To cite this article: H C Rosu and S C Mancas 2020 *J. Phys.: Conf. Ser.* 1540 012005 https://doi.org/10.1088/1742-6596/1540/1/012005

Journal of Physics: Conference Series

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Factorization of the Riesz-Feller Fractional Quantum Harmonic Oscillators

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Abstract. Using the Riesz-Feller fractional derivative, we apply the factorization algorithm to the fractional quantum harmonic oscillator along the lines previously proposed by Olivar-Romero and Rosas-Ortiz, extending their results. We solve the non-Hermitian fractional eigenvalue problem in the k space by introducing in that space a new class of Hermite 'polynomials' that we call Riesz-Feller Hermite 'polynomials'. Using the inverse Fourier transform in Mathematica, interesting analytic results for the same eigenvalue problem in the x space are also obtained. Additionally, a more general factorization with two different Lévy indices is briefly introduced.

1. Introduction

A type of fractional quantum harmonic oscillator has been first discussed by Laskin in one of his breakthrough papers [1] on fractional quantum mechanics, but he tackled only a semiclassical approximation. Since then, several authors have dealt with the spatial fractional Schrödinger equation with different types of fractional derivatives and various potentials presenting contradictory results and arguments [2–7].

Some years ago, Olivar-Romero and Rosas-Ortiz [8] were first ones to apply the factorization method [9, 10] to a fractional differential equation choosing precisely the fractional quantum harmonic oscillator as the case study for their considerations. In line with Laskin, they used the Riesz fractional derivative reporting some interesting results and making suggestions for future work. This motivated us to proceed with a substantial extension of their results, which we present in this paper.

In Section 2, we briefly review the factorization method for the standard quantum harmonic oscillator. In Section 3, where the main results of this work can be found, we present the factorization algorithm for the fractional quantum harmonic oscillator with Riesz-Feller derivatives instead of the Riesz ones as employed in [8]. We have been encouraged to work with the non-Hermitian Riesz-Feller kinetic energy in the Hamiltonian for this case by the recent physical results reported by Berman and Moiseyev [11] for the same type of Hamiltonian in the case of impenetrable rectangular potential. In Section 4, we briefly address the factorization with different fractional indices, and we end up stating the conclusions of this work.

Journal of Physics: Conference Series **1540** (2020) 012005 doi:10.1088/1742-6596/1540/1/012005

2. Factorization of the standard quantum harmonic oscillator revisited

Setting $\hbar = m = \omega_0 = 1$, the eigenvalue problem for the standard Hamiltonian operator of the quantum harmonic oscillator is

$$H_{\text{h.o.}}\psi_n \equiv \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\psi_n = \lambda_n\psi_n \ , \qquad n = 0, 1, 2\ldots, \tag{1}$$

where λ_n is the spectral parameter (dimensionless energy), is a basic quantum mechanical eigenvalue problems. Suppose we take $\lambda_0 = \epsilon$, where ϵ is a constant to be specified later. Then, as will be shown next, $\lambda_1 = 1 + \epsilon$, $\lambda_2 = 2 + \epsilon$,..., $\lambda_n = n + \epsilon$, ..., i.e., for a given $\lambda > \lambda_0$, the nearest spectral neighbors from below and from above are $\lambda - 1$ and $\lambda + 1$, respectively, and one can write (1) in the form

$$H_{\text{h.o.}}\psi = \lambda\psi. \tag{2}$$

By means of the factoring operators

$$a_1 = \tfrac{1}{\sqrt{2}}\left(-\tfrac{d}{dx} + x\right) = -\tfrac{1}{\sqrt{2}}e^{\frac{x^2}{2}}\tfrac{d}{dx}e^{-\frac{x^2}{2}}, \tag{3}$$

$$a_2 = \tfrac{1}{\sqrt{2}}\left(\tfrac{d}{dx} + x\right) = \tfrac{1}{\sqrt{2}}e^{-\frac{x^2}{2}}\tfrac{d}{dx}e^{\frac{x^2}{2}}, \tag{4}$$

the Hamiltonian $H_{\text{h.o.}}$ can be expressed as

$$H_{\text{h.o.}} = a_1a_2 + \epsilon = a^\dagger a + \epsilon, \tag{5}$$

where ϵ is a factorization reminder known as the factorization constant, which for the quantum harmonic oscillator is $\epsilon = 1/2$, while the commutator of the factoring operators is $[a_2, a_1] = 2\epsilon = 1$. These factoring operators have been introduced in quantum mechanics by Fock and Dirac already in the 1930's, but as complex conjugated expressions of a_1 and a_2 called creation and annihilation operators, respectively,

$$a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \equiv a_1, \qquad a = \frac{1}{\sqrt{2}}(x + ip) \equiv a_2,$$

where $p = -i\frac{d}{dx}$ is the quantum mechanical momentum (recall that $\hbar = 1$). Then, $a_1a_2 = H_{\text{h.o.}} - \frac{1}{2}$ and $a_2a_1 = H_{\text{h.o.}} + \frac{1}{2}$. Hence, the following intertwining formulas

$$H_{\text{h.o.}}a_1 = a_1(H_{\text{h.o.}} + 1), \qquad H_{\text{h.o.}}a_2 = a_2(H_{\text{h.o.}} - 1), \tag{6}$$

allow an algebraic solution method (factorization algorithm) for the eigenvalue problem (2). If ψ is an eigenfunction for eigenvalue λ, then the intertwining relationships show that $a_1\psi$ and $a_2\psi$ are the neighbor eigenfunctions at $\lambda + 1$ and $\lambda - 1$, respectively. The first step of the algorithm is to find the ground state eigenfunction from the kernel of a_2,

$$a_2\psi_0 = 0, \quad \frac{d}{dx}e^{\frac{x^2}{2}}\psi_0 = 0, \quad \psi_0 = N_0e^{-\frac{x^2}{2}},$$

for which $\lambda_0 = \epsilon = \frac{1}{2}$ as can be checked in (2). The integration constant N_0 is fixed to $N_0 = 1/\sqrt[4]{\pi}$ through the normalization condition $\int_{-\infty}^{\infty} |\psi_0|^2 dx = 1$.

In the second step, one can find each of the excited eigenfunctions ψ_n, with eigenvalues $\lambda_n = n + \frac{1}{2}$, by applying n times a_1 to ψ_0,

$$\psi_n = C_n(a_1)^n\psi_0 = N_n(-1)^n\left(e^{\frac{x^2}{2}}\frac{d}{dx}e^{-\frac{x^2}{2}}\right)^n e^{-\frac{x^2}{2}} = N_nH_n(x)e^{-\frac{x^2}{2}}, \tag{7}$$

where $H_n(x)$ are the Hermite polynomials,

$$H_n = (-1)^n \left(e^{\frac{x^2}{2}} \frac{d}{dx} e^{-\frac{x^2}{2}} \right)^n e^{-\frac{x^2}{2}}. \tag{8}$$

The normalization constants $N_n = N_0/\sqrt{2^n n!}$ are obtained by the normalization conditions for ψ_n.

3. The fractional factorization method

We follow Olivar-Romero and Rosas-Ortiz and consider a pair of operators A_α and B_α such that

$$H_\alpha \equiv \frac{1}{\alpha}\left(-\frac{d^\alpha}{dx^\alpha} + x^2 \right) = B_\alpha A_\alpha + \epsilon_\alpha, \tag{9}$$

where the factorization remainder ϵ_α can be either a number (as in the conventional factorization) or a fractional-differential operator. The parameter α defines the fractional order of the derivative and is also known as the Lévy stability index because for positive values $\alpha \leq 2$ characterizes the Lévy stable probability distributions, see [1] for more details.

Usually, in fractional quantum mechanics, one works with α in the interval (1,2] because for $\alpha \leq 1$ the Lévy distributions have undefined mean. However, in the factorization method, there is no essential change in the formal results for subunit values of α.

The simplest expressions for the factoring operators are

$$A_\alpha = \frac{1}{\sqrt{\alpha}}\left(\frac{d^{\alpha/2}}{dx^{\alpha/2}} + x \right) \quad \text{and} \quad B_\alpha = \frac{1}{\sqrt{\alpha}}\left(-\frac{d^{\alpha/2}}{dx^{\alpha/2}} + x \right) . \tag{10}$$

These are the same as proposed in [8] up to the scaling $1/\sqrt{\alpha}$ and provide

$$B_\alpha A_\alpha = \frac{1}{\alpha}\left(-\frac{d^\alpha}{dx^\alpha} - \frac{\alpha}{2}\frac{d^{\alpha/2-1}}{dx^{\alpha/2-1}} + x^2 \right). \tag{11}$$

Comparing (11) with (9), one obtains

$$\epsilon_\alpha = \frac{1}{2}\frac{d^{\alpha/2-1}}{dx^{\alpha/2-1}}, \tag{12}$$

which shows that for $\alpha \neq 2$ the factorization remainder ϵ_α is a fractional differential derivative of order $\frac{\alpha}{2} - 1$ while the case $\alpha = 2$ leads to the constant $\epsilon_2 = 1/2$ and the factoring operators A_2 and B_2 reduce to the usual annihilation and creation operators of the standard harmonic oscillator.

According to the factorization algorithm, we have to solve the kernel equation of A_α,

$$A_\alpha \psi_0^{(\alpha)}(x) = 0 \quad \longrightarrow \quad \left[\frac{d^{\alpha/2}}{dx^{\alpha/2}} + x \right] \psi_0^{(\alpha)}(x) = 0 . \tag{13}$$

Since this is a fractional derivative equation, we will solve it in the k-space by taking into account that the Fourier transform $\mathcal{F}$ of the (quantum) Riesz-Feller derivative d^α/dx^α of a function is characterized by its specific symbol Ψ_α^θ

$$\mathcal{F}\{d^\alpha \psi(x)/dx^\alpha\} := -\Psi_\alpha^\theta \, \phi(k), \qquad \Psi_\alpha^\theta = |k|^\alpha e^{i\,\mathrm{sgn}(k)\,\theta\frac{\pi}{2}}, \tag{14}$$

where $\phi(k) = \mathcal{F}\{\psi(x)\}$ and θ is the skewness (asymmetry) parameter. The latter is usually restricted to numerical values located at the so-called Takayasu–Feller diamond domain, $|\theta| \leq \min[\alpha, 2-\alpha]$ [11]. The factoring operators in the k dual coordinate are

$$A_{k,\alpha} = \Psi^{\theta}_{\alpha/2} + i\frac{d}{dk}, \qquad B_{k,\alpha} = \Psi^{\theta}_{\alpha/2} - i\frac{d}{dk}, \tag{15}$$

and the kernel solution of $A_{k,\alpha}$ is the function

$$\phi_0^{(\alpha)}(k) = \exp\left(-\frac{|k|^{\alpha/2+1}}{\alpha/2+1}\right), \tag{16}$$

as shown in Appendix A. We will call this ground state wave–function in the dual k coordinate as the sub-Gaussian function for any $\alpha < 2$ which turns Gaussian for $\alpha = 2$. All the other excited states are obtained by the repeated usage of the creation operator $B_{k,\alpha}$. For example, the first three excited state in the k coordinate will be

$$\phi_1^{(\alpha)}(k) = B_{k,\alpha}\phi_0^{(\alpha)}(k) = 2i\,\mathrm{sgn}(k)\ |k|^{\frac{\alpha}{2}}\phi_0^{(\alpha)}, \tag{17}$$

$$\phi_2^{(\alpha)}(k) = B_{k,\alpha}\phi_1^{(\alpha)}(k) = \left[\alpha|k|^{\frac{\alpha}{2}-1} - 4|k|^{\alpha}\right]\phi_0^{(\alpha)}, \tag{18}$$

$$\phi_3^{(\alpha)}(k) = B_{k,\alpha}\phi_2^{(\alpha)}(k) = i\,\mathrm{sgn}(k)\left[-8|k|^{\frac{3\alpha}{2}} + 6\alpha|k|^{\alpha-1} - \alpha\left(\tfrac{\alpha}{2}-1\right)|k|^{\frac{\alpha}{2}-2}\right]\phi_0^{(\alpha)}. \tag{19}$$

The eigenfunctions in the x coordinate can be obtained by performing the inverse Fourier transforms of the ϕ functions. In Figs. 1 and 2, we present the ground state eigenfunctions and the first three excited eigenfunctions in the k and x coordinates, respectively. All even eigenfunctions ϕ_{2n} are real and all the odd eigenfunctions ϕ_{2n+1} are purely imaginary, but nevertheless their inverse Fourier transforms, ψ_{2n+1}, are real.

Regarding the bell-shaped ψ_0 wave–functions as obtained from the fractional sub–Gaussians ϕ_0 for different values of α by the inverse Fourier transform, they can be expressed analytically in terms of a small set of generalized hypergeometric functions according to Mathematica. The explicit expressions of ψ_0 for $\alpha = 3/2$ and $\alpha = 1$ are provided in Appendix B. For both ϕ_0 and ψ_0 functions, one can define a degree of non–Gaussianity simply as

$$\tilde{\eta}_\alpha = \frac{\phi_0^{(2)} - \phi_0^{(\alpha)}}{\phi_0^{(2)}} = 1 - \frac{\phi_0^{(\alpha)}}{\phi_0^{(2)}} \quad \text{and} \quad \eta_\alpha = 1 - \frac{\psi_0^{(\alpha)}}{\psi_0^{(2)}}, \tag{20}$$

respectively. For $\alpha = 2$, we have $\tilde{\eta}_2 = \eta_2 = 0$. It is easy to calculate $\tilde{\eta}_\alpha$ and η_α from

$$\tilde{\eta}_\alpha = 1 - e^{\frac{|k|^2}{2}}\phi_0^{(\alpha)}, \qquad \eta_\alpha = 1 - e^{\frac{x^2}{2}}\psi_0^{(\alpha)}. \tag{21}$$

Both non-Gaussian deformations are displayed in Fig. 3 for the three illustrative values of α used in this paper. The plots are up to the intersection points with the pure Gaussians, i.e., only for positive $\tilde{\eta}_\alpha$ and η_α, since for the small negative values in the tail regions there are some numerical problems related to the generalized hypergeometric functions.

In general, one can write

$$\phi_n(k) = i^n \widetilde{H}_n \phi_0^{(\alpha)}, \tag{22}$$

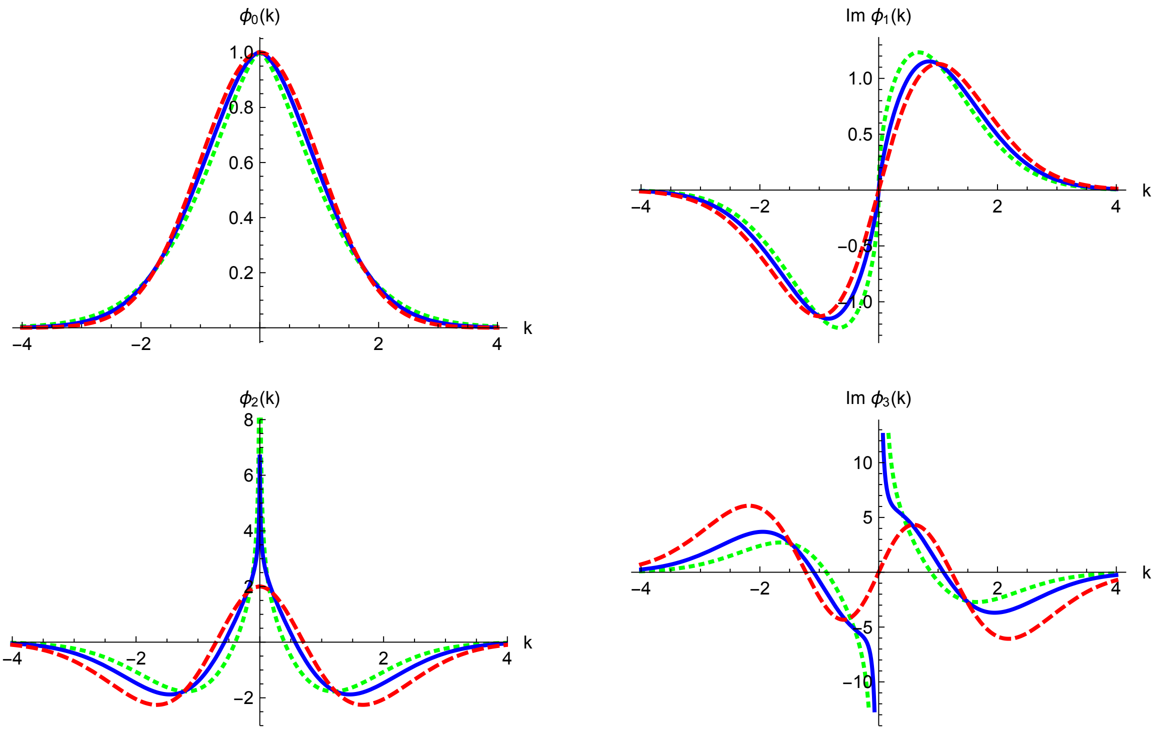

Figure 1. The 'ground state' wave–function and the first three excited wave–functions in the k space for $\alpha = 2$ (red color), $\alpha = 3/2$ (blue color), and $\alpha = 1$ (green color). The odd wave–functions are purely imaginary.

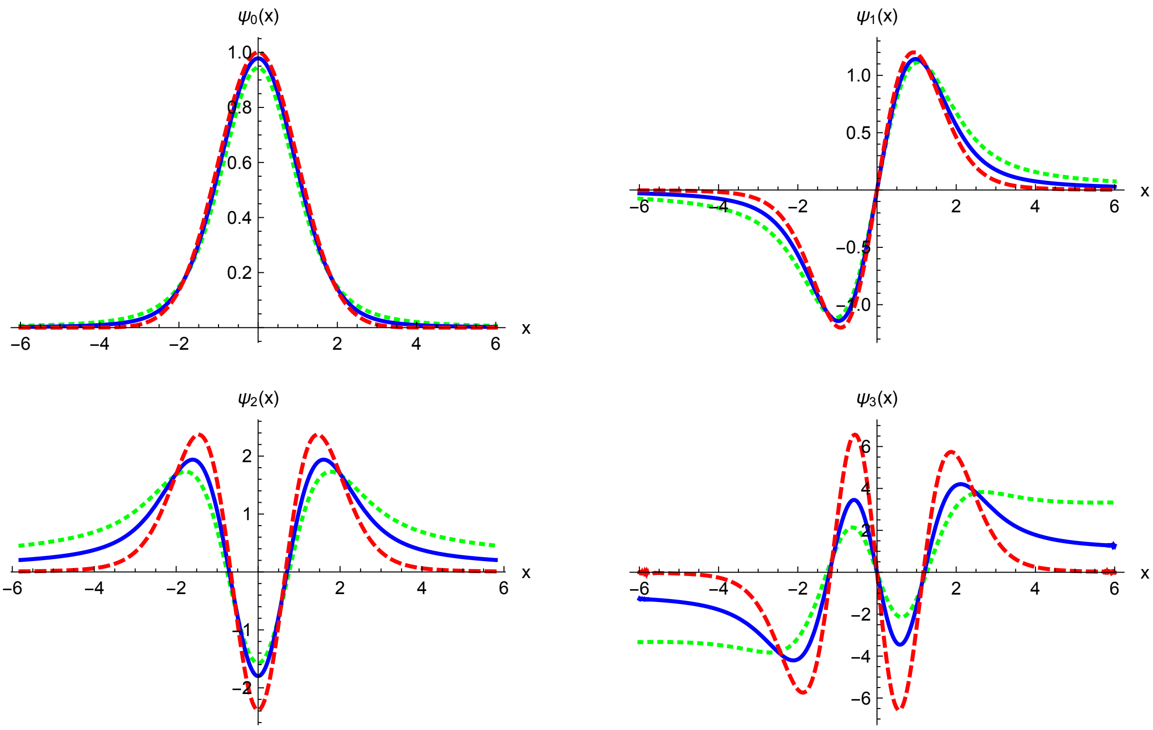

Figure 2. The wave–functions in the x space obtained by inverse Fourier transforms of the wave–functions from the previous figure.

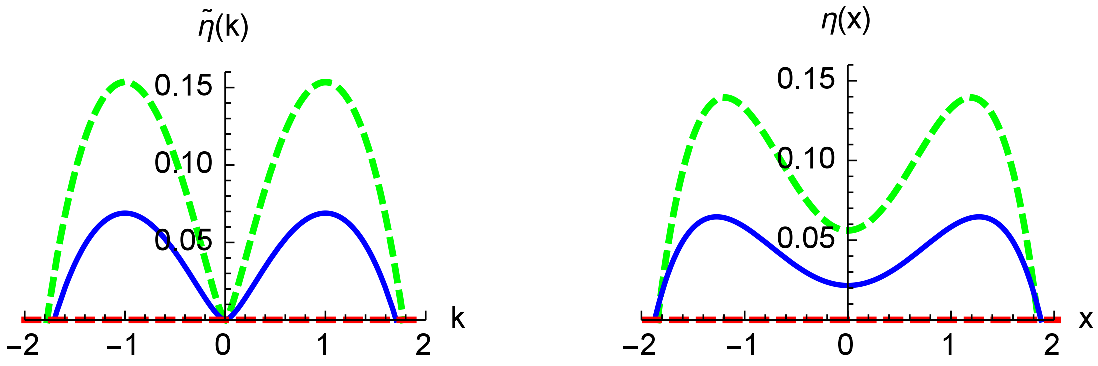

Figure 3. Graphs of the non-Gaussian deformations in the k and x spaces for α =1, 3/2 and 2 from top to bottom, respectively.

where $\widetilde{H}_n(k)$ are the fractionally-deformed Hermite 'polynomials':

$$\widetilde{H}_0 = 1,$$

$$\widetilde{H}_1(k) = 2\ \mathrm{sgn}(k)|k|^{\frac{\alpha}{2}},$$

$$\widetilde{H}_2(k) = 4|k|^{\frac{2\alpha}{2}} - \alpha|k|^{\frac{\alpha}{2}-1},$$

$$\widetilde{H}_3(k) = \mathrm{sgn}(k)\left[8|k|^{\frac{3\alpha}{2}} - 6\alpha|k|^{\frac{2\alpha}{2}-1} + 2\tfrac{\alpha}{2}\left(\tfrac{\alpha}{2}-1\right)|k|^{\frac{\alpha}{2}-2}\right],$$

$$\begin{aligned}\widetilde{H}_4(k) = {} & 16|k|^{\frac{4\alpha}{2}} - 24\alpha|k|^{\frac{3\alpha}{2}-1} + 6\alpha(\alpha-1)|k|^{\frac{2\alpha}{2}-2} \\ & +2\tfrac{\alpha}{2}\left(\tfrac{\alpha}{2}-1\right)|k|^{\frac{2\alpha}{2}-2} - 2\tfrac{\alpha}{2}\left(\tfrac{\alpha}{2}-1\right)\left(\tfrac{\alpha}{2}-2\right)|k|^{\frac{\alpha}{2}-3} \\ & \vdots \end{aligned} \tag{23}$$

that we also call Riesz-Feller Hermite 'polynomials'. For $\alpha = 2$, they turn into the standard Hermite polynomials up to a negative sign for the odd ones, though in the $|k|$ variable. The first five, leaving aside the trivial case of $\widetilde{H}_0$, are plotted in Fig. 4. Due to the centrifugal type terms (negative powers) present in their expressions for $\alpha < 2$, they are singular at the origin unless $\widetilde{H}_1$ which is only discontinuous there.

The general expression for $\widetilde{H}_n(k)$ is

$$\begin{aligned}\widetilde{H}_n(k) \quad &= \mathrm{sgn}(k)^n\Big[2^n|k|^{\frac{n\alpha}{2}} - p_1(\alpha)|k|^{\frac{(n-1)\alpha}{2}-1} + p_2(\alpha)|k|^{\frac{(n-2)\alpha}{2}-2} \\ & -p_3(\alpha)|k|^{\frac{(n-3)\alpha}{2}-3} + \ldots + (-1)^{n-1}p_{n-1}(\alpha)|k|^{\frac{\alpha}{2}-(n-1)}\Big],\end{aligned} \tag{24}$$

where $p_i(\alpha)$ are polynomials of order i in α that can be determined from the following counterpart of the Rodrigues formula

$$\widetilde{H}_n(k) = (-1)^n \mathrm{sgn}(k)^n\, e^{2\frac{|k|^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}} \frac{d^n}{dk^n} e^{-2\frac{|k|^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}}, \tag{25}$$

which for $\alpha = 2$, turns into

$$\widetilde{H}_n(k) = (-1)^n \mathrm{sgn}(k)^n\, e^{k^2} \frac{d^n}{dk^n} e^{-k^2}, \tag{26}$$

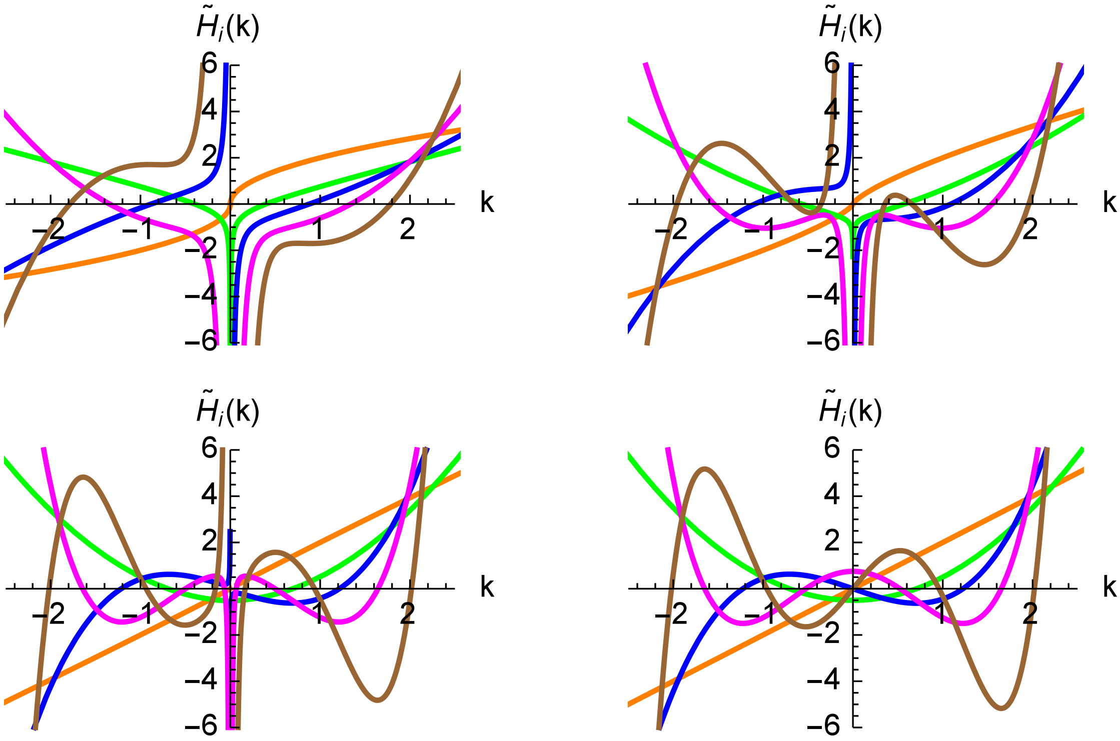

Figure 4. Graphs of the first five Riesz–Feller Hermite 'polynomials' scaled by the inverse square of their 'degree' in the k space for $\alpha = 1$ (top left), 1.5 (top right), 1.95 (bottom left), and 2 (bottom right).

which can be compared with the standard x space formula

$$H_n(x) = (-1)^n \, e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \; .$$

Moving to the calculation of the eigenvalues $\lambda_n(k)$ in the k space, it can be shown that the expressions reported in [8] correspond to the asymmetry parameter $\theta = 0$, (Riesz derivative). In particular, the first three eigenvalues are:

$$\lambda_0(k) = \tfrac{1}{2}|k|^{\frac{\alpha}{2}-1}, \tag{27}$$

$$\lambda_1(k) = \tfrac{3}{2}|k|^{\frac{\alpha}{2}-1} - \tfrac{1}{2}\left(\tfrac{\alpha}{2} - 1\right)|k|^{-2}, \tag{28}$$

$$\lambda_2(k) = \frac{\left(\frac{11\alpha}{2}-6\right)|k|^{\frac{\alpha}{2}-1} - 10|k|^{\alpha} - \left(\frac{\alpha}{2}-1\right)\left(\frac{\alpha}{2}-2\right)|k|^{-2}}{\alpha - 4|k|^{\frac{\alpha}{2}+1}}. \tag{29}$$

However, for $\theta = 1$ an additional complex term adds up to each eigenvalue. This term is given by

$$\left(\Psi_\alpha^1 - 1\right)|k|^{(2n+2)\frac{\alpha}{2}}\,, \qquad n = 0, 1, 2, \ldots, \tag{30}$$

for the even eigenvalues $\lambda_{2n}(k)$ and

$$\left(\Psi_\alpha^1 - 1\right)|k|^{(2n+1)\frac{\alpha}{2}}\,, \qquad n = 1, 2, \ldots, \tag{31}$$

for the odd eigenvalues $\lambda_{2n-1}(k)$. This means that all eigenfunctions, including the bell-shaped sub-Gaussians, correspond to metastable states and we do not expect exceptional points like in the case of impenetrable rectangular wells [11].

Journal of Physics: Conference Series **1540** (2020) 012005 doi:10.1088/1742-6596/1540/1/012005

4. Factorization with operators of different fractionallity

The use of different Lévy indices in the factoring operators has been another suggestion in [8]. Although this issue is beyond the scope of this work, here we briefly show how to do it leaving its full consideration for future work.

Let us consider the following factoring operators

$$A_\delta = D^{\delta/2} + x, \qquad B_\gamma = -D^{\gamma/2} + x, \tag{32}$$

where the D's stand for the derivatives of the indicated Lévy fractional orders. Then, assuming $\alpha = \frac{\delta+\gamma}{2}$, we obtain

$$\mathcal{H}_\alpha = B_\gamma A_\delta + \epsilon_{\gamma\delta}. \tag{33}$$

Thus, the remainder operator has the more complicated dissipative form

$$\epsilon_{\gamma\delta} = \frac{\gamma}{2} D^{\gamma/2-1} + x\left(D^{\gamma/2} - D^{\delta/2}\right). \tag{34}$$

Notice that $\epsilon_{\gamma\delta} = \epsilon_\alpha$ when $\delta = \gamma \equiv \alpha$.

Besides, one can also use the reverted factorization

$$\mathcal{H}_\alpha = -D^\alpha + x^2 = A_\delta B_\gamma - \epsilon_{\delta\gamma}, \tag{35}$$

which displays a remainder operator given by

$$\epsilon_{\delta\gamma} = -\frac{\delta}{2} D^{\delta/2-1} + x\left(D^{\delta/2} - D^{\gamma/2}\right). \tag{36}$$

One can see that the second dissipative term is opposite in sign to the corresponding dissipative term in (34).

Again, for the eigenvalue problems of the factored Hamiltonians, one should work in the Fourier k-space and come back to the x-space by the inverse Fourier transform. To move these operators in the k-space, i.e. to obtain their Fourier counterparts, the following Fourier transforms are needed

$$\mathcal{F}\{D^\alpha \psi_0(x)\} = -\Psi^\theta_\alpha \phi_0(k),$$

$$\mathcal{F}\{x^2 \psi_0(x)\} = -\tfrac{d^2\phi_0(k)}{dk^2},$$

$$\mathcal{F}\{D^{\frac{\delta}{2}-1}\psi_0(x)\} = -\Psi^\theta_{\frac{\delta}{2}-1}\phi_0(k),$$

$$\mathcal{F}\{xD^{\frac{\delta}{2}}\psi_0(x)\} = i\tfrac{d}{dk}\left(\Psi^\theta_{\frac{\delta}{2}}\phi_0(k)\right) = i\phi_0(k)\tfrac{d}{dk}\left(\Psi^\theta_{\frac{\delta}{2}}\right) + i\Psi^\theta_{\frac{\delta}{2}}\tfrac{d\phi_0(k)}{dk}.$$

We also need:

$$\frac{d}{dk}\left(\Psi^\theta_\alpha\right) = \frac{d}{dk}\left[|k|^\alpha e^{i\,\mathrm{sgn}(k)\theta\frac{\pi}{2}}\right] = \alpha|k|^{\alpha-1}\mathrm{sgn}(k)e^{i\,\mathrm{sgn}(k)\theta\frac{\pi}{2}} = \alpha\,\mathrm{sgn}(k)\Psi^\theta_{\alpha-1}.$$

Using the last two equations, we obtain

$$\mathcal{F}\{xD^{\frac{\delta}{2}}\psi_0(x)\} = i\frac{\delta}{2}\,\mathrm{sgn}(k)\Psi^\theta_{\frac{\delta}{2}-1}\phi_0(k) + i\Psi^\theta_{\frac{\delta}{2}}\frac{d\phi_0(k)}{dk}.$$

Here, we provide the result for the remainder operator $\tilde{\epsilon}_{\gamma\delta}$ in the Fourier space

$$\tilde{\epsilon}_{\gamma\delta} = -\frac{\gamma}{2}\Psi^\theta_{\frac{\gamma}{2}-1} + i\left[\left(\frac{\gamma\,\mathrm{sgn}(k)}{2}\Psi^\theta_{\frac{\gamma}{2}-1} + \Psi^\theta_{\frac{\gamma}{2}}\frac{d}{dk}\right) - \left(\frac{\delta\,\mathrm{sgn}(k)}{2}\Psi^\theta_{\frac{\delta}{2}-1} + \Psi^\theta_{\frac{\delta}{2}}\frac{d}{dk}\right)\right].$$

5. Conclusion

We have used the quantum Riesz-Feller derivative in the factorization of the fractional quantum harmonic oscillator as proposed by Olivar-Romero and Rosas-Ortiz in [8]. We have obtained more results in analytic form as counterparts of the standard factorization of the quantum harmonic oscillator. We confirm the expressions for the fractional wave–functions in [8] that we obtain when the value of the asymmetry parameter is taken $\theta = 1$. On the other hand, we have found that the eigenvalues have a supplementary complex term with respect to the formulas for $\theta = 0$. Therefore all the 'eigenstates' are metastable despite the impenetrability of the parabolic well. A factorization with different Lévy parameters has been also sketched up.

Acknowledgement The organizers of the QuantFest-2019 workshop are acknowledged for the excellent conditions they offered during the event. Both authors wish to thank Dr. Oscar Rosas-Ortiz for invitation and the occasion to share memories about Bogdan Mielnik. Thanks are also due to the referee for interesting comments.

Appendix A: Effective calculation of ϕ_0 for any α

The calculation of ϕ_0 proceeds as follows. The kernel equation

$$A_\alpha \psi_0(x) \equiv \left(D^{\alpha/2} + x\right) \psi_0(x) = 0$$

is Fourier transformed by taking into account that the fractional derivative is a (quantum) Riesz–Feller derivative

$$\mathcal{F}\{D^{\alpha/2}\psi_0\} + \mathcal{F}\{x\psi_0\} = 0 \quad \longrightarrow \quad -\Psi^\theta_{\alpha/2}\phi_0 - i\frac{d\phi_0(k)}{dk} = 0.$$

Separating variables and formally integrating, we obtain

$$\ln\phi_0 = i\int \Psi^\theta_{\alpha/2} dk = i\int |k|^{\frac{\alpha}{2}} e^{i\,\mathrm{sgn}(k)\,\theta\frac{\pi}{2}} dk.$$

The integral in the right hand side is evaluated separately for the two possible cases:

(i) $k > 0$, then $|k| = k$, $\mathrm{sgn}(k) = +1$.

$$\ln\phi_0 = i\int k^{\alpha/2} e^{i\theta\frac{\pi}{2}} dk = ie^{i\theta\frac{\pi}{2}}\int k^{\alpha/2} dk = ie^{i\theta\frac{\pi}{2}}\frac{k^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}.$$

Let $\theta = 1$, then $e^{i\theta\frac{\pi}{2}} = i$, so

$$\ln\phi_0 = -\frac{k^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1} \quad \longrightarrow \quad \phi_0 = Ce^{-\frac{k^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}}.$$

(ii) $k < 0$, let $k = -p$, then $p > 0$ and $|p| = p$. Then

$$\ln\phi_0 = -i\int |p|^{\alpha/2} e^{i\mathrm{sgn}(-p)\theta\frac{\pi}{2}} dp = -ie^{-i\theta\frac{\pi}{2}}\int p^{\alpha/2} dp.$$

For $\theta = 1$:

$$\ln\phi_0 = -\int p^{\alpha/2} dp \quad \longrightarrow \quad \phi_0 = e^{-\frac{p^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}} \quad \longrightarrow \quad \phi_0 = Ce^{-\frac{(-k)^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}}.$$

From (i) and (ii), we conclude that for $\theta = 1$:

$$\phi_0 = Ce^{-\frac{|k|^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}}.$$

Appendix B: ψ_0 for $\alpha = 1$ and $\alpha = 3/2$

For $\alpha = 1$, the k-space ground state wave-function is

$$\phi_0 = e^{-\frac{2}{3}|k|^{3/2}}.$$

This is a fractional sub–Gaussian function, whose inverse Fourier transform can be written as the following summation

$$\psi_0 = \sum_{m=0}^{2} a_{2m} x^{2m} f_{2m}(-x^6/6^2),$$

where f_{2m} are generalized hypergeometric functions which together with the coefficients a_{2m} are:

$$a_0 = 2^{1/3} \cdot 3^{2/3} \cdot \Gamma\left(\frac{5}{3}\right), \quad f_0 = {}_2F_3\left(\frac{5}{12}, \frac{11}{12}; \frac{2}{6}, \frac{3}{6}, \frac{5}{6}; -\frac{x^6}{6^2}\right),$$

$$a_2 = -\frac{3}{2}, \quad f_2 = {}_3F_4\left(\frac{3}{4}, \frac{4}{4}, \frac{5}{4}; \frac{4}{6}, \frac{5}{6}, \frac{7}{6}, \frac{8}{6}; -\frac{x^6}{6^2}\right)$$

$$a_4 = \frac{7}{16} \cdot \left(\frac{3}{2}\right)^{1/3} \cdot \Gamma\left(\frac{7}{3}\right), \quad f_4 = {}_2F_3\left(\frac{13}{12}, \frac{19}{12}; \frac{7}{6}, \frac{9}{6}, \frac{10}{6}; -\frac{x^6}{6^2}\right).$$

For $\alpha = 3/2$, the k-space ground state wave-function is

$$\phi_0 = e^{-\frac{4}{7}|k|^{7/4}}.$$

This is a fractional sub–Gaussian function, whose inverse Fourier transform can be written as the following summation

$$\psi_0 = \sum_{m=0}^{6} a_{2m} x^{2m} f_{2m}(-x^{14}/14^6),$$

where f_{2m} are generalized hypergeometric functions which together with the coefficients a_{2m} are given next:

$$f_0 = {}_6F_{11}\left(\frac{11}{56}, \frac{18}{56}, \frac{25}{56}, \frac{39}{56}, \frac{46}{56}, \frac{53}{56}; \frac{2}{14}, \frac{3}{14}, \frac{4}{14}, \frac{5}{14}, \frac{6}{14}, \frac{7}{14}, \frac{9}{14}, \frac{10}{14}, \frac{11}{14}, \frac{12}{14}, \frac{13}{14}; -\frac{x^{14}}{14^6}\right),$$

$$a_0 = \frac{7^3 \cdot 7^{4/7}}{11 \cdot 2 \cdot 2^{1/7} \cdot 3^2 \cdot 5^2} \Gamma\left(\frac{32}{7}\right),$$

$$f_2 = {}_6F_{11}\left(\frac{19}{56}, \frac{26}{56}, \frac{33}{56}, \frac{47}{56}, \frac{54}{56}, \frac{61}{56}; \frac{4}{14}, \frac{5}{14}, \frac{6}{14}, \frac{7}{14}, \frac{8}{14}, \frac{9}{14}, \frac{11}{14}, \frac{12}{14}, \frac{13}{14}, \frac{15}{14}, \frac{16}{14}; -\frac{x^{14}}{14^6}\right),$$

$$a_2 = \frac{5 \cdot 2^{4/7}}{2^3 \cdot 7^{11/14}} \frac{\Gamma\left(-\frac{2}{7}\right)}{\sin^2\left(\frac{\pi}{7}\right)} \frac{\sin\left(\frac{\pi}{14}\right)}{\cos\left(\frac{3\pi}{14}\right)},$$

$$f_4 = {}_6F_{11}\left(\frac{27}{56}, \frac{34}{56}, \frac{41}{56}, \frac{55}{56}, \frac{62}{56}, \frac{69}{56}; \frac{6}{14}, \frac{7}{14}, \frac{8}{14}, \frac{9}{14}, \frac{10}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}, \frac{16}{14}, \frac{17}{14}, \frac{18}{14}; -\frac{x^{14}}{14^6}\right),$$

$$a_4 = \frac{13 \cdot 2^{2/7} \cdot 7^{5/14}}{2^7} \frac{\Gamma\left(\frac{6}{7}\right)}{\sin^2\left(\frac{\pi}{7}\right)} \frac{\sin\left(\frac{\pi}{14}\right)}{\cos\left(\frac{3\pi}{14}\right)},$$

$$f_6 = {}_7F_{12}\left(\frac{35}{56}, \frac{42}{56}, \frac{49}{56}, \frac{56}{56}, \frac{63}{56}, \frac{70}{56}, \frac{77}{56}; \frac{8}{14}, \frac{9}{14}, \frac{10}{14}, \frac{11}{14}, \frac{12}{14}, \frac{13}{14}, \frac{15}{14}, \frac{16}{14}, \frac{17}{14}, \frac{18}{14}, \frac{19}{14}, \frac{20}{14}; -\frac{x^{14}}{14^6}\right),$$

$$a_6 = -\frac{7^3 \cdot 7^{1/7}}{2^8 \cdot 3 \cdot 5} \cot\left(\frac{\pi}{7}\right) \tan\left(\frac{\pi}{14}\right) \tan\left(\frac{3\pi}{14}\right),$$

$$f_8 = {}_6F_{11}\left(\frac{43}{56},\frac{50}{56},\frac{57}{56},\frac{71}{56},\frac{78}{56},\frac{85}{56};\frac{10}{14},\frac{11}{14},\frac{12}{14},\frac{13}{14},\frac{15}{14},\frac{17}{14},\frac{18}{14},\frac{19}{14},\frac{20}{14},\frac{21}{14},\frac{22}{14};-\frac{x^{14}}{14^6}\right),$$

$$a_8 = \frac{7^7 \cdot 7^{1/7}}{19 \cdot 43 \cdot 3^5 \cdot 5^3 \cdot 2^{14} \cdot 2^{2/7}} \Gamma\left(\frac{64}{7}\right),$$

$$f_{10} = {}_6F_{11}\left(\frac{51}{56},\frac{58}{56},\frac{65}{56},\frac{79}{56},\frac{86}{56},\frac{93}{56};\frac{12}{14},\frac{13}{14},\frac{15}{14},\frac{16}{14},\frac{17}{14},\frac{19}{14},\frac{20}{14},\frac{21}{14},\frac{22}{14},\frac{23}{14},\frac{24}{14};-\frac{x^{14}}{14^6}\right),$$

$$a_{10} = -\frac{7^8 \cdot 7^{2/7}}{11 \cdot 13 \cdot 17 \cdot 29 \cdot 3^5 \cdot 5^3 \cdot 2^{17} \cdot 2^{4/7}} \Gamma\left(\frac{72}{7}\right),$$

$$f_{12} = {}_6F_{11}\left(\frac{59}{56},\frac{66}{56},\frac{73}{56},\frac{87}{56},\frac{94}{56},\frac{101}{56};\frac{15}{14},\frac{16}{14},\frac{17}{14},\frac{18}{14},\frac{19}{14},\frac{21}{14},\frac{22}{14},\frac{23}{14},\frac{24}{14},\frac{25}{14},\frac{26}{14};-\frac{x^{14}}{14^6}\right),$$

$$a_{12} = \frac{7^9 \cdot 7^{3/7}}{11^2 \cdot 13 \cdot 59 \cdot 73 \cdot 3^5 \cdot 5^2 \cdot 2^{24} \cdot 2^{6/7}} \Gamma\left(\frac{80}{7}\right).$$

Counter to these expressions, the hypergeometric formula for the Gaussian function is

$$e^{-x^2/2} = {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x^2/2\right) - \frac{x^2}{3}\,{}_1F_1\left(\frac{3}{2},\frac{5}{2};-x^2/2\right), \tag{37}$$

which can be obtained from the confluent hypergeometric form of the erf function

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(x) = \int^x e^{-t^2} dt = x\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x^2\right),$$

the chain rule, and the formula (see, e.g., NIST Handbook of Mathematical Functions)

$$\frac{d}{dz}\,{}_1F_1(a,b;z) = \frac{a}{b}\,{}_1F_1(a+1,b+1;z),$$

for $z = -x^2$.

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