the control objective was achieved through an auxiliary subsystem with a simple generalized continuous structure through which the required dissipation is dynamically injected by means of the position error as input variable. Simulation and experimental tests show the applicability of the proposed method and confirm the analytical results. The experiments further showed that, regardless of the modelling imprecisions, the finite-time stabilizer presents a considerably smaller (almost imperceptible) steady-state error than that obtained with the exponential controller, giving an idea about the finite-time controller performance against uncertainties.

Desired conservative-force compensation versions of the previous control schemes were further designed. Far from what one could have expected, such conservative-force compensation versions are not a simple extension of the on-line compensation cases but they have rather proven to need more involved requirements resulting from a closed-loop analysis with a considerably higher degree of complexity. Moreover, they have overcome the impossibility to apply Lemma 2.3 on their transition from finite-time to exponential stabilization, which could not be solved keeping the local-homogeneity criterion of the former in view of the open-loop conservative force which is kept acting on the closed loop. Experimental and simulation results, for both the state-feedback and output-feedback controllers, have shown the actual ability of the proposed approaches to guarantee the considered types of convergence avoiding input saturation. Furthermore, both the on-line and desired conservative-force compensation versions of the developed schemes were tested and actually compared when the only difference among them is on the type of the referred compensation term. They both gave rise to suitable results with very small differences among the corresponding closed-loop responses. Thus, the implementation simplifications earned through the desired compensation are concluded to have a negligible cost on the system performance, passing the bill rather to the closed-loop analysis.

Thus, the regulation-under-bounded-input problem was successfully solved through the accomplishment of the stated objectives. Moreover, such a development made possible to investigate (numerically) about the stabilization time of the proposed controllers, which (in this case) brings to the fore that finite-time controllers not always have faster stabilization than exponential ones, which will depend of the functions involved in the control schemes. Furthermore, through the implementation of the proposed controllers, it was observed that (in almost all the cases) finite-time stabilizers present a more efficient ability to counteract the inertial effect than the exponential controllers, *i.e.*, the control signals obtained from the finite-time controllers showed considerably less and lower variations during the transient responses, concluding that such a nice feature is related not only to the finite-time nature of the controller but also to its generalized-function structure.

After studying the regulation case, the problem of finite-time tracking continuous control of constrained-input mechanical systems was solved through a strict Lyapunov function, under the consideration of linear damping terms in the open-loop dynamics.

While the construction of such a strict Lyapunov function and the corresponding analytical support constitute by themselves an innovative analytical finding, the proposed approach gathers (analog to the position control approaches) the following properties: it keeps a continuous structure; it gives the freedom to choose among finite-time and (local) exponential convergence through a simple design parameter; it guarantees the control objective for any initial conditions; it avoids input saturation along the closed loop trajectories, and it keeps a Saturating-Proportional (SP) Saturating-Derivative (SD) type action based structure. In addition, there is no tuning restriction on the control gains. The applicability of such controller was shown through simulation implementations, where it was observed that both the finite-time and exponential tracking objectives as well as the input saturation avoidance were accomplished. Moreover, the experimental tests have shown that the control scheme forces the system trajectories to approach the objective in both (finite-time and exponential) convergence cases. A small post-transient tracking-error was perceived due to model imprecisions, which motivated the robustness analysis presented in Chapter 3.

Based on the proposed finite-time tracking controller, a robustness analysis was developed under the additional consideration of an input-matching bounded perturbation term. The result developed has formally confirmed (within the considered context) that, for a sufficiently small perturbation bound, the closed-loop error variable trajectories converge in finite-time to an origin-centered ball whose radius is not only directly related to the perturbation term bound (in such a way that the greater, resp. smaller, is the perturbation bound, the greater, resp. smaller, is the ball radius) but it also becomes smaller for the finite-time controllers than for their analog exponential versions. Furthermore, beyond such a meaningful confirmation, the result developed here has overcome important analytical limitations from previous works by suitably addressing the time-varying nature of the unperturbed dynamics, avoiding the restriction of any of the parameters involved in the control design and applying for any initial condition on the system (position and velocity) variables. Simulation and experimental tests have corroborated the analytical result. Moreover, further tests under adverse perturbation conditions showed that the so-cited robustness-related superiority of finite-time controllers over asymptotical ones cannot be taken to be universal, but that it effectively holds as long as the perturbation term matching the input be sufficiently small, whose opposite case is generally intended to be avoided or attenuated in actual applications.

Beyond the results achieved in this work, there are still aspects that may be considered for future research. For instance, the continuous tracking control scheme developed for fully-damped mechanical systems achieves the control objective globally (for any initial condition), however this becomes a challenging issue to solve when the under-damped and undamped cases of such systems are considered. Additionally, an output-feedback tracking approach should be considered in order to provide a controller that achieves the control objectives disregarding velocity measurements. Finally, an interesting problem still to solve is on the development of the finite-time adaptive versions of the controllers proposed in this work in order to reduce their parameter-dependence.

APPENDIX A

On properties of robot manipulators

The constant bounds shown in Property 1.1 and Assumptions 1.1–1.3 —corresponding to $H(q)$, $C(q,\dot{q})$, and $g(q)$—, listed in Section 1.4, are calculated through the following developments. Particularly, such bounds are obtained for the robot manipulators described in Section 1.5.

A.1 2-DOF Robot manipulator

The dynamical model of the 2-DOF robot manipulator exposed in Subsection 1.5.1 is rewritten here as

$$H(q) = \begin{pmatrix} \delta_1 + \delta_2 \cos q_2 & \delta_3 + \delta_4 \cos q_2 \\ \delta_3 + \delta_4 \cos q_2 & \delta_3 \end{pmatrix} \tag{A.1}$$

$$C(q,\dot{q}) = \begin{pmatrix} -\delta_4 \dot{q}_2 \sin q_2 & -\delta_4(\dot{q}_1 + \dot{q}_2) \sin q_2 \\ \delta_4 \dot{q}_1 \sin q_2 & 0 \end{pmatrix} \tag{A.2}$$

$$g(q) = \begin{pmatrix} \delta_5 \sin q_1 + \delta_6 \sin(q_1 + q_2) \\ \delta_6 \sin(q_1 + q_2) \end{pmatrix} \tag{A.3}$$

$$F = \begin{pmatrix} \delta_7 & 0 \\ 0 & \delta_8 \end{pmatrix} \tag{A.4}$$

where $\delta_1 = 2.351$, $\delta_2 = 0.168$, $\delta_3 = 0.102$, $\delta_4 = 0.084$, $\delta_5 = 38.465$, $\delta_6 = 1.825$, $\delta_7 = 2.288$, and $\delta_8 = 0.175$.

A.1.1 Inertia matrix boundedness

Let us recall Property 1.1 and Assumption 1.1 from Section 1.4, where the existence of the inertia matrix bounds (μ_m and μ_M) was stated. In the following an approximation of such bounds is obtained. The bounds of $H(q)$ are estimated as $\mu_m = \lambda_m(H(q))$ and

$\mu_M = \lambda_M(H(q))$. Then, λ_m and λ_M are obtained from the characteristic equation of $H(q)$,

$$\lambda^2 - \big(\delta_1 + \delta_3 + \delta_2 \cos q_2\big)\lambda + \delta_3\big(\delta_1 + \delta_2 \cos q_2\big) - \big(\delta_3 + \delta_4 \cos q_2\big)^2 = 0$$

whence we have that

$$\begin{aligned} \mu_m &= \min_{q_2 \in \mathbb{R}} \left\{ \delta_a - \sqrt{\delta_a^2 + \big(\delta_3 + \delta_4 \cos q_2\big)^2 - \delta_3\big(\delta_1 + \delta_2 \cos q_2\big)} \right\} \\ \mu_M &= \max_{q_2 \in \mathbb{R}} \left\{ \delta_a + \sqrt{\delta_a^2 + \big(\delta_3 + \delta_4 \cos q_2\big)^2 - \delta_3\big(\delta_1 + \delta_2 \cos q_2\big)} \right\} \end{aligned} \tag{A.5}$$

with

$$\delta_a \triangleq \left(\frac{\delta_1 + \delta_3 + \delta_2 \cos q_2}{2} \right)$$

Through a numerical evaluation of expressions (A.5), which considered values of $q_2 \in [0, 2\pi]$, were obtained $\mu_m = 0.088\,\mathrm{kg\,m^2}$ and $\mu_M = 2.533\,\mathrm{kg\,m^2}$. Furthermore, Assumption 1.1 specifies the existence of bounds $\mu_{M,i}$, $i = 1, 2$, such that $\|H_i(q)\| \leq \mu_{M,i}$. In this case, such bounds are calculated as follows:

$$\begin{aligned} \mu_{M,1} &= \max_{q_2 \in \mathbb{R}} \left\{ \left[\big(\delta_1 + \delta_2 \cos q_2\big)^2 + \big(\delta_3 + \delta_4 \cos q_2\big)^2 \right]^{1/2} \right\} \\ \mu_{M,2} &= \max_{q_2 \in \mathbb{R}} \left\{ \left[\delta_3^2 + \big(\delta_3 + \delta_4 \cos q_2\big)^2 \right]^{1/2} \right\} \end{aligned} \tag{A.6}$$

From the numerical implementation of expressions (A.6), we obtained $\mu_{M1} = 2.5259\,\mathrm{kg\,m^2}$ and $\mu_{M2} = 0.2121\,\mathrm{kg\,m^2}$.

A.1.2 Coriolis and centrifugal effect matrix boundedness

In the following we develop a semi-analytical procedure to obtain an estimation of k_C. Consider the matrix $C(q, \dot{q})$ defined in (A.2), from which we have that

$$\begin{aligned} \|C(q,\dot{q})\dot{q}\|^2 &= \delta_4^2 \sin^2(q_2)\Big[\dot{q}_1^4 + \dot{q}_2^4 + 4(\dot{q}_1\dot{q}_2)^2 + 4(\dot{q}_1\dot{q}_2)\dot{q}_2^2\Big] \\ &\leq \delta_4^2 \sin^2(q_2)\Big[\dot{q}_1^4 + \dot{q}_2^4 + 4(\dot{q}_1\dot{q}_2)^2 + 2(\dot{q}_1^2 + \dot{q}_2^2)\dot{q}_2^2\Big] \\ &= \delta_4^2 \sin^2(q_2)\Big[\dot{q}_1^4 + 3\dot{q}_2^4 + 6(\dot{q}_1\dot{q}_2)^2\Big] \\ &\leq 3\delta_4^2 \sin^2(q_2)\Big[\dot{q}_1^2 + \dot{q}_2^2\Big]^2 = 3\delta_4^2 \sin^2(q_2)\|\dot{q}\|^4 \end{aligned}$$

whence $\|C(q,\dot{q})\dot{q}\| \leq \sqrt{3}\,\delta_4 |\sin q_2| \|\dot{q}\|^2 \leq \sqrt{3}\,\delta_4 \|\dot{q}\|^2$. Recalling that $\|C(q,\dot{q})\| \leq k_C \|\dot{q}\|$ (from Property 1.2.4 and Assumption 1.2), therefore $k_C = \sqrt{3}\,\delta_4 = 0.1454$. Further, $k_{C,i}$, $i = 1, 2$ (see Assumption 1.2), are obtained from considering that $C(q,\dot{q})\dot{q}$ can be rewritten as

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} \dot{q}^T \bar{C}_1(q) \dot{q} \\ \dot{q}^T \bar{C}_2(q) \dot{q} \end{bmatrix} \tag{A.7}$$

where

$$\bar{C}_1(q) = \begin{pmatrix} 0 & -\delta_4 \sin q_2 \\ -\delta_4 \sin q_2 & -\delta_4 \sin q_2 \end{pmatrix}, \quad \bar{C}_2(q) = \begin{pmatrix} \delta_4 \sin q_2 & 0 \\ 0 & 0 \end{pmatrix}$$

Observe, from (A.7), that $\dot{q}^T \bar{C}_i(q)\dot{q} \leq \|\bar{C}_i(q)\|\|\dot{q}\|^2$, with $\|\bar{C}_i(q)\| = \sqrt{\lambda_M\big(\bar{C}_i^T(q)\bar{C}_i(q)\big)} \leq k_{C,i}$, $i = 1, 2$. Thus, for $\bar{C}_1(q)$, we have that

$$\bar{C}_1^T(q)\bar{C}_1(q) = \begin{bmatrix} \delta_4^2 \sin^2(q_2) & \delta_4^2 \sin^2(q_2) \\ \delta_4^2 \sin^2(q_2) & 2\delta_4^2 \sin^2(q_2) \end{bmatrix}$$

$\lambda_i\big(\bar{C}_1^T(q)\bar{C}_1(q)\big)$, $i = 1, 2$, are obtained from

$$\lambda^2 - 3\delta_4^2 \sin^2(q_2)\lambda + \delta_4^4 \sin^4(q_2) = 0$$

whence, $\lambda_M\big(\bar{C}_1^T(q)\bar{C}_1(q)\big) = \frac{3+\sqrt{5}}{2}\delta_4^2 \sin^2(q_2)$ and $\|\bar{C}_1(q)\| = \delta_4|\sin(q_2)|\big((3 + \sqrt{5})/2\big)^{1/2} \leq \delta_4\big((3 + \sqrt{5})/2\big)^{1/2}$, consequently,

$$k_{C,1} = \delta_4 \left(\frac{3 + \sqrt{5}}{2}\right)^{1/2} \tag{A.8}$$

Finally, from (A.8), we get $k_{C,1} = 0.1359\,\text{kg}\,\text{m}^2$. Furthermore, for $\bar{C}_2(q)$, we have that

$$\bar{C}_2^T(q)\bar{C}_2(q) = \begin{bmatrix} \delta_4^2 \sin^2(q_2) & 0) \\ 0 & 0 \end{bmatrix}$$

whence, $\lambda_M\big(\bar{C}_2^T(q)\bar{C}_2(q)\big) = \delta_4^2 \sin^2(q_2)$ and $\|\bar{C}_2(q)\| = \delta_4|\sin(q_2)| \leq \delta_4$, consequently,

$$k_{C,2} = \delta_4$$

from which $k_{C,2} = 0.084\,\text{kg}\,\text{m}^2$.

A.1.3 Conservative force vector and damping-effect matrix boundedness

From the expression of $g(q)$ in (A.3) note that

$$g_1(q) \leq \max_{q\in\mathbb{R}^2}\Big\{\delta_5 \sin q_1 + \delta_6 \sin(q_1 + q_2)\Big\} = B_{g1}$$
$$g_2(q) \leq \max_{q\in\mathbb{R}^2}\Big\{\delta_6 \sin(q_1 + q_2)\Big\} = B_{g2}$$

From a numerical implementation (where we considered values on $\{(q_1, q_2) \in \mathbb{R}^2 : q_1 \in [0, 2\pi], q_2 \in [0, 2\pi]\}$), we get $B_{g1} = 40.29$ and $B_{g2} = 1.825$. On the other hand, recalling F as defined in (A.4), note that

$$\|F_1\| = f_M = \delta_7 = 2.288$$
$$\|F_2\| = f_m = \delta_8 = 0.175$$

A.2 Anthropomorphic-type arm

The dynamical model of the first 3-DOF robot manipulator exposed in Subsection 1.5.2 is recalled here,

$$H(q) = \begin{pmatrix} h_{11}(q) & h_{12}(q) & h_{13}(q) \\ h_{12}(q) & h_{22}(q) & h_{23}(q) \\ h_{13}(q) & h_{23}(q) & \zeta_{13} \end{pmatrix} \tag{A.9}$$

with

$$\begin{aligned}
h_{11}(q) &= \zeta_1 + \zeta_2 \cos^2 q_2 + \zeta_3 \sin 2q_2 + \zeta_4 \sin(2q_2 + 2q_3) + \zeta_5 \cos^2(q_2 + q_3) \\
&\quad + 2l_2\zeta_6 \sin q_2 \sin(q_2 + q_3) + 2l_2\zeta_7 \sin q_2 \cos(q_2 + q_3) \\
h_{12}(q) &= h_{13}(q) + \zeta_{10} \cos q_2 + \zeta_{11} \sin q_2 \\
h_{13}(q) &= \zeta_8 \cos(q_2 + q_3) + \zeta_9 \sin(q_2 + q_3) \\
h_{22}(q) &= \zeta_{12} + 2l_2\zeta_6 \cos q_3 - 2l_2\zeta_7 \sin q_3 \\
h_{23}(q) &= \zeta_{13} + l_2\zeta_6 \cos q_3 - l_2\zeta_7 \sin q_3
\end{aligned}$$

$$C(q, \dot{q}) = \begin{pmatrix} a_1(q)\dot{q}_2 + a_2(q)\dot{q}_3 & a_1(q)\dot{q}_1 + a_3(q)\dot{q}_2 + a_4(q)\dot{q}_3 & a_2(q)\dot{q}_1 + a_4(q)\big(\dot{q}_2 + \dot{q}_3\big) \\ -a_1(q)\dot{q}_1 & -a_5(q)\dot{q}_3 & -a_5(q)\big(\dot{q}_2 + \dot{q}_3\big) \\ -a_2(q)\dot{q}_1 & a_5(q)\dot{q}_2 & 0 \end{pmatrix} : \tag{A.10}$$

$$\begin{aligned}
a_1(q) &= -\frac{\zeta_2}{2} \sin 2q_2 + \zeta_3 \cos 2q_2 + \zeta_4 \cos(2q_2 + 2q_3) - \frac{\zeta_5}{2} \sin(2q_2 + 2q_3) + l_2\zeta_6 \sin(2q_2 + q_3) \\
&\quad + l_2\zeta_7 \cos(2q_2 + q_3) \\
a_2(q) &= \zeta_4 \cos(2q_2 + 2q_3) - \frac{\zeta_5}{2} \sin(2q_2 + 2q_3) + l_2\zeta_6 \sin(q_2) \cos(q_2 + q_3) \\
&\quad - l_2\zeta_7 \sin(q_2) \sin(q_2 + q_3) \\
a_3(q) &= a_4(q) - \zeta_{10} \sin q_2 + \zeta_{11} \cos q_2 \\
a_4(q) &= -\zeta_8 \sin(q_2 + q_3) + \zeta_9 \cos(q_2 + q_3) \\
a_5(q) &= l_2\big(\zeta_6 \sin q_3 + \zeta_7 \cos q_3\big)
\end{aligned}$$

$$g(q) = \begin{pmatrix} 0 \\ g_0\big(\zeta_6 \sin(q_2 + q_3) + \zeta_7 \cos(q_2 + q_3)\big) + \zeta_{14} \sin q_2 + \zeta_{14} \cos q_2 \\ g_0\big(\zeta_6 \sin(q_2 + q_3) + \zeta_7 \cos(q_2 + q_3)\big) \end{pmatrix} \tag{A.11}$$

$$F = \begin{pmatrix} \zeta_{16} & 0 & 0 \\ 0 & \zeta_{17} & 0 \\ 0 & 0 & \zeta_{18} \end{pmatrix} \tag{A.12}$$

In the expressions above: $l_2 = 0.35$ m and $g_0 = 9.81\,\mathrm{m/s^2}$. The system parameters ζ_i, $i = \overline{1, 18}$, were identified (see 1.5.2) as

$\zeta_1 = 2.8968$	$\zeta_7 = 6.828 \times 10^{-3}$	$\zeta_{13} = 8.9485 \times 10^{-2}$
$\zeta_2 = -0.35456$	$\zeta_8 = 7.9808 \times 10^{-2}$	$\zeta_{14} = 16.4906$
$\zeta_3 = 7.6885 \times 10^{-4}$	$\zeta_9 = -4.5168 \times 10^{-3}$	$\zeta_{15} = 0.202478$
$\zeta_4 = -6.5428 \times 10^{-5}$	$\zeta_{10} = 0.89111$	$\zeta_{16} = 0.4$
$\zeta_5 = -5.5442 \times 10^{-3}$	$\zeta_{11} = 2.5675 \times 10^{-3}$	$\zeta_{17} = 1.2806$
$\zeta_6 = 0.11152$	$\zeta_{12} = 1.2607$	$\zeta_{18} = 0.64$

A.2.1 Inertia matrix boundedness

Similar to Subsection A.1.1, the goal is to approximate the bounds stated through Property 1.1 and Assumption 1.1 corresponding to $H(q)$ defined in (A.9). With this goal in mind, the eigenvalues of $H(q)$, $\lambda_i(H(q))$, $i = 1, 2, 3$, are obtained from its characteristic equation,

$$\begin{aligned}\lambda^3 - \Big[h_{11}(q) + h_{22}(q) + h_{33}(q)\Big]\lambda^2 \\ + \Big[h_{11}(q)h_{22}(q) + h_{11}(q)h_{33}(q) + h_{22}(q)h_{33}(q) - h_{12}^2(q) - h_{13}^2(q) - h_{23}^2(q)\Big]\lambda \\ - \Big[2h_{12}(q)h_{13}(q)h_{23}(q) + h_{11}(q)h_{22}(q)h_{33}(q) - h_{11}(q)h_{23}^2(q) - h_{22}(q)h_{13}^2(q) - h_{33}(q)h_{12}^2(q)\Big] = 0\end{aligned}$$

whence $\mu_m = \min_{q \in \mathbb{R}^3}\{\lambda_i(H(q))\}$ and $\mu_M = \max_{q \in \mathbb{R}^3}\{\lambda_i(H(q))\}, i = 1, 2, 3$. Thus, from a numerical estimation (by taking values on $\{(q_1, q_1, q_3) \in \mathbb{R}^3 : q_1 \in [0, 2\pi], q_2 \in [0, 2\pi], q_3 \in [0, 2\pi]\}$), we get $\mu_m = 0.0761$ and $\mu_M = 3.0846$. In this case, the bounds $\mu_{M,i}$, $i = 1, 2, 3$ are defined as

$$\begin{aligned}\mu_{M,1} &= \max_{q \in \mathbb{R}^3}\left\{\Big[h_{11}^2(q) + h_{12}^2(q) + h_{13}^2(q)\Big]^{1/2}\right\} \\ \mu_{M,2} &= \max_{q \in \mathbb{R}^3}\left\{\Big[h_{12}^2(q) + h_{22}^2(q) + h_{23}^2(q)\Big]^{1/2}\right\} \\ \mu_{M,3} &= \max_{q \in \mathbb{R}^3}\left\{\Big[h_{13}^2(q) + h_{23}^2(q) + h_{33}^2(q)\Big]^{1/2}\right\}\end{aligned} \tag{A.13}$$

From the numerical estimation of the expressions in (A.13), we have that $\mu_{M,1} = 2.975$, $\mu_{M,2} = 1.6589$ and $\mu_{M,3} = 0.1757$.

A.2.2 Coriolis and centrifugal effect matrix boundedness

In the following, a k_C estimation is obtained through developments similar to those made in Subsection A.1.2. Consider the matrix $C(q, \dot{q})$ defined in (A.10), from which we have

$$
\begin{aligned}
\|C(q,\dot{q})\dot{q}\|^2 = & \Big[a_1^2(q)+a_2^2(q)\Big]\dot{q}_1^4 + \Big[a_3^2(q)+a_5^2(q)\Big]\dot{q}_2^4 + \Big[a_4^2(q)+a_5^2(q)\Big]\dot{q}_3^4 \\
& + 2\Big[2a_1^2(q) - a_2(q)a_5(q)\Big](\dot{q}_1\dot{q}_2)^2 + 2\Big[2a_2^2(q)+a_1(q)a_5(q)\Big](\dot{q}_1\dot{q}_3)^2 \\
& + 2\Big[2\big(a_4^2(q)+a_5^2(q)\big) + a_3(q)a_4(q)\Big](\dot{q}_2\dot{q}_3)^2 + 4a_1(q)\Big[2a_2(q)+a_5(q)\Big](\dot{q}_2\dot{q}_3)\dot{q}_1^2 \\
& + 4\Big[a_1(q)a_3(q)(\dot{q}_1\dot{q}_2) + \big(2a_1(q)a_4(q)+a_2(q)a_3(q)\big)(\dot{q}_1\dot{q}_3) + a_3(q)a_4(q)(\dot{q}_2\dot{q}_3)\Big]\dot{q}_2^2 \\
& + 4\Big[a_4(q)\big(a_1(q)+2a_2(q)\big)(\dot{q}_1\dot{q}_2) + a_2(q)a_4(q)(\dot{q}_1\dot{q}_3) + \big(a_4^2(q)+a_5^2(q)\big)(\dot{q}_2\dot{q}_3)\Big]\dot{q}_3^2 \\
\le & \Big[a_1^2(q)+a_2^2(q)\Big]\dot{q}_1^4 + \Big[a_3^2(q)+a_5^2(q)+2\big(|a_1(q)a_3(q)|+|a_3(q)a_4(q)|\big)\Big]\dot{q}_2^4 \\
& + \Big[3\big(a_4^2(q)+a_5^2(q)\big)+2|a_2(q)a_4(q)|\Big]\dot{q}_3^4 \\
& + 2\Big[2a_1^2(q)+\bar{\beta}_1(q)+\bar{\beta}_2(q)+|a_1(q)a_3(q)|-a_2(q)a_5(q)\Big](\dot{q}_1\dot{q}_2)^2 \\
& + 2\Big[2a_2^2(q)+\bar{\beta}_2(q)+\bar{\beta}_3(q)+|a_2(q)a_4(q)|+a_1(q)a_5(q)\Big](\dot{q}_1\dot{q}_3)^2 \\
& + 2\Big[3\big(a_4^2(q)+a_5^2(q)\big)+\bar{\beta}_1(q)+\bar{\beta}_3(q)+|a_3(q)a_4(q)|+a_3(q)a_4(q)\Big](\dot{q}_2\dot{q}_3)^2 \\
\le & \bar{k}_C\Big[\dot{q}_1^4+\dot{q}_2^4+\dot{q}_3^4+2(\dot{q}_1\dot{q}_2)^2+2(\dot{q}_1\dot{q}_3)^2+2(\dot{q}_2\dot{q}_3)^2\Big] \\
= & \bar{k}_C\Big[\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2\Big]^2 = \bar{k}_C\|\dot{q}\|^4
\end{aligned}
\tag{A.14}
$$

with

$$
\begin{aligned}
\bar{k}_C \triangleq & \max_{q\in\mathbb{R}^3}\Big\{a_1^2(q)+a_2^2(q),\ a_3^2(q)+a_5^2(q)+2\big(|a_1(q)a_3(q)|+|a_3(q)a_4(q)|\big), \\
& 3\big(a_4^2(q)+a_5^2(q)\big)+2|a_2(q)a_4(q)|, \\
& 2a_1^2(q)+\bar{\beta}_1(q)+\bar{\beta}_2(q)+|a_1(q)a_3(q)|-a_2(q)a_5(q), \\
& 2a_2^2(q)+\bar{\beta}_2(q)+\bar{\beta}_3(q)+|a_2(q)a_4(q)|+a_1(q)a_5(q), \\
& 3\big(a_4^2(q)+a_5^2(q)\big)+\bar{\beta}_1(q)+\bar{\beta}_3(q)+|a_3(q)a_4(q)|+a_3(q)a_4(q)\Big\} \\
\bar{\beta}_1(q) \triangleq & |2a_1(q)a_4(q)+a_2(q)a_3(q)| \\
\bar{\beta}_2(q) \triangleq & |a_1(q)(2a_2(q)+a_5(q))| \\
\bar{\beta}_3(q) \triangleq & |a_4(q)(a_1(q)+2a_2(q))|
\end{aligned}
$$

Note, from (A.14), that $\|C(q,\dot{q})\dot{q}\| \le \sqrt{\bar{k}_C}\|\dot{q}\|^2$, whence $k_C = \sqrt{\bar{k}_C}$ (recalling that $\|C(q,\dot{q})\| \le k_C\|\dot{q}\|$ from Assumption 1.2.4). Then, we obtain from a numerical estimation $k_C = 1.1116$. Further, $k_{C,i}$, $i=1,2,3$, are obtained from considering that $C(q,\dot{q})\dot{q}$ can be rewritten as

$$
C(q,\dot{q})\dot{q} = \begin{pmatrix} \dot{q}^T\hat{C}_1(q)\dot{q} \\ \dot{q}^T\hat{C}_2(q)\dot{q} \\ \dot{q}^T\hat{C}_3(q)\dot{q} \end{pmatrix}
$$

where

$$\hat{C}_1(q) \triangleq \begin{bmatrix} 0 & a_1(q) & a_2(q) \\ a_1(q) & a_3(q) & a_4(q) \\ a_2(q) & a_4(q) & a_4(q) \end{bmatrix}, \quad \hat{C}_2(q) \triangleq \begin{bmatrix} -a_1(q) & 0 & 0 \\ 0 & 0 & -a_5(q) \\ 0 & -a_5(q) & -a_5(q) \end{bmatrix}$$

$$\hat{C}_3(q) \triangleq \begin{bmatrix} -a_2(q) & 0 & 0 \\ 0 & a_5(q) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that $\dot{q}^T \hat{C}_i(q)\dot{q} \leq \|\hat{C}_i(q)\|\|\dot{q}\|^2$, $i = 1, 2, 3$, with $\|\hat{C}_i(q)\| = \sqrt{\lambda_M\big(\bar{C}_i^T(q)\hat{C}_i(q)\big)} \leq k_{C,i}$, $i = 1, 2, 3$. Thus, for $\hat{C}_1(q)$, we have that

$$\hat{C}_1^T(q)\hat{C}_1(q) = \begin{pmatrix} \hat{c}_{11}(q) & \hat{c}_{12}(q) & \hat{c}_{13}(q) \\ \hat{c}_{12}(q) & \hat{c}_{22}(q) & \hat{c}_{23}(q) \\ \hat{c}_{13}(q) & \hat{c}_{23}(q) & \hat{c}_{33}(q) \end{pmatrix}$$

with

$$\begin{array}{ll} \hat{c}_{11}(q) = a_1^2(q) + a_2^2(q) & \hat{c}_{22}(q) = a_1^2(q) + a_3^2(q) + a_4^2(q) \\ \hat{c}_{12}(q) = a_1(q)a_3(q) + a_2(q)a_4(q) & \hat{c}_{23}(q) = a_1(q)a_2(q) + a_4(q)\big(a_3(q) + a_4(q)\big) \\ \hat{c}_{13}(q) = a_4(q)\big(a_1(q) + a_2(q)\big) & \hat{c}_{33}(q) = a_2^2(q) + 2a_4^2(q) \end{array}$$

then, $\lambda_i\big(\hat{C}_1^T(q)\hat{C}_1(q)\big)$ are obtained from

$$\begin{aligned} &\lambda^3 - \Big[\hat{c}_{11}(q) + \hat{c}_{22}(q) + \hat{c}_{33}(q)\Big]\lambda^2 \\ &\quad + \Big[\hat{c}_{11}(q)\hat{c}_{22}(q) + \hat{c}_{11}(q)\hat{c}_{33}(q) + \hat{c}_{22}(q)\hat{c}_{33}(q) - \hat{c}_{12}^2(q) - \hat{c}_{13}^2(q) - \hat{c}_{23}^2(q)\Big]\lambda \\ &\quad - \Big[2\hat{c}_{12}(q)\hat{c}_{13}(q)\hat{c}_{23}(q) + \hat{c}_{11}(q)\hat{c}_{22}(q)\hat{c}_{33}(q) - \hat{c}_{11}(q)\hat{c}_{23}^2(q) - \hat{c}_{22}(q)\hat{c}_{13}^2(q) - \hat{c}_{33}(q)\hat{c}_{12}^2(q)\Big] = 0 \end{aligned} \tag{A.15}$$

A numerical implementation was run in order to get the solutions of (A.15) and we get $\Big(\lambda_M\big(\hat{C}_1^T(q)\hat{C}_1(q)\big)\Big)^{1/2} \leq 0.98 = k_{C,1}$. Further, for $\hat{C}_2(q)$, we have that

$$\hat{C}_2^T(q)\hat{C}_2(q) = \begin{pmatrix} a_1^2(q) & 0 & 0 \\ 0 & a_5^2(q) & a_5^2(q) \\ 0 & a_5^2(q) & 2a_5^2(q) \end{pmatrix}$$

from the expression of $\hat{C}_2^T(q)\hat{C}_2(q)$, we calculate $\lambda_i\big(\hat{C}_2^T(q)\hat{C}_2(q)\big)$, $i = 1, 2, 3$, from

$$\lambda^3 - \Big[a_1^2(q) + 3a_5^2(q)\Big]\lambda^2 + a_5^2(q)\Big[3a_1^2(q) + a_5^2(q)\Big]\lambda - a_1^2(q)a_5^4(q) = 0$$

from the calculations we obtained $\Big(\lambda_M\big(\hat{C}_2^T(q)\hat{C}_2(q)\big)\Big)^{1/2} \leq 0.39 = k_{C,2}$. Finally, from the definition of $\hat{C}_3(q)$, we get that

$$\hat{C}_3^T(q)\hat{C}_3(q) = \begin{pmatrix} a_2^2(q) & 0 & 0 \\ 0 & a_5^2(q) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The values of $\lambda_i(\hat{C}_3(q)\hat{C}_3^T(q))$ are $\lambda_1 = a_2^2(q)$, $\lambda = a_5^2(q)$ and $\lambda_3 = 0$, whence $\lambda_M(\hat{C}_3^T(q)\hat{C}_3(q)) = \max_{(q\,\in\,\mathbb{R}^3)}\{a_2^2(q),\ a_5^2(q)\}$. The numerical calculation showed that $\left(\lambda_M(\hat{C}_3^T(q)\hat{C}_3(q))\right)^{1/2} \leq 0.04 = k_{C,3}$.

A.2.3 Conservative force vector and damping-effect matrix boundedness

From the expression of $g(q)$ in (A.11) note that $B_{g1} = 0$ and

$$\begin{aligned} g_2(q) &\leq \max_{q\in\mathbb{R}^3}\Big\{g_0\big(\zeta_6\sin(q_2+q_3)+\zeta_7\cos(q_2+q_3)\big)+\zeta_{14}\sin q_2+\zeta_{14}\cos q_2\Big\} = B_{g2} \\ g_3(q) &\leq \max_{q\in\mathbb{R}^3}\Big\{g_0\big(\zeta_6\sin(q_2+q_3)+\zeta_7\cos(q_2+q_3)\big)\Big\} = B_{g3} \end{aligned}$$

whence, after numerical calculations (by taking values on $\{(q_1,q_1,q_3)\in\mathbb{R}^3 : q_1\in[0,2\pi], q_2\in[0,2\pi], q_3\in[0,2\pi]\}$), $B_{g2} = 17.5879$ and $B_{g3} = 1.0961$. On the other hand, from the definition of F in (A.12), note that $\|F_1\| = \zeta_{16} = 0.4$, $\|F_2\| = \zeta_{17} = 1.2806$, $\|F_3\| = \zeta_{18} = 0.64$, $f_M = \zeta_{17}$ and $f_m = \zeta_{16}$.

A.3 Phantom haptic interface robot

We rewrite here the dynamical model of the Phantom robot exposed in Subsection 1.5.2,

$$H(q) = \begin{pmatrix} h_{11}(q) & 0 & 0 \\ 0 & h_{22}(q) & h_{23}(q) \\ 0 & h_{23}(q) & h_{33}(q) \end{pmatrix} \tag{A.16}$$

with

$$\begin{aligned} h_{11}(q) &= [28.33+11.32\cos(2q_2)-3.91\cos(2q_3)+9.12\cos(q_2)\sin(q_3)]\times 10^{-4} \\ h_{22}(q) &= 24.26\times 10^{-4} \\ h_{23}(q) &= -[4.56\sin(q_2-q_3)]\times 10^{-4} \\ h_{33}(q) &= 9.32\times 10^{-4} \end{aligned}$$

$$C(q,\dot q) = \begin{pmatrix} a_1(q)\dot q_2 + a_2(q)\dot q_3 & a_1(q)\dot q_1 & a_2(q)\dot q_1 \\ -a_1(q)\dot q_1 & 0 & a_3(q)\dot q_3 \\ -a_2(q)\dot q_1 & -a_3(q)\dot q_2 & 0 \end{pmatrix} : \tag{A.17}$$

$$\begin{aligned} a_1(q) &= -[11.32\sin(2q_2) + 4.56\sin(q_2)\sin(q_3)] \times 10^{-4} \\ a_2(q) &= [3.91\sin(2q_3) + 4.56\cos(q_2)\cos(q_3)] \times 10^{-4} \\ a_3(q) &= [4.56\cos(q_2 - q_3)] \times 10^{-4} \end{aligned}$$

$$g(q) = \begin{pmatrix} 0 \\ -162.98\cos(q_2) \\ -737.55\sin(q_3) \end{pmatrix} \times 10^{-4} \tag{A.18}$$

A.3.1 Inertia matrix boundedness

Similar to the previously developed, we focused on the calculation of the bounds μ_m, μ_M and $\mu_{M,i}$, $i = 1, 2, 3$, for $H(q)$ defined in (A.16). With this goal in mind, the eigenvalues of $H(q)$, $\lambda_i(H(q))$, $i = 1, 2, 3$, are obtained from the characteristic equation,

$$\lambda^3 - \Big[h_{11}(q) + h_{22}(q) + h_{33}(q)\Big]\lambda^2 + \Big[h_{11}(q)h_{22}(q) + h_{11}(q)h_{33}(q) + h_{22}(q)h_{33}(q) - h_{23}^2(q)\Big]\lambda \\ - \Big[h_{11}(q)h_{22}(q)h_{33}(q) - h_{11}(q)h_{23}^2(q)\Big] = 0$$

whence $\mu_m = \min_{q\in\mathbb{R}^3}\{\lambda_i(H(q))\}$ and $\mu_M = \max_{q\in\mathbb{R}^3}\{\lambda_i(H(q))\}, i = 1, 2, 3$, thus, from a numerical estimation (similar to that made in the previous subsections), we get $\mu_m = 8.04 \times 10^{-4}$ and $\mu_M = 25 \times 10^{-4}$. In this case, the bounds $\mu_{M,i}$, $i = 1, 2, 3$ are defined as

$$\begin{aligned} \mu_{M,1} &= \max_{q\,\in\,\mathbb{R}^3}\big\{|h_{11}(q)|\big\} \\ \mu_{M,2} &= \max_{q\,\in\,\mathbb{R}^3}\Big\{\Big[h_{22}^2(q) + h_{23}^2(q)\Big]^{1/2}\Big\} \\ \mu_{M,3} &= \max_{q\,\in\,\mathbb{R}^3}\Big\{\Big[h_{23}^2(q) + h_{33}^2(q)\Big]^{1/2}\Big\} \end{aligned} \tag{A.19}$$

from the numerical estimation of the expressions in (A.19), we have that $\mu_{M,1} = 53\times10^{-4}$, $\mu_{M,2} = 25 \times 10^{-4}$ and $\mu_{M,3} = 10 \times 10^{-4}$.

A.3.2 Coriolis and centrifugal effect matrix boundedness

An estimation of k_C is obtained through the following developments, which are similar to those made in the previous subsection. Consider the matrix $C(q,\dot q)$ defined in (A.17), from which we have

$$
\begin{aligned}
\|C(q,\dot q)\dot q\|^2 &= 2a_1^2(q)\dot q_1^4 + a_3^2(q)\dot q_2^4 + a_3^2(q)\dot q_3^4 + 2a_1(q)\big[2a_1(q)+a_3(q)\big](\dot q_1\dot q_2)^2 \\
&\quad + 2\Big[2a_2^2(q) - a_1(q)a_3(q)\Big](\dot q_1\dot q_3)^2 + 8a_1(q)a_2(q)(\dot q_2\dot q_3)\dot q_1^2 \\
&\le 2a_1^2(q)\dot q_1^4 + a_3^2(q)\dot q_2^4 + a_3^2(q)\dot q_3^4 \\
&\quad + 2\big[a_1(q)(2a_1(q)+a_3(q)) + 2|a_1(q)a_2(q)|\big](\dot q_1\dot q_2)^2 \\
&\quad + 2\Big[2(a_2^2(q) + |a_1(q)a_2(q)|) - a_1(q)a_3(q)\Big](\dot q_1\dot q_3)^2 \\
&\le \hat k_C\big[\dot q_1^4 + \dot q_2^4 + \dot q_3^4 + 2(\dot q_1\dot q_2)^2 + 2(\dot q_1\dot q_3)^2 + 2(\dot q_2\dot q_3)^2\big] \\
&= \hat k_C\|\dot q\|^4
\end{aligned}
\tag{A.20}
$$

with

$$
\begin{aligned}
\hat k_C =& \max_{q\in\mathbb{R}^3}\Big\{2a_1^2(q),\ a_3^2(q),\ a_1(q)(2a_1(q)+a_3(q)) + 2|a_1(q)a_2(q)|, \\
& 2(a_2^2(q) + |a_1(q)a_2(q)|) - a_1(q)a_3(q)\Big\}
\end{aligned}
$$

Note, from (A.20), that $\|C(q,\dot q)\dot q\| \le \sqrt{\hat k_C}\|\dot q\|^2$, whence $k_C = \sqrt{\hat k_C}$ (recalling that $\|C(q,\dot q)\| \le k_C\|\dot q\|$ from Assumption 1.2.4). Then, we obtained from a numerical estimation $k_C = 24\times 10^{-4}$. Further, $k_{C,i}$, $i=1,2,3$, are obtained from considering that $C(q,\dot q)\dot q$ can be rewritten as

$$
C(q,\dot q)\dot q = \begin{pmatrix} \dot q^T C_1(q)\dot q \\ \dot q^T C_2(q)\dot q \\ \dot q^T C_3(q)\dot q \end{pmatrix}
$$

where

$$
C_1(q) \triangleq \begin{bmatrix} 0 & a_1(q) & a_2(q) \\ a_1(q) & 0 & 0 \\ a_2(q) & 0 & 0 \end{bmatrix}, \quad C_2(q) \triangleq \begin{bmatrix} -a_1(q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3(q) \end{bmatrix}
$$

$$
C_3(q) \triangleq \begin{bmatrix} -a_2(q) & 0 & 0 \\ 0 & -a_3(q) & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Observe that $\dot q^T C_i(q)\dot q \le \|C_i(q)\|\|\dot q\|^2$, $i=1,2,3$, with $\|C_i(q)\| = \sqrt{\lambda_M\big(C_i^T(q)C_i(q)\big)} \le k_{C,i}$, $i=1,2,3$. Thus, for $C_1(q)$, we have that

$$
C_1^T(q)C_1(q) = \begin{pmatrix} a_1^2(q)+a_2^2(q) & 0 & 0 \\ 0 & a_1^2(q) & a_1(q)a_2(q) \\ 0 & a_1(q)a_2(q) & a_2^2(q) \end{pmatrix}
$$

then, $\lambda_i\big(C_1^T(q)C_1(q)\big)$, $i=1,2,3$, are obtained as $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = a_1^2(q) + a_2^2(q)$. From this, we have that $\lambda_M\big(C_1^T(q)C_1(q)\big) = \max_{q\in\mathbb{R}^3}\{a_1^2(q)+a_2^2(q)\}$. A numerical implementation was run and we get $\Big(\lambda_M\big(C_1^T(q)C_1(q)\big)\Big)^{1/2} \leq 15 \times 10^{-4} = k_{C,1}$. Further, for $C_2(q)$, we have that

$$C_2^T(q)C_2(q) = \begin{pmatrix} a_1^2(q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3^2(q) \end{pmatrix}$$

from the expression of $C_2^T(q)C_2(q)$, we get $\lambda_1 = a_1^2(q)$, $\lambda_2 = 0$ and $\lambda_3 = a_3^2(q)$. From this, we have that $\lambda_M\big(C_2^T(q)C_2(q)\big) = \max_{q\in\mathbb{R}^3}\{a_1^2(q),\, a_3^2(q)\}$ and numerically we get $\Big(\lambda_M\big(C_2^T(q)C_2(q)\big)\Big)^{1/2} \leq 15 \times 10^{-4} = k_{C,2}$. Finally, from the definition of $C_3(q)$, we get that

$$C_3^T(q)C_3(q) = \begin{pmatrix} a_2^2(q) & 0 & 0 \\ 0 & a_3^2(q) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The values of $\lambda_i(C_3^T(q)C_3(q))$, $i = 1, 2, 3$, result to be $\lambda_1 = a_2^2(q)$, $\lambda_2 = a_3^2(q)$ and $\lambda_3 = 0$, whence $\lambda_M(C_3^T(q)C_3(q)) = \max_{q\,\in\,\mathbb{R}^3}\{a_2^2(q),\, a_3^2(q)\}$. The numerical calculation showed that $\Big(\lambda_M(C_3^T(q)C_3(q))\Big)^{1/2} \leq 7.41 \times 10^{-4} = k_{C,3}$.

A.3.3 Conservative force vector

From the expression of $g(q)$ in (A.18) note that $B_{g1} = 0$ and

$$g_2(q) \leq \max_{q_2\in\mathbb{R}}\Big\{ -162.98\cos(q_2)\Big\} \times 10^{-4} = B_{g2}$$
$$g_3(q) \leq \max_{q_3\in\mathbb{R}}\Big\{ -737.55\sin(q_3)\Big\} \times 10^{-4} = B_{g3}$$

whence $B_{g2} = 162.98 \times 10^{-4}$ and $B_{g3} = 737.55 \times 10^{-4}$.

APPENDIX B

On the conditions of the desired conservative force compensation control scheme

B.1 On inequalities (5.1)

Note from (3.11) that $\beta_0\beta_1 = (2-\gamma)/\gamma$ and (in accordance to Corollaries 3.3 and 3.4) that

$$1 \le \gamma < 2 \iff 0 < \frac{2-\gamma}{\gamma} \le 1 \iff 0 < \beta_0\beta_1 \le 1$$

Observe that on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \le 2B_{gj}/k_g\}$ we have that

$$|\varsigma| \le \frac{2B_{gj}}{k_g} \iff |\varsigma|^{1-\beta_0\beta_1} \le \Big(\frac{2B_{gj}}{k_g}\Big)^{1-\beta_0\beta_1} \iff k_{1j}^{\beta_0\beta_1}\Big(\frac{2B_{gj}}{k_g}\Big)^{\beta_0\beta_1-1}|\varsigma| \le |k_{1j}\varsigma|^{\beta_0\beta_1}$$

while from (5.1a) we have, for all $\varsigma \neq 0$, that:

$$\begin{aligned}
k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} &\iff k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1}|\varsigma|^{1/\beta_0\beta_1} < k_{1j}|\varsigma|^{1/\beta_0\beta_1} \\
&\iff k_g^{\beta_0\beta_1}(2B_{gj})^{1-\beta_0\beta_1}|\varsigma| < k_{1j}^{\beta_0\beta_1}|\varsigma| \\
&\iff k_g|\varsigma| < k_{1j}^{\beta_0\beta_1}\Big(\frac{2B_{gj}}{k_g}\Big)^{\beta_0\beta_1-1}|\varsigma|
\end{aligned}$$

From these developments we thus get, on

$$\Big\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \le \min\Big\{\frac{2B_{gj}}{k_g},\ \frac{L_0^{1/\beta_1}}{k_{1j}}\Big\}\Big\}$$

that

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \iff k_g|\varsigma| < |k_{1j}\varsigma|^{\beta_0\beta_1}$$

and consequently, for all $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \le L_0^{1/\beta_1}/k_{1j}\}$ that

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \iff \min\{k_g|\varsigma|, 2B_{gj}\} < |k_{1j}\varsigma|^{\beta_0\beta_1}$$

whence, under the additional consideration of (5.1b), we get that:

$$\begin{aligned}
(5.1) \Longrightarrow \min\{k_g|\varsigma|, 2B_{gj}\} &< \begin{cases} |k_{1j}\varsigma|^{\beta_0\beta_1} & \text{if} \quad |\varsigma| \le \frac{L_{0j}^{1/\beta_1}}{k_{1j}} \\ L_{0j}^{\beta_0} + (M_{0j} - L_{0j}^{\beta_0})\tanh\left(\frac{|k_{1j}\varsigma|^{\beta_0\beta_1} - L_{0j}^{\beta_0}}{M_{0j} - L_{0j}^{\beta_0}}\right) & \text{if} \quad |\varsigma| > \frac{L_{0j}^{1/\beta_1}}{k_{1j}} \end{cases} \\
&= \begin{cases} |\sigma_{1j}(k_{1j}\varsigma)|^{\beta_0} & \text{if} \quad |\sigma_{1j}(k_{1j}\varsigma)| \le L_{0j} \\ \sigma_{bs}^{+}(|\sigma_{1j}(k_{1j}\varsigma)|; \beta_0, 0, L_{0j}, M_{0j}) & \text{if} \quad |\sigma_{1j}(k_{1j}\varsigma)| > L_{0j} \end{cases} \\
&= \left|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\right|
\end{aligned}$$

$\forall \varsigma \ne 0$.

B.2 On inequalities (5.3)

Since (5.1a) and (5.3a) are analog inequalities, we have on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \le 2B_{gj}/k_g\}$ that

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \iff k_g|\varsigma| < |k_{1j}\varsigma|^{\beta_0\beta_1}$$

and consequently, for all $\varsigma \ne 0$ that

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \iff \min\{k_g|\varsigma|, 2B_{gj}\} < |k_{1j}\varsigma|^{\beta_0\beta_1}$$

whence, under the additional consideration of (5.3b), we get that

$$(5.3) \iff \min\{k_g|\varsigma|, 2B_{gj}\} < \min\{|k_{1j}\varsigma|^{\beta_0\beta_1}, M_{1j}^{\beta_0}\} = |\sigma_{1j}(k_{1j}\varsigma)|^{\beta_0} = |\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)| \quad \forall \varsigma \ne 0$$

B.3 On inequalities (4.5) and (5.4)

Observe that on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \le 2B_{gj}/k_g\}$ we have that (recall from Corollaries 3.5 and 3.6):

$$|\varsigma| \le \frac{2B_{gj}}{k_g} \iff |\varsigma|^{1-\beta_1} \le \left(\frac{2B_{gj}}{k_g}\right)^{1-\beta_1} \iff k_{1j}^{\beta_1}\left(\frac{2B_{gj}}{k_g}\right)^{\beta_1-1}|\varsigma| \le |k_{1j}\varsigma|^{\beta_1}$$

while from (5.4a) we have, for all $\varsigma \ne 0$, that:

$$
\begin{aligned}
k_{1j} > k_g(2B_{gj})^{1-\beta_1/\beta_1} &\iff k_g(2B_{gj})^{(1-\beta_1)/\beta_1}|\varsigma|^{1/\beta_1} < k_{1j}|\varsigma|^{1/\beta_1} \\
&\iff k_g^{\beta_1}(2B_{gj})^{1-\beta_1}|\varsigma| < k_{1j}^{\beta_1}|\varsigma| \\
&\iff k_g|\varsigma| < k_{1j}^{\beta_1}\left(\frac{2B_{gj}}{k_g}\right)^{\beta_1-1}|\varsigma|
\end{aligned}
$$

From these developments we thus get, on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \leq 2B_{gj}/k_g\}$, that:

$$
k_{1j} > k_g(2B_{gj})^{1-\beta_1/\beta_1} \iff k_g|\varsigma| < |k_{1j}\varsigma|^{\beta_1}
$$

and consequently, for all $\varsigma \neq 0$, that:

$$
k_{1j} > k_g(2B_{gj})^{1-\beta_1/\beta_1} \iff \min\{k_g|\varsigma|,\, 2B_{gj}\} < |k_{1j}\varsigma|^{\beta_1}
$$

whence, under the additional consideration of (5.4b), we get that:

$$
(5.4) \Longrightarrow \min\{k_g|\varsigma|,\, 2B_{gj}\} < \min\{|k_{1j}\varsigma|^{\beta_1},\, M_{1j}\} = |\sigma ij(k_{1j}\varsigma)| \quad \forall\varsigma \neq 0
$$

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APPENDIX C

Published results

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Local-homogeneity-based global continuous control for mechanical systems with constrained inputs: finite-time and exponential stabilisation

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ABSTRACT

A global continuous control scheme for the finite-time or (local) exponential stabilisation of mechanical systems with constrained inputs is proposed. The approach is formally developed within the theoretical framework of local homogeneity. This has permitted to solve the formulated problem not only guaranteeing input saturation avoidance but also giving a wide range of design flexibility. The proposed scheme is characterised by a saturating-proportional-derivative type term with generalised saturating and locally homogeneous structure that permits multiple design choices on both aspects. The work includes a simulation implementation section where the veracity of the so-cited argument claiming that finite-time stabilisers are faster than asymptotical ones is studied. In particular, a way to carry out the design so as to, indeed, guarantee faster stabilisation through finite-time controllers (beyond their finite-time convergence) is shown.

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Finite-time stabilisation; local homogeneity; mechanical systems; constrained inputs; saturation

1. Introduction

Continuous control aiming at the finite-time convergence of an equilibrium being (simultaneously) rendered stable has been a topic of increasing interest in the last decades. Inspired by the seminal work of Haimo (1986), several researchers have devoted efforts to settle down a suitable underlying analytical framework for the subject. Important contributions in this direction are those due to Bhat and Bernstein (1995, 1997, 1998, 2000, 2005), by formally stating a precise definition of *finite-time stability* that gathers both the (Lyapunov) stability and finite-time convergence, thoroughly developing Lyapunov-based criteria for its determination, and clearly characterising its relationship with homogeneous vector fields. This latter characterisation has been particularly attractive in view of its simplicity: for a homogeneous vector field with asymptotically stable equilibrium at the origin, verifying negativity of the homogeneity degree suffices to conclude finite-time stability (of the origin). This naturally leads to the idea of involving homogeneity in control design to readily achieve finite-time stabilisation. Nevertheless, such a strategy is tied to the requirements imposed by homogeneity, which is (conventionally) a *global* property. For instance, in a coordinate-dependent framework, a vector field with bounded components cannot be homogeneous (Bhat & Bernstein, 2005). Consequently, within such a framework, the referred strategy cannot be applied under bounded input constraints. Nevertheless, such a design restriction has been proven to be relaxed through alternative notions of homogeneity (Zavala-Río & Fantoni, 2014).

Based on the theoretical framework of local homogeneity (details are given in Section 2), this work proposes a bounded continuous control design method for constrained-input mechanical systems, guaranteeing global stabilisation with either finite-time or (local) expo nential convergence. The choice upon the type of convergence is simply stated through a design parameter involved in the control scheme. Such a choice is made possible through a suitable extension (stated in this paper) of the theoretical framework of local homogeneity; interesting enough, within the design context developed in this work, such an extension permits exponential stabilisation through unconventional control structures. The finite-time stabilisation choice of the proposed approach – achieved through bounded inputs – remains, however, the main motivation and original goal of the present work. This is motivated by the advantages of finite-time controllers that are generally claimed in relation to asymptotic ones – such as faster convergence and improved robustness to uncertainties (Hong, Wang, & Cheng, 2006; Huang, Lin, & Yang, 2005; Qian & Li, 2005) – as well as their conceptual suitability for certain tasks such as *consensus* (Wang & Xiao, 2010) and *formation* (Xiao, Wang, Chen, & Gao, 2009) of multi-agent systems.

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Observer-less output-feedback global continuous control for the finite-time and exponential stabilization of mechanical systems with constrained inputs

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ABSTRACT

An observer-less output-feedback global continuous control scheme for the finite-time or (local) exponential stabilization of mechanical systems with constrained inputs is proposed. The approach is formally developed within the theoretical framework of local homogeneity. The closed-loop analysis incorporates a complementary insight on the control-induced motion dissipation through an *ad hoc* feedback-system passivity theorem. The work includes a simulation implementation section where the performance difference of the proposed scheme with previous observer-based and differentiation algorithms is brought to the fore.

1. Introduction

The last decades have witnessed an increasing interest on stabilization with finite-time convergence through continuous feedback. Such an intriguing topic is traced back to the seminal work of Haimo in [13], where finite-time stability on second-order (*double integrator*) systems of the form

$$\ddot{x} = u \tag{1}$$

with $u = u(x, \dot{x})$ continuous, was studied, particularly proving the referred stability property for

$$u = -k_1|x|^a \text{sign}(x) - k_2|\dot{x}|^b \text{sign}(\dot{x}) \tag{2}$$

$k_1 = k_2 = 1$, with $b \in (0, 1)$ and $a > b/(2-b)$ —or equivalently $a \in (0, 1)$ and $b < 2a/(1+a)$— [13, Corollary 1], and even stating finite-time stability preservation under (some type of) additional vanishing terms [13, Corollary 2]. Later on, useful foundations were settled down by Bhat and Bernstein [3–7], who stated —for continuous autonomous systems— a formal definition of *finite-time stable equilibrium*, proposed a Lyapunov-based criterion for its determination, and developed its characterization for *homogeneous* vector fields. This last contribution has been particularly appealing in view of its simplicity since, provided that the origin is an asymptotically stable equilibrium of a homogeneous vector field, finite-time stability is concluded by simply verifying that the degree of homogeneity is negative. Such a simplicity is perceived for instance by comparing the (rather involved) analysis developed in the proof of [13, Corollary 1] against [2, Example 5.6], where finite-time stability on (1)–(2) is analyzed through homogeneity, whence the referred stability property is concluded for $a \in (0, 1)$ and $b = 2a/(1+a)$, or equivalently $b \in (0, 1)$ and $a = b/(2-b)$.[1] However, for finite-time control design purposes, such a simple criterion might be restrictive in view of the requirements naturally imposed by homogeneity, which is conventionally a *global* property (see for instance [2] for a formal definition of homogeneous (scalar) functions and vector fields in a coordinate-dependent framework). For instance, in a constrained-input context, the closed-loop system would include bounded components which would preclude the corresponding vector field to be homogeneous [7] (in a coordinate-dependent framework). Nevertheless, such a restriction has been proven to be relaxed through alternative notions of homogeneity [40].

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[1] The analyses in [2, Example 5.6] and the proof of [13, Corollary 1] are actually valid for any $k_1 > 0$ and $k_2 > 0$.

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Further results on the global continuous control for finite-time and exponential stabilisation of constrained-input mechanical systems: desired conservative-force compensation and experiments

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Abstract: Saturating-proportional-derivative-type global continuous control for the finite-time or (local) exponential stabilisation of mechanical systems with bounded inputs is achieved involving the *desired* conservative-force compensation. Far from what one could expect, the proposed controller is not a simple extension of the *on-line* compensation case but it rather proves to entail a closed-loop analysis with a considerably higher degree of complexity. This gives rise to more involved requirements to guarantee its successful performance and implementability. Interesting enough, the proposal even shows that actuators with higher power-supply capabilities than in the on-line compensation case are required. Other important analytical limitations are further overcome through the developed algorithm. Experimental tests on a two-degree-of-freedom robotic arm corroborate the efficiency of the proposed scheme.

1 Introduction

A global continuous state-feedback scheme for the finite-time and exponential stabilisation of mechanical systems with bounded inputs has been recently proposed and thoroughly motivated in [1]. Giving a formal solution to the corresponding formulated problem under the explicit consideration of input constraints and the explicit choice on the system trajectory convergence (among finite-time and exponential) constitute the main distinctions of such an approach with respect to continuous finite-time controllers developed for mechanical systems before their appearance [2–4] (which were developed in an unconstrained input context; see for instance [1, Section 1] for a brief description of such previous works). However, the distinctive features do not stop there: while the cited previous approaches mainly rely on the *dynamic inversion* technique – or exact compensation of the whole dynamics – (except for one of the two controllers presented in [2]), the scheme in [1] benefits from the inherent passive nature of mechanical systems. This is done by keeping a (saturating) proportional-derivative type structure with exclusive compensation of the conservative-force (vector) term as a direct way to suitably reshape the closed-loop potential energy so as to set the desired posture as the only equilibrium position on the whole configuration space (of course, with the required stability property). Through such an *online* compensation of the conservative-force term, exclusively (instead of compensating the whole dynamics), the system model dependence on the designed scheme is considerably reduced, consequently simplifying the control structure and decreasing the inherent inconveniences of modelling inaccuracies as well as the implied computation burden. However, these improvements could still be potentiated if the on-line compensation term could be replaced by the conservative-force term exclusively evaluated at the desired position. Such a *desired conservative-force compensation* idea was first introduced in an unconstrained-input conventional (infinite-time) stabilisation context by Takegaki and Arimoto [5] and, ever since its introduction in the literature, it has been much appreciated in view of its simplicity and simplification improvements. This constitutes the main motivation of this work which aims at developing a *desired-conservative-force-compensation* extension of the saturating-proportional-derivative (SPD)-type finite-time/exponential stabilisation scheme from [1]. Far from what one could expect, such a design task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one. Such a replacement happens to keep the required (desired) closed-loop equilibrium position but not its uniqueness. Contrarily to the on-line compensation case [where the open-loop conservative forces are (ideally) cancelled out], in the desired compensation case further design requirements prove to be needed so as to guarantee that the control-induced potential energy component *dominates* the open-loop one (in order to guarantee uniqueness of the desired closed-loop equilibrium configuration). This was already pointed out in the unconstrained-input conventional case [5], where such a domination goal was shown to be achieved through a P control (vector) term with an absolutely stronger growing rate than that of the open-loop conservative force term in any direction (at every point) on the configuration space; in particular, under the simple consideration of uncoupled linear P and D control actions, this was shown to be achieved by simply fixing P gains higher than the highest (induced) norm value of the Jacobian matrix of the conservative force term (assuming that such a Jacobian matrix is bounded) [6]. However, the solution of the referred uniqueness issue cannot be that simple in the analytical context considered here – under the consideration of input constraints, the contemplated type of trajectory convergence (finite-time or exponential) and the generalised form of the SPD controller component – in view of the special functions involved in the SPD term to guarantee the achievement of the formulated stabilisation goal. This represents an important analytical challenge to which this work succeeds to give a solution enjoying the technical benefits from the desired conservative-force compensation. Interesting enough, the exhaustive analysis developed here further brings to the fore that actuators with higher power-supply capabilities than in the on-line-compensation case are required. This results from the *worst-case* type design

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Further advancements on the output-feedback global continuous control for the finite-time and exponential stabilisation of bounded-input mechanical systems: desired conservative-force compensation and experiments

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ABSTRACT

Global Saturating-Proportional Saturating-Derivative (SP-SD) type continuous control for the finite-time or (local) exponential stabilisation of mechanical systems with bounded inputs is achieved avoiding velocity variables in the feedback, and further simplified through *desired* conservative-force compensation. The proposed output-feedback controller is not a simple extension of the *on-line* compensation case but it rather proves to entail a closed-loop analysis with considerably higher degree of complexity that gives rise to more involved requirements. Interestingly, the proposal even shows that actuators with higher power-supply capabilities than in the on-line compensation case are required. Other important analytical limitations are further overcome through the developed algorithm. Experimental tests on a multi-degree-of-freedom robot corroborate the efficiency of the proposed approach.

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Output feedback; finite-time stabilisation; mechanical systems; desired conservative-force compensation; bounded inputs

1. Introduction

An output-feedback global continuous control scheme for the finite-time and exponential stabilisation of mechanical systems with bounded inputs has been recently proposed and thoroughly motivated in Zamora-Gómez, Zavala-Río, and López-Araujo (2017). Guaranteeing the corresponding formulated control objective under the explicit consideration of input constraints and the explicit choice on the system trajectory convergence, under the exclusive consideration of position variables in the feedback, are among the main characteristics that distinguish such an approach from continuous finite-time controllers developed for mechanical systems before its appearance: (Hong, Xu, & Huang, 2002; Sanyal & Bohn, 2015; Zhao, Li, Zhu, & Gao, 2010) (see for instance Zamora-Gómez et al., 2017, §1 for a brief description of such previous works). But there is still an important distinction: while the cited previous works are mainly state-feedback approaches that rely on the *dynamic inversion* technique – or exact compensation of the whole dynamics – (except for one of the two controllers presented in Hong et al., 2002), and the only output-feedback extension (formulated in Hong et al., 2002) is based on (model-based) finite-time observers, the scheme in Zamora-Gómez et al. (2017) exploits the inherent passive nature of mechanical systems, avoiding state reconstruction. This is done by keeping a (saturating) Proportional-Derivative type structure with exclusive compensation of the conservative-force (vector) term as a direct way to suitably reshape the closed-loop potential energy so as to set the desired posture as the only equilibrium position on the whole configuration space; damping is further injected through a (model-free) dynamic dissipation subsystem whose output is involved in the feedback as a *damped-derivative* action. Through such a control scheme (which avoids reproduction of any other term of the open-loop dynamics apart from the described *on-line* compensation of the conservative forces), the system model dependence of the designed algorithm is considerably reduced, consequently simplifying the control structure and decreasing the inherent inconveniences of modelling inaccuracies as well as the implied computation burden. But these advantages could still be potentiated by replacing the (unique) on-line compensation term by the conservative-force term exclusively evaluated at the desired position (Kelly, Santibáñez, & Loría, 2005, Chapter 8). Such a *desired conservative-force compensation* idea was first developed in an unconstrained-input conventional (infinite-time) stabilisation framework by Takegaki and Arimoto (1981) and, ever since its introduction in the literature, it has been the subject of diverse studies (Kelly, 1997), been at the core of control design advancements (Zavala-Río & Santibáñez, 2007), and proven to be widely appreciated in view of its simplicity and simplification improvements. This constitutes the main motivation of this work which aims at developing a *desired-conservative-force-compensation* extension of the output-feedback SP-SD-type (Saturating-Proportional Saturating-Derivative) finite-time/exponential stabilisation scheme from Zamora-Gómez et al. (2017). Far from what one could expect, such a design task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one. Such a replacement turns out to keep the required (desired) closed-loop equilibrium position but not

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Continuous Control for Fully Damped Mechanical Systems With Input Constraints: Finite-Time and Exponential Tracking

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***Abstract*—A motion continuous control scheme for fully damped mechanical systems with constrained inputs is proposed. It gives the freedom to choose among finite-time and (local) exponential convergence through a simple design parameter. The control objective is achieved from any initial conditions, for desired trajectories that can be physically tracked avoiding actuator saturation and loss of motion error dissipation, globally induced through the aid of the natural damping terms explicitly considered in the open-loop dynamics. The stability analysis is based on a strict Lyapunov function and is formally developed within an appropriate analytical framework that takes into account the time-varying character naturally adopted by the closed loop. Simulation tests are further included.**

***Index Terms*—Input constraints, motion continuous control, mechanical systems, strict Lyapunov function, uniform finite-time tracking.**

I. Introduction

Finite-time control through continuous feedback has been a research topic of increasing interest in the last few years. Such an intriguing topic has attracted attention on its need for a suitable analytical framework around its conceptualization and characterization. In this direction, important contributions have been developed for autonomous systems in the works of Bhat and Bernstein [2], [3], by stating a precise definition of a finite-time stable equilibrium, a Lyapunov-function-based criterion for its determination, and a useful characterization for homogeneous vector fields.

Finite-time stability and stabilization for time-varying vector fields has evolved more slowly and is still in progress. Important extensions and generalizations of the previously cited works from Bhat and Bernstein have been developed, for instance, in [9], by stating precise definitions and Lyapunov-type characterizations for nonautonomous systems. Uniform stability has been very recently studied within the framework of homogeneity in [13] where, in particular, the characterization of the global uniform finite-time stability has been extended to time-varying vector fields. These contributions show the complexity entailed in the nonautonomous case in relation to the previously cited time-invariant case. For instance, the existence of a homogeneous Lyapunov function characterized for autonomous vector fields in [12] does not apply for time-varying ones, and a similar extension for the latter case does not exist. Consequently, results based on such a fundamental work of Rosier [12], such as the finite-time-stability-preservation *approximation* approach of Hong *et al.* [5], do not apply in the nonautonomous case. Stability/stabilization studies in the time-varying context shall take into account such important analytical limitations and consequently entail a more complex analysis.

Finite-time continuous control of mechanical systems has been treated, for instance, in [4], [6], [14], and [16]. These works mainly give rise to diverse finite-time regulators and are consequently developed within the framework of autonomous systems. Once we move on to the tracking control problem, which naturally implies a time-varying closed-loop dynamics, the stability analysis suffers from the above-mentioned impossibility to involve analytical tools exclusively addressed to time-invariant vector fields, and shall consequently be developed within the framework of nonautonomous systems, for instance, through the use of a suitable *strict* Lyapunov function. Strict Lyapunov functions have hardly been very recently constructed in [4] to support finite-time control of robot manipulators disregarding input constraints, leaving the more complex tracking-under-bounded-input case unsolved.

This work gives a solution to the—up to our knowledge—open problem of (uniform) finite-time tracking continuous control of constrained-input mechanical systems, under the consideration of linear damping terms in the open-loop dynamics. The proposed approach actually gives the freedom to choose the type of trajectory convergence, among finite-time and exponential, through a simple control parameter. The stability analysis is based on a suitable strict Lyapunov function, and is formally developed within an appropriate analytical framework that takes into account the inherent time-varying nature of the closed loop. The design relies on the consideration of the natural damping terms, which are directly involved in the characterization of the subset of desired trajectories for which the control objective is achieved from any initial conditions (by ensuring motion error dissipation *globally*, as will be made clear later on in Remark 3.4). Such a characterization further restricts the choice to desired motions generating open-loop (reaction and inherent force/torque) terms whose addition remains within the actuator bounds; those transgressing such a restriction would not even be physically possible to be accurately tracked. The control synthesis thus guarantees the formulated goal through control signals evolving within

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RESEARCH ARTICLE

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A robustness study of a finite-time/exponential tracking continuous control scheme for constrained-input mechanical systems: Analysis and experiments

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Summary

The closed-loop analysis of a recently proposed continuous scheme for the finite-time or exponential tracking control of constrained-input mechanical systems is reformulated under the consideration of an input-matching bounded perturbation term. This is motivated by the poor number of works devoted to support the so-cited argument claiming that continuous finite-time controllers are more robust than asymptotical (infinite-time) ones under uncertainties and the limitations of their results. We achieve to analytically prove that, for a perturbation term with sufficiently small bound, the considered tracking continuous control scheme leads the closed-loop error variable trajectories to get into an origin-centered ball whose radius becomes smaller in the finite-time convergence case, entailing smaller posttransient variations than in the exponential case. Moreover, this is shown to be achieved for any initial condition, avoiding to restrain any of the parameters involved in the control design, and under the suitable consideration of the nonautonomous nature of the closed loop. The study is further corroborated through experimental tests on a multi-degree-of-freedom robotic manipulator, which do not only confirm the analytical result but also explore the scope or limitations of its conclusions under adverse perturbation conditions.

KEYWORDS

constrained inputs, input-matching perturbation, mechanical systems, robustness, tracking continuous control, uniform finite-time stability

1 | INTRODUCTION

Control synthesis aiming at the accomplishment of a regulation or trajectory tracking goal in finite time through continuous feedback has been the subject of intensive research in the last years. Numerous works with such a design objective formulation have been motivated arguing benefits of the finite-time algorithms over the asymptotic (infinite-time) ones, such as faster convergence and improved robustness under uncertainties.[1-4] However, this has not yet been exhaustively explored or brought to the fore through formal analysis or implementation tests. The only analysis treating one of those aspects, that the authors are aware of, was developed by Bhat and Bernstein,[5] who studied the robustness issue. More

