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# Stability Analysis of PD AQM control for delays models of TCP networks

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**ABSTRACT**
This paper focuses on the local stability analysis of nonlinear delay models of transmission control protocol/active queue management (TCP/AQM) networks by using a Proportional-Derivative (PD) controller as AQM strategy. More precisely, we derive the complete set of PD controllers that exponentially stabilizes the corresponding linear delay system. Nonlinear sufficient stability conditions for the local asymptotic stability of the equilibrium are derived by means of the Lyapunov-Krasovskii functional approach. The robustness issue to uncertainty in the network parameters is also addressed. The results are illustrated by means of some numerical examples.

**KEYWORDS**
PD AQM controllers; Stability; Robust Stability; Time-Delay Systems



## 1. Introduction

During the last decades a lot of efforts have been focused to the analysis and design of Active Queue Management (AQM) schemes for supporting end-to-end Transmission Control Protocol (TCP) congestion control in networks. The fluid-flow delay model introduced in (Hollot *et al.* 2002) for describing the behavior of TCP/AQM networks has become a reference for investigating the qualitative properties of TCP/AQM networks and developing control theoretic design and analysis for the AQM. Thus, based on such a model, Proportional (P) and Proportional-Integral (PI) controllers were proposed in (Hollot *et al.* 2002); Proportional Derivative (PD) control are introduced in (Sun *et al.* 2004), (Kim 2006) and (Azadegan *et al.* 2013); $\mathcal{H}^\infty$ controllers in (Quet and Ozbay 2004); while (Yan *et al.* 2004) considers a variable structure control as AQM scheme. **Due to their simplicity and easy implementation, several works have been devoted to compare and modify the P,PI and PD controllers for improving stability, robustness, and performance properties, see the survey papers (Ryu *et al.* 2003) and (Adams 2012).**

**Among these works, it is interesting to note the state-space feedback formulation of the TCP/AQM control proposed in (Kim 2006), where it is shown that a PD-type control structure in terms of the queue length is the natural state feedback control to fully support TCP dynamics. The**

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**capabilities of the PD AQM control on regulating the queue length under different network scenarios as well as comparisons with other AQM strategies have been illustrated by simulations in (Azadegan *et al.* 2013) and (Sun *et al.* 2004).**

Leaving the advantages (increases of speed response) or drawbacks (steady-state error) of a PD control aside, it appears that its stability and robust stability properties as AQM strategy have not been sufficiently investigated in the literature in counterpart with the studies made for P and PI AQM controllers. **For instance, for the linearization of the delay model introduced by (Hollot *et al.* 2002), the existing designs of PD AQM controllers are based on sufficient conditions for closed-loop stability, whereas the designs are made by means of some heuristic rules in (Sun *et al.* 2004), the minimization of a linear quadratic cost function in (Kim 2006), and in terms of linear matrix inequalities (Azadegan *et al.* 2013). Thus, the existing designs do not provide the set of all stabilizing PD controllers. On the other hand, for P and PI AQM controllers the complete characterization of the set of all stabilizing gain values have been reported in (Michiels *et al.* 2006) and (Melchor-Aguilar and Niculescu 2009), respectively, but, to the best of the authors' knowledge, there are no specific results for the problem of finding all stabilizing PD controllers.** Furthermore, one does not find a nonlinear stability analysis of the delay model with a PD control as AQM scheme.

These lack on the stability analysis of PD AQM control motivate the current paper, where we present linear and nonlinear stability analyses of a simplified version of the model introduced in (Hollot *et al.* 2002). More precisely, for a given set of network parameters (round-trip time, number of TCP loads and link capacity), we firstly present the complete characterization of the set of all PD controllers that exponentially stabilize the linearized delay model. Some preliminary results in this direction were given in (Puerto-Piña and Melchor-Aguilar 2016). Also, we investigate the geometric properties of the set of all PD stabilizing controllers with respect to variations on network parameters of delay, TCP loads, and link capacity. Secondly, we perform a nonlinear stability analysis and obtain a sufficient condition for local asymptotic stability along with some estimates of the attraction region of the closed-loop equilibrium by using the Lyapunov-Krasovskii functional approach. Finally, we investigate the robustness issue of the PD control. We show that a new closed-loop equilibrium point exists when a PD stabilizing control, designed for some nominal network parameters, is implemented in a system with new different network parameters, and obtain an analytic expression for such a new equilibrium. By using the linear stability analysis along with the geometric properties of the set of all stabilizing PD controllers, as well as the nonlinear stability analysis, we derive robust stability results for assuring the local asymptotic stability of the whole family of equilibria point generated from uncertainty network parameters of interval type.

The remaining part of the paper is organized as follows. Section 2 presents the mathematical model and PD AQM control. We show that the corresponding closed-loop system is a neutral delay system and give a formal justification of the transformation proposed in (Kim 2006) and (Azadegan *et al.* 2013) to convert the neutral delay system in a retarded delay system. Additionally, we introduce a scaling transformation of time and state to the model, which will help us to simplify the subsequent analysis. The linear stability analysis is given in Section 3, and the nonlinear stability analysis in Section 4. The robust stability analysis is presented in section 5. Numerical examples illustrating the main results are given in Section 6, and concluding remarks in section

7 end the paper.

## 2. Mathematical model and PD AQM controllers

We consider the dynamic delay model introduced in (Hollot *et al.* 2002) for describing the behavior of TCP/AQM networks. **Such a model is a simplified version of the one developed in (Misra *et al.* 2000) by using fluid-flow and stochastic differential equation analysis, which relates the average value of key network variables of $n$ homogeneous TCP-controlled sources and a single congested router. This model is described by the following coupled non-linear differential equations including time-varying delays**:

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau(t)} - \frac{w(t)w(t-\tau(t))}{2\tau(t-\tau(t))} p(t-\tau(t)), \\ \dot{q}(t) = \frac{n(t)}{\tau(t)} w(t) - c(t), \end{cases} \tag{1}$$

where $w(t)$ denotes the average of TCP windows size (packets), $q(t)$ is the average queue length (packets), $\tau(t) = \frac{q(t)}{c} + \tau_p$ is the round-trip time (secs) with $\tau_p$ representing the propagation delay, $c(t)$ is the link capacity (packets/sec), $n(t)$ is the number of TCP sessions and $p(\cdot)$ is the probability of a packet marking which represents the AQM control strategy **and takes values only in** $[0,1]$. **The queue length** $q(t)$ **and window-size** $w(t)$ **are positive and bounded functions. i.e.,** $q(t) \in [0, q_{\max}]$ **and** $w(t) \in [0, w_{\max}]$.

By assuming that $n(t) = n, \tau(t) = \tau$ and $c(t) = c$, the model (1) is approximated by the following system:

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau} - \frac{w(t)w(t-\tau)}{2\tau} p(t-\tau), \\ \dot{q}(t) = \frac{n}{\tau} w(t) - c. \end{cases} \tag{2}$$

For a desired equilibrium $q_d$ the unique equilibrium point $(w_0, q_0, p_0)$ of (2) is defined by

$$w_0 = \frac{\tau c}{n}, q_0 = q_d \text{ and } p_0 = \frac{2}{w_0^2}.$$

Most works on analysis and design of AQM controllers are based on the linear version of (2), and there are only a few works that address their nonlinearities, see (Hollot and Chait 2002) and (Michiels *et al.* 2006), where a nonlinear stability analysis for a P AQM controller is performed by means of the Lyapunov-Razumikhin and Lyapunov-Krasovskii approaches, respectively. As it was motivated in (Hollot and Chait 2002) and (Michiels *et al.* 2006), in order to address the nonlinearities of the system (2) it is convenient to consider the following simplified system:

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau} - \frac{w^2(t)}{2\tau} p(t-\tau), \\ \dot{q}(t) = \frac{n}{\tau} w(t) - c, \end{cases} \tag{3}$$

which is a good local approximation of the system (2) around the equilibrium point when $w_0 \gg 1$, see (Michiels *et al.* 2006) for a mathematical justification. Although the condition $w_0 \gg 1$ imposes a restriction on the network parameters for considering (3) as a good approximation of (2), it is satisfied for typical range of parameters arising in practice as argued in (Hollot and Chait 2002) and (Michiels *et al.* 2006).

We here consider the simplified model (3) and a PD type control as AQM strategy, and perform linear and nonlinear stability analyses of the corresponding closed-loop systems. Thus, for a desired queue equilibrium point $q_d$ let us consider the following PD-type controller:

$$p(t) = K_p\left(q(t) - q_d\right) + K_d\dot{q}\left(t\right) + p_0. \tag{4}$$

**In (Kim 2006) was shown that the natural state-feedback to fully support the TCP dynamics is a PD control. The main reasoning on this is that the windows size $w(t)$ and queue length $q(t)$ are the state variables of the system (3) and, therefore, they need to be in a state feedback for completely controlling the TCP dynamics. Now, since the second equation of (3) expresses the queue dynamic as a function of the windows size, then it appears that $q(t)$ may be used instead of $w(t)$ thus leading to a PD-type control structure. On the other hand, the use of a PD control overcomes the implementation restriction of having a measure or estimation of $w(t)$ which is not accessible at the router's side in real networks, see (Azadegan *et al.* 2013) for discussions.**

The closed-loop system (3)-(4) is

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau} - \frac{w^2(t)}{2\tau}\left[K_p\left(q(t-\tau) - q_d\right) + K_d\dot{q}\left(t-\tau\right) + p_0\right], \\ \dot{q}(t) = \frac{n}{\tau}w(t) - c. \end{cases} \tag{5}$$

The delay system (5) is of *neutral* type as involves the time derivative of past values of $q(t)$, see (Vyhlidal *et al.* 2004) for discussions when a PD-type feedback is applied to a system in the presence of delays in the input signal. By following the approaches presented in (Azadegan *et al.* 2013) and (Kim 2006), let us differentiate the second equation of (5) and substitute the right-hand sides of the first and second equation of (5) to obtain

$$\ddot{q}(t) = \frac{n}{\tau^2} - \frac{1}{2n}\left(\dot{q}(t) + c\right)^2\left(K_p\left(q(t-\tau) - q_d\right) + K_d\dot{q}(t-\tau) + p_0\right), \tag{6}$$

a delay system of the *retarded* type in the variable $q(t)$, which is considered equivalent to the neutral delay system (5) in (Azadegan *et al.* 2013) and (Kim 2006) for designing the PD controller's gains. The process of converting the coupled dynamics in (5) to a single dynamic in (6) is a special system transformation only valid for particular initial functions, as we demonstrate in the following subsection.

### 2.1. *Transformation from neutral to retarded closed-loop systems*

Let us start by revising the initial value problem for the neutral delay system (5) and the retarded delay system (6). On one hand, in order to define a solution of the neutral delay system (5) one needs to know an initial differentiable function defined in the interval $[-\tau, 0]$ for $q(t)$, and an initial value for $w(t)$. Let $\mathcal{C}^1\left([-\tau, 0], \mathbb{R}\right)$ be the space of real-valued functions having a continuos derivative in $(-\tau, 0)$, a right-hand continuous derivative at $-\tau$ and a left-hand continuous derivative at 0. For an initial function $\varphi_q \in \mathcal{C}^1\left([-\tau, 0], \mathbb{R}\right)$ and value $\varphi_w = \varphi_w(0)$, let $q(t, \varphi_q, \varphi_w)$ and $w(t, \varphi_q, \varphi_w)$

be the corresponding solutions of (5). On the other hand, in order to define a solution of the retarded delay system (6) one needs to know an initial differentiable function defined in the interval $[-\tau, 0]$ for $q(t)$. Let $\psi \in \mathcal{C}^1([-\tau, 0], \mathbb{R})$ be such an initial function satisfying

$$\psi = \varphi_q \text{ and } \dot{\psi}(0) = \frac{n}{\tau}\varphi_w - c, \tag{7}$$

and let $\bar{q}(t, \psi)$ be the corresponding solution of (6).

**Lemma 1.** $q(t, \varphi_q, \varphi_w) = \bar{q}(t, \psi)$.

***Proof.*** Function $q(t, \varphi_q, \varphi_w)$ clearly satisfies (6) for $t \geq 0$. By definition $\varphi_q$ coincides with $\psi$. The second equation of (5) implies that

$$\dot{\varphi}_q(0) = \dot{q}(0) = \frac{n}{\tau}w(0) - c = \frac{n}{\tau}\varphi_w - c = \dot{\psi}(0).$$

Consider now the function $\bar{q}(t, \psi)$. Clearly, $\bar{q}(t, \psi)$ is differentiable for $t \geq 0$. Let us define the function

$$w(t) = \frac{\tau}{n}\left(\dot{\bar{q}}(t, \psi) + c\right). \tag{8}$$

Since $\bar{q}(t, \psi)$ is twice differentiable for $t \geq 0$ then $w(t)$ is differentiable for $t \geq 0$. By differentiating both sides of the equation (8) and substituting the right-hand side of (6) one obtains that $w(t)$ satisfies the first equation of (5) while that from (8) follows that $\dot{\bar{q}}(t, \psi)$ satisfies the second equation of (5). Again, by definition $\varphi_q$ coincides with $\psi$ and from (8) one obtains the initial function for $w(t)$, that is,

$$\varphi_w = w(0) = \frac{\tau}{n}\left(\dot{\psi}(0) + c\right) = \frac{\tau}{n}\left(\dot{\varphi}_q(0) + c\right).$$

$\square$

The Lemma 1 shows that under the restriction (7) on the initial functions, the neutral delay system (5) is equivalent to the retarded system (6) as proposed by (Kim 2006) and (Azadegan *et al.* 2013), but not properly justified.

Now, it makes sense to revise if the restriction (7) is satisfied in a real TCP/AQM scenario. **Recall that TCP consists of the slow start and congestion avoidance phases and that the system (1), then the system (5), models the congestion avoidance phase only**. Thus, the initial conditions for (5) are determined by the slow start phase of the TCP. In (Srikant 2012) is shown that the dynamics of the queue length in the router are the same in both the slow start phase and the congestion avoidance phase, i.e., the following equation holds: $\dot{q}(\bar{t}) = \frac{nw(\bar{t})}{\tau} - c$, where $\bar{t}$ denotes the time variable for the two phases of the TCP algorithm. Thus, it follows that the restriction (7) on the initial functions is always satisfied.

### 2.2. *Scaling transformations*

Consider the following scaling transformations:

$$t^{(New)} = \frac{t^{(Old)}}{\tau}, \text{ and } \tilde{q}(t^{(New)}) = \frac{q(t^{(Old)})}{n}, \tag{9}$$

These transformations of time and state were proposed in (Michiels *et al.* 2006) in order to simplify the stability analysis of P AQM controllers. We will also use and exploit them in our stability investigation of PD AQM controllers. Thus, by applying the transformation (9) to the retarded delay system (6) one obtains

$$\ddot{\tilde{q}}(t) = 1 - 0.5\left(\dot{\tilde{q}}(t) + w_0\right)^2 \left(K_p\left(n\tilde{q}(t-1) - q_d\right) + \frac{K_d n}{\tau}\dot{\tilde{q}}(t-1) + p_0\right). \tag{10}$$

In the new coordinates the unique equilibrium of (10) is

$$\tilde{q}_e = \frac{q_d}{n}. \tag{11}$$

Let $y(t) = \tilde{q}(t) - \tilde{q}_e$. Then, we get the following nonlinear system:

$$\ddot{y}(t) = 1 - 0.5\left(\dot{y}(t) + w_0\right)^2 \left(nK_p y(t-1) + \frac{n}{\tau}K_d\dot{y}(t-1) + p_0\right) \tag{12}$$

which has trivial solution. In the following we will use the system (12) to investigate the local asymptotic stability of the equilibrium of the system (3) in closed-loop with the PD control (4).

## 3. Linear Stability Analysis

### 3.1. *Complete stability region*

Linearizing the system (12) around the zero solution we arrive at the following linear delay system:

$$\ddot{y}(t) = -\frac{2n}{\tau c}\dot{y}(t) - \left(\frac{(\tau c)^2}{2n}\right)K_p y(t-1) - \left(\frac{\tau c^2}{2n}\right)K_d\dot{y}(t-1). \tag{13}$$

It is well known that the system (13) is exponentially stable if and only if its characteristic function (quasipolynomial)

$$f(s) = s^2 + \frac{2n}{\tau c}s + \frac{\tau c^2}{2n}K_d s e^{-s} + \frac{(\tau c)^2}{2n}K_p e^{-s},$$

has no zeros with non-negative real parts, see, e.g. (Gu *et al.* 2003). The following result provides the complete characterization of the controller's gains $(K_p, K_d)$ for which (13) is exponentially stable.

**Proposition 2.** *Given network parameters $(n, \tau, c)$, the linear system (13) is exponentially stable if and only if the controller's gains $(K_p, K_d)$ belong to the stability*

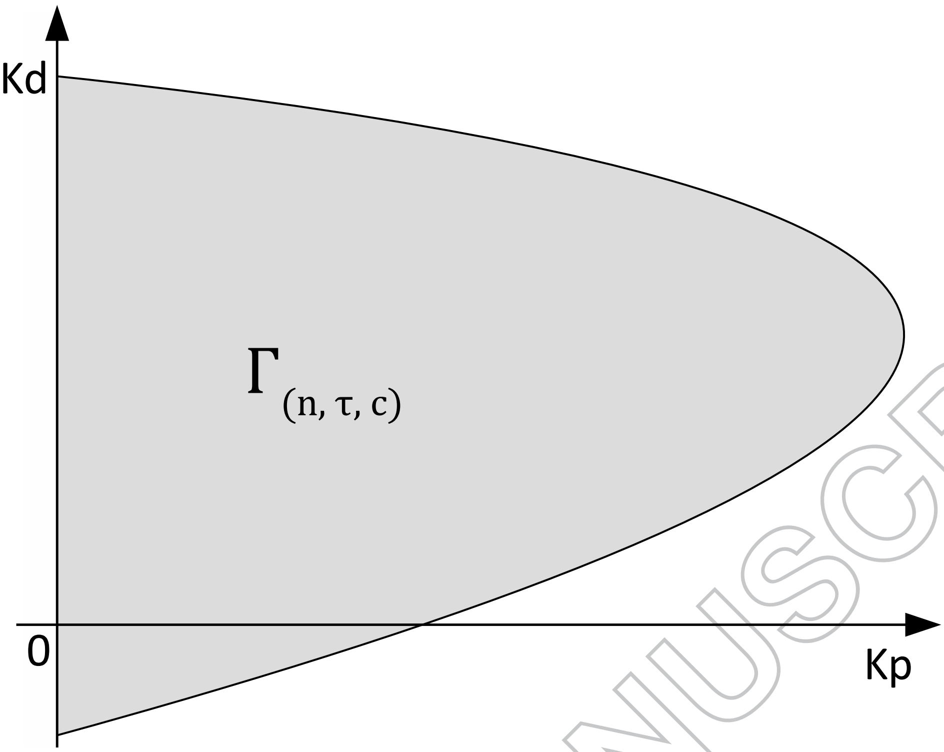


Figure 1.: Stability region $\Gamma_{(n,\tau,c)}$

*region* $\Gamma_{(n,\tau,c)}$*, plotted in Fig. 1, whose boundary* $\partial\Gamma_{(n,\tau,c)}$ *in the controller's gains space* $(K_p, K_d)$ *is defined by*

$$\begin{aligned} \partial\Gamma_{(n,\tau,c)} &= \{(K_p(\omega,n,\tau,c), K_d(\omega,n,\tau,c)) : \omega \in (0,\bar{\omega})\} \cup \\ &\quad \{(K_p, K_d) : K_p = 0, K_d \in [K_d(0), K_d(\bar{\omega})]\}, \end{aligned} \tag{14}$$

*where*

$$K_p(\omega,n,\tau,c) = \frac{2n}{(\tau c)^2}\left(\frac{2n}{\tau c}\omega\sin(\omega) + \omega^2\cos(\omega)\right), \tag{15}$$

$$K_d(\omega,n,\tau,c) = \frac{2n}{\tau c^2}\left(\omega\sin(\omega) - \frac{2n}{\tau c}\cos(\omega)\right), \tag{16}$$

*and* $\bar{\omega}$ *is the solution of the equation*

$$-\frac{c\tau\omega}{2n} = \tan(\omega), \tag{17}$$

*for* $\omega \in \left(\frac{\pi}{2}, \pi\right)$.

***Proof.*** The proof follows the same line of arguments used in the proof of Proposition 2 in (Puerto-Piña and Melchor-Aguilar 2016). Note that (17) is a transcendental equation for which the solution $\bar{\omega} \in \left(\frac{\pi}{2}, \pi\right)$ can be found numerically by plotting the two functions $-\frac{c\tau\omega}{2n}$ and $\tan(\omega)$, see Fig. 2. □

**Corollary 3.** *Given network parameters $(n, \tau, c)$, the linear system (13) with $K_d = 0$ is exponentially stable if and only if*

$$0 < K_p < \frac{4n^2}{(\tau c)^3} \frac{\tilde{\omega}}{\sin(\tilde{\omega})},$$

*where $\tilde{\omega}$ is the solution of the equation*

$$\tan(\omega) = \frac{2n}{(\tau c)\,\omega}, \tag{18}$$

*for $\omega \in \left(0, \frac{\pi}{2}\right)$.*

***Proof.*** From the parametrization (15) follows that $K_d(\omega, n, \tau, c) = 0$ if and only if the equation (18) holds. This transcendental equation has a unique solution $\tilde{\omega}$ in the interval $\left(0, \frac{\pi}{2}\right)$, see Fig. 2. For this $\tilde{\omega}$ we have that $K_p(\tilde{\omega}, n, \tau, c) = \frac{4n^2}{(\tau c)^3} \frac{\tilde{\omega}}{\sin(\tilde{\omega})}$. It follows that a necessary and sufficient condition for exponential stability of (13), with $K_d = 0$, is $0 < Kp < K_p(\tilde{\omega}, n, \tau, c)$ from which follows the result. □

The result in Corollary 3 coincides with the result in Theorem 1 of (Michiels *et al.* 2006), where necessary and sufficient conditions for exponential stability of P AQM controllers are provided. Thus, the Proposition 2 generalizes the linear stability results presented in (Michiels *et al.* 2006).

### 3.2. *Geometric properties of the stability regions*

We here present some properties of the boundary of the stability region $\Gamma_{(n,\tau,c)}$ that will play an essential role in deriving robust stability results for PD controllers in section 5. To simplify the notation we define $\rho = (n, \tau, c)$. Then, $K_p(\omega, n, \tau, c)$, $K_d(\omega, n, \tau, c)$ and $\Gamma_{(n,\tau,c)}$ are written as $K_p(\omega, \rho)$, $K_d(\omega, \rho)$ and $\Gamma_\rho$, respectively.

**Lemma 4.** *The continuos function $K_d(\omega, \rho)$ satisfies:*

1) *$K_d(\omega, \rho) < 0$ for $\omega \in [0, \tilde{\omega})$, $K_d(\omega, \rho) = 0$ for $\omega = \tilde{\omega}$ and $K_d(\omega, \rho) > 0$ for $\omega \in (\tilde{\omega}, \bar{\omega}]$.*
2) *$K_d(\omega, \rho)$ is strictly increasing of $\omega$ in the interval $(0, \bar{\omega})$.*

***Proof.***

1) From the proof of Corollary 3 follows that $K_d(\tilde{\omega}, \rho) = 0$. Furthermore, it holds that $\frac{2n}{c\tau\omega} > \tan(\omega)$ and $\cos(\omega) > 0, \forall\omega \in [0, \tilde{\omega})$, that imply $K_d(\omega, \rho) < 0, \forall\omega \in [0, \tilde{\omega})$. On the other hand, it holds that $\frac{2n}{c\tau\omega} < \tan(\omega)$ and $\cos(\omega) > 0$ for $\left(\tilde{\omega}, \frac{\pi}{2}\right)$ that in turn imply $K_d(\omega, \rho) > 0, \forall\omega \in \left(\tilde{\omega}, \frac{\pi}{2}\right)$. For $\omega \in \left(\frac{\pi}{2}, \bar{\omega}\right)$ we have $\frac{2n}{c\tau\omega} >$ $\tan(\omega)$ and $\cos(\omega) < 0$ which implies that $K_d(\omega, \rho) > 0, \forall\omega \in \left(\frac{\pi}{2}, \bar{\omega}\right)$. Evidently, $K_d(\frac{\pi}{2}, \rho) > 0$ which ends the proof of the statement.
2) We have $\frac{d}{d\omega}\left(K_d(\omega, \rho)\right) = \frac{2n}{\tau c^2}\left(m(\omega) + \sin(\omega)\right)$, where

$$m(\omega) = \omega\cos(\omega) + \frac{2n}{\tau c}\sin(\omega). \tag{19}$$

For $\omega = \frac{\pi}{2}$ we have $m(\omega) = \frac{2n}{\tau c} > 0$. For $\omega \neq \frac{\pi}{2}$, $m(\omega)$ can be rewritten as

$m(\omega) = \frac{2n}{\tau c}\cos(\omega)\left(\frac{\tau c\omega}{2n} + \tan(\omega)\right)$. When $\omega \in \left(0, \frac{\pi}{2}\right)$ we have $\cos(\omega) > 0$ and $\tan(\omega) > -\frac{\tau c\omega}{2n}$, which imply that $m(\omega) > 0$ for $\omega \in \left(0, \frac{\pi}{2}\right)$. When $\omega \in \left(\frac{\pi}{2}, \bar{\omega}\right)$ we have $\cos(\omega) < 0$ and $\tan(\omega) < -\frac{\tau c\omega}{2n}$ implying that $m(\omega) > 0$ for $\omega \in \left(\frac{\pi}{2}, \bar{\omega}\right)$. Taking into account the above and the fact that $\sin(\omega) > 0$ for all $\omega \in (0, \bar{\omega})$ one arrives at the conclusion that $\frac{d}{d\omega}\left(K_d(\omega, \rho)\right) > 0$ for $\omega \in (0, \bar{\omega})$ as required.

□

**Lemma 5.** *The continuos function $K_p(\omega, \rho)$ satisfies:*

*1) $K_p(\omega, \rho) > 0$, $\forall \omega \in (0, \bar{\omega})$.*
*2) $K_p(\omega, \rho)$ has a* ***unique*** *local maximum at $\omega_m \in (\tilde{\omega}, \bar{\omega})$, where $\omega_m$ is the solution of the equation*

$$n(\omega) = \tan(\omega), \ \forall \omega \in (\tilde{\omega}, \bar{\omega}), \tag{20}$$

*where*

$$n(\omega) = \frac{\left(\frac{2n}{\tau c} + 2\right)\omega}{\left(\omega^2 - \frac{2n}{\tau c}\right)}. \tag{21}$$

***Proof.***

1) The statement follows by observing that $K_p(\omega, \rho)$ can be written as $K_p(\omega, \rho) = \frac{2n\omega}{(\tau c)^2} m(\omega)$, where $m(\omega)$ is given by (19). Since $m(\omega) > 0$ for all $\omega \in (0, \bar{\omega})$ then $K_p(\omega, \rho) > 0$, $\forall \omega \in (0, \bar{\omega})$.
2) We have

$$\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) = \frac{2n}{(\tau c)^2}\left(\left(\frac{2n}{\tau c} + 2\right)\omega\cos(\omega) + \left(\frac{2n}{\tau c} - \omega^2\right)\sin(\omega)\right)$$

Since $\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) \neq 0$ for $\omega^2 = \frac{2n}{\tau c}$ and $\cos(\omega) = 0$ then the above expression can be rewritten as

$$\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) = \frac{2n}{(\tau c)^2}\left(\omega^2 - \frac{2n}{\tau c}\right)\cos(\omega)\left(n(\omega) - \tan(\omega)\right),$$

where $n(\omega)$ is given by (21). It follows that $\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) = 0$ if and only if $n(\omega) = \tan(\omega)$. The behavior of the function $n(\omega)$ is shown in Fig. 2. The transcendental equation $n(\omega) = \tan(\omega)$ has an infinite number of solutions and we are interested in those in the interval $(0, \bar{\omega})$. Taking into account that $w_0 = \frac{\tau c}{n} \geq 1$ we have that $\sqrt{\frac{2n}{\tau c}} \leq \sqrt{2} < \frac{\pi}{2}$ and, therefore, there exists a solution $\omega_m \in \left(\sqrt{\frac{2n}{\tau c}}, \frac{\pi}{2}\right)$ of $n(\omega) = \tan(\omega)$. Clearly, $n(\omega) > \tan(\omega)$ for $\omega \in \left(\sqrt{\frac{2n}{\tau c}}, \omega_m\right)$ while $n(\omega) < \tan(\omega)$ for $\omega \in \left(\omega_m, \frac{\pi}{2}\right)$ and, therefore, $\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) > 0$ when $\omega \in \left(\sqrt{\frac{2n}{\tau c}}, \omega_m\right)$ and $\frac{d}{d\omega}\left(K_p(\omega, \rho)\right) < 0$ when $\omega \in \left(\omega_m, \frac{\pi}{2}\right)$ that implies $K_p(\omega, \rho)$ has a local maximum at $\omega_m$.

Now, we show that in fact $\omega_m \in (\tilde{\omega}, \bar{\omega})$. Clearly, $\omega_m < \bar{\omega}$ since $\bar{\omega} \in \left(\frac{\pi}{2}, \pi\right)$. To show that $\omega_m > \tilde{\omega}$ let us suppose that $\omega_m \leq \tilde{\omega}$. Because of $\tan(\omega)$ is a strictly increasing function of $\omega$ then $\tan(\omega_m) \leq \tan(\tilde{\omega})$. Since the inequalities

$$n(\omega_m) = \frac{\left(\frac{2n}{\tau c} + 2\right)\omega_m}{\left(\omega_m^2 - \frac{2n}{\tau c}\right)} \geq \frac{2\omega_m}{\omega_m^2} = \frac{2}{\omega_m} \text{ and } \frac{2n}{(\tau c)\,\tilde{\omega}} < \frac{2}{\tilde{\omega}}$$

hold, then

$$\frac{2}{\omega_m} \leq n(\omega_m) = \tan(\omega_m) \leq \tan(\tilde{\omega}) = \frac{2n}{(\tau c)\,\tilde{\omega}} < \frac{2}{\tilde{\omega}}.$$

The inequality $\frac{2}{\omega_m} < \frac{2}{\tilde{\omega}}$ implies that $\omega_m > \tilde{\omega}$ which contradicts the assumption $\omega_m \leq \tilde{\omega}$ and proves the desired result.

□

**Remark 1.** ***The stability region $\Gamma_\rho$ is convex. In fact, from the parametrization (15)-(16) we have that*** $K_p(\omega,\rho) = \frac{2n}{(\tau c)^2 \sin(\omega)}\left[\frac{2n}{\tau c}\omega + \frac{\tau c^2}{2n}\omega\cos(\omega)K_d(\omega,\rho)\right]$. ***Direct calculations show that*** $K_p(\omega,\rho)$ ***is unimodal w.r.t.*** $K_d(\omega,\rho)$, ***when*** $\omega \in (0,\bar{\omega})$, ***which implies $\Gamma_\rho$ is convex.***

Now, we derive some geometric properties of the boundary of the stability region with respect to variations in the network parameters. Let us consider network parameters $\rho = (n,\tau,c)$ and $\rho_0 = (n_0,\tau_0,c_0)$ satisfying the following condition:

$$n \geq n_0, \tau \leq \tau_0 \text{ and } c \leq c_0, \tag{22}$$

Let $\bar{\omega}_0, \omega_{m0}$ and $\tilde{\omega}_0$ be the solutions of the equations (17), (18), and (20) corresponding to the parameters $\rho_0$, respectively. Similarly, let $\bar{\omega}, \omega_m$ and $\tilde{\omega}$ be the solutions of the equations (17), (18), and (20) corresponding to the parameters $\rho$, respectively.

**Remark 2.** *For networks parameters $\rho$ and $\rho_0$ satisfying (22) the following inequalities hold:*

$$\tilde{\omega}_0 \leq \tilde{\omega}, \omega_{m0} \leq \omega_m \text{ and } \bar{\omega}_0 \leq \bar{\omega} \tag{23}$$

*The result can be validated by simple inspecting the plots of the functions involved in the equations (17), (18), and (20) corresponding to the parameters $\rho_0$ and $\rho$, respectively, see Fig 2.*

**Lemma 6.** *Given parameters $\rho$ and $\rho_0$ satisfying (22) the following hold:*

*1)* $K_d(\omega,\rho) \leq K_d(\omega,\rho_0) \leq 0, \forall\omega \in [0,\tilde{\omega}_0]$.
*2)* $0 < K_d(\omega,\rho_0) \leq K_d(\omega,\rho), \forall\omega \in (\tilde{\omega}_0,\bar{\omega}_0]$.
*3)* $0 < K_p(\omega,\rho_0) \leq K_p(\omega,\rho), \forall\omega \in (0,\bar{\omega}_0)$.

***Proof.***

1) From the Lemma 4 and $\tilde{\omega}_0 \leq \tilde{\omega}$ follow that $K_d(\omega,\rho_0) \leq 0$ and $K_d(\omega,\rho) \leq 0$ for all $\omega \in [0,\tilde{\omega}_0]$. Taking into account that $\cos(\omega) > 0$ for $\omega \in [0,\tilde{\omega}_0]$ and $\frac{n_0}{\tau_0 c_0} \leq \frac{n}{\tau c}$

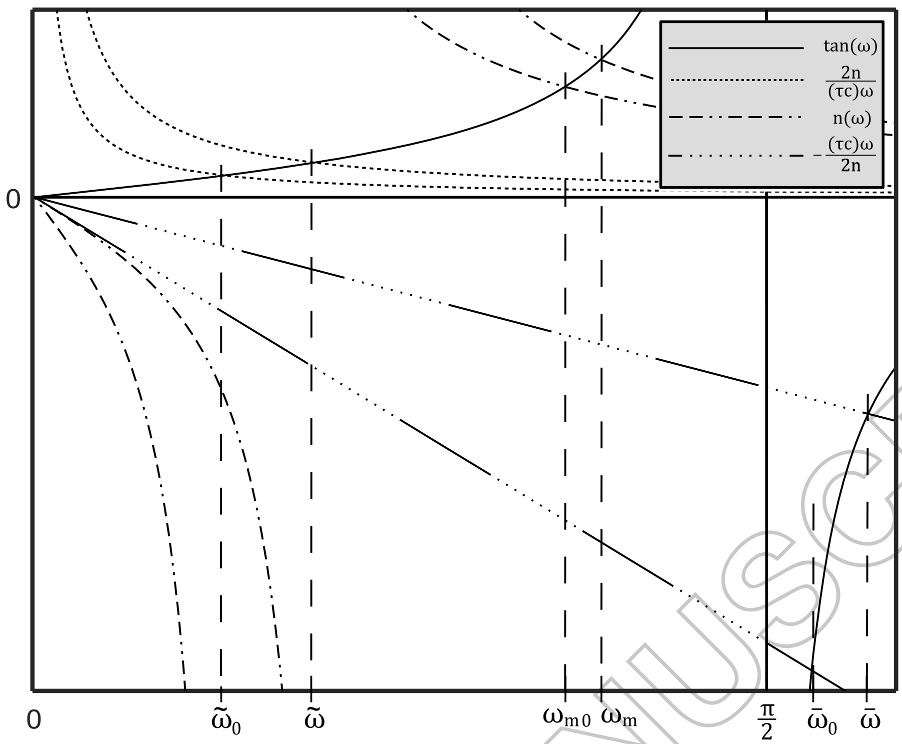


Figure 2.: Numerical solution of (17), (18) and (20) for network parameters $(n, \tau, c)$ and $(n_0, \tau_0, c_0)$ satisfying (22)

we have

$$\frac{2n}{\tau c^2}\left(\omega \sin(\omega) - \frac{2n}{\tau c}\cos(\omega)\right) \leq \frac{2n_0}{\tau_0 c_0^2}\left(\omega \sin(\omega) - \frac{2n_0}{\tau_0 c_0}\cos(\omega)\right) \leq 0, \forall \omega \in [0, \tilde{\omega}_0],$$

and then $K_d(\omega, \rho) \leq K_d(\omega, \rho_0) \leq 0, \forall \omega \in [0, \tilde{\omega}_0]$.

2) From the Lemma 4, $\tilde{\omega}_0 \leq \tilde{\omega}$ and $\bar{\omega}_0 \leq \bar{\omega}$ follow that $K_d(\omega, \rho_0) > 0$ and $K_d(\omega, \rho) > 0$ for all $\omega \in (\tilde{\omega}_0, \bar{\omega}_0]$. By using the inequality $\frac{n_0}{\tau_0 c_0} \leq \frac{n}{\tau c}$ direct calculations lead to

$$\frac{2n}{\tau c^2}\left(\omega \sin(\omega) - \frac{2n}{\tau c}\cos(\omega)\right) \geq \frac{2n_0}{\tau_0 c_0^2}\left(\omega \sin(\omega) - \frac{2n_0}{\tau_0 c_0}\cos(\omega)\right) > 0, \omega \in (\tilde{\omega}_0, \bar{\omega}_0],$$

and then $K_d(\omega, \rho) \geq K_d(\omega, \rho_0) > 0, \forall \omega \in (\tilde{\omega}_0, \bar{\omega}_0]$.

3) From the Lemma 5 we have that $K_p(\omega, \rho_0) = \frac{2n_0\omega}{(\tau_0 c_0)^2} m(\omega, \rho_0)$ and $K_p(\omega, \rho) = \frac{2n\omega}{(\tau c)^2} m(\omega, \rho)$, where $m(\omega, \rho_0)$ and $m(\omega, \rho)$ are determined by (19) corresponding to the parameters $\rho_0$ and $\rho$, respectively. Also, from the Lemma 5 and the fact that $\bar{\omega}_0 \leq \bar{\omega}$ one has that $m(\omega, \rho_0) > 0$ and $m(\omega, \rho) > 0$ for all $\omega \in (0, \bar{\omega}_0)$. Since $\sin(\omega) > 0$ for $\omega \in (0, \bar{\omega}_0)$ and $\frac{n_0}{\tau_0 c_0} \leq \frac{n}{\tau c}$ then it is easy to see that $m(\omega, \rho) \geq m(\omega, \rho_0)$ from which follows that $K_p(\omega, \rho) \geq K_p(\omega, \rho_0)$ for all $\omega \in (0, \bar{\omega}_0)$ as required.

□

**Proposition 7.** *Given networks parameters $\rho_0$ and $\rho$ satisfying (22) the following property holds:*

$$\Gamma_{\rho_0} \subseteq \Gamma_\rho.$$

***Proof.*** Obviously, $\Gamma_{\rho_0} = \Gamma_\rho$ when $\rho_0 = \rho$. Note that in order to demonstrate the non-trivial general result it suffices to prove that the stability region inclusion holds for some parameters $\rho_1 = (n_1, \tau_1, c_1)$ satisfying (22), since we can use a continuation procedure by redefining $n_1 = n_0, \tau_1 = \tau_0$ and $c_1 = c_0$. Without any loss of generality we can assume that $\tilde{\omega}_0 < \tilde{\omega} < \omega_1$ and $\omega_{m0} < \omega_m < \omega_2$ since one can choose parameters $\rho_1$ satisfying (22) such that these inequalities hold.

Let $(K_{p0}, K_{d0}) \in \Gamma_{\rho_0}$. From the convexity of $\Gamma_{\rho_0}$ there exist $\omega_1 \in (0, \omega_{m0})$ and $\omega_2 \in (\omega_{m0}, \bar{\omega}_0)$ such that

$$K_{p0} = K_p(\omega_1, \rho_0) = K_p(\omega_2, \rho_0) \text{ and } K_d(\omega_1, \rho_0) < K_{d0} < K_d(\omega_2, \rho_0). \tag{24}$$

Let us observe the point $(K_p(\omega_1, \rho_0), K_d(\omega_1, \rho_0)) \in \partial\Gamma_{\rho_0}$. Based on the properties of the functions $K_p(\omega_1, \rho)$ and $K_d(\omega_1, \rho)$ w.r.t. parameters variations we have to consider the following two cases:

1) $\omega_1 \in (0, \tilde{\omega}_0)$. From the Lemma 6 we have

$$0 < K_p(\omega_1, \rho_0) < K_p(\omega_1, \rho_1) \text{ and } K_d(\omega_1, \rho_1) < K_d(\omega_1, \rho_0) < 0.$$

These inequalities and the facts that $K_d(\omega, \rho_1)$ is increasing for $\omega \in (0, \bar{\omega}_0)$ and $K_p(\omega, \rho_1)$ is increasing for $\omega \in (0, \omega_1) \subset (0, \tilde{\omega}_0)$ imply there exists $\omega_3 < \omega_1$ such that

$$K_p(\omega_1, \rho_0) = K_p(\omega_3, \rho_1) \text{ and } K_d(\omega_3, \rho_1) < K_d(\omega_1, \rho_1)$$

2) $\omega_1 \in (\tilde{\omega}_0, \omega_{m0})$. In this case, from the Lemma 6 we have that

$$0 < K_p(\omega_1, \rho_0) < K_p(\omega_1, \rho_1) \text{ and } 0 < K_d(\omega_1, \rho_0) < K_d(\omega_1, \rho_1).$$

Since $K_d(\omega, \rho_1)$ is increasing for $\omega \in (0, \bar{\omega}_0)$ then there exists $\omega_4 < \omega_1$ such that $K_d(\omega_1, \rho_0) = K_d(\omega_4, \rho_1)$. On the other hand, since $K_p(\omega, \rho_1)$ is increasing for $\omega \in (0, \omega_1)$ then there exists $\omega_5 < \omega_1$ such that $K_p(\omega_1, \rho_0) = K_p(\omega_5, \rho_1)$. The continuity and increasing behavior of $K_p(\omega, \rho_1)$ and $K_d(\omega, \rho_1)$ in the interval $(0, \omega_1)$ imply that $\omega_5 < \omega_4$. Thus, we have that there exists $\omega_5 < \omega_1$ such that

$$K_p(\omega_1, \rho_0) = K_p(\omega_5, \rho_1) \text{ and } K_d(\omega_5, \rho_1) < K_d(\omega_1, \rho_0).$$

From the above analysis in the two cases and the inequality (24) follow that there exists $\omega_1^{'} < \omega_1$, where $\omega_1^{'}$ is either $\omega_3$ or $\omega_5$ in each one of the cases, such that

$$K_{p0} = K_p(\omega_1^{'}, \rho_1) \text{ and } K_d(\omega_1^{'}, \rho_1) < K_{d0}. \tag{25}$$

Now, let us observe the point $(K_p(\omega_2, \rho_0), K_d(\omega_2, \rho_0)) \in \partial\Gamma_{\rho_0}$. From the Lemma 6 we have

$$0 < K_p(\omega_2, \rho_0) < K_p(\omega_2, \rho_1) \text{ and } 0 < K_d(\omega_2, \rho_0) < K_d(\omega_2, \rho_1).$$

From the above inequalities and the facts that $K_p(\omega, \rho_1)$ is decreasing for $\omega \in (\omega_2, \bar{\omega}_0)$, because of $\omega_2 > \omega_m$, while $K_d(\omega, \rho_1)$ is increasing for $\omega \in (\omega_2, \bar{\omega}_0)$ follow that there exists $\omega_2^{'} > \omega_2$ such that

$$K_p(\omega_2, \rho_0) = K_p(\omega_2^{'}, \rho_1) \text{ and } K_d(\omega_2, \rho_1) < K_d(\omega_2^{'}, \rho_1).$$

It follows from (24) that

$$K_p(\omega_2, \rho_0) = K_p(\omega_2^{'}, \rho_1) \text{ and } K_{d0} < K_d(\omega_2^{'}, \rho_1) \tag{26}$$

From (25) and (26) one arrives at the conclusion

$$K_{p0} = K_p(\omega_2^{'}, \rho_1) = K_p(\omega_1^{'}, \rho_1) \text{ and } K_d(\omega_1^{'}, \rho_1) < K_{d0} < K_d(\omega_2^{'}, \rho_1)$$

which implies that $(K_{p0}, K_{d0}) \in \Gamma_{\rho_1}$ and, therefore, $\Gamma_{\rho_0} \subset \Gamma_{\rho_1}$ as required. □

## 4. Nonlinear Analysis

The nonlinear system (12) can be written as:

$$\begin{aligned} \ddot{y}(t) &= -0.5\left(\dot{y}^2(t) + 2w_0\dot{y}(t)\right)\left(K_1 y(t-1) + K_2\dot{y}(t-1) + p_0\right) \\ &\quad -0.5w_0^2\left(K_1 y(t-1) + K_2\dot{y}(t-1)\right), \end{aligned} \tag{27}$$

where $K_1 = nK_p$ and $K_2 = \frac{n}{\tau}K_d$. In order to formulate a Lyapunov-Krasovskii functional approach for the stability of the nonlinear system (27) we need to introduce a little of terminology. As usual, for a solution $y(t,\varphi), t \geq 0$, we define the natural state of (27) by $y_t(\varphi)(\theta) = y(t+\theta,\varphi), \theta \in [-1,0]$. For simplicity of the notation one writes $y_t(\varphi)$ instead of $y_t(\varphi)(\theta)$. Now, in a Lyapunov-Krasovskii functional framework for (27) one needs to define a functional $(y_t(\varphi), \dot{y}_t(\varphi)) \to v\left(y_t(\varphi), \dot{y}_t(\varphi)\right) \in \mathbb{R}_+$. Since $y(t,\varphi)$ is continuously differentiable for all $t \geq -h$ then $y_t(\varphi) \in \mathcal{C}^1\left([-1,0], \mathbb{R}\right)$ for all $t \geq 0$. As a consequence, the Lyapunov functionals should be defined on the vector $(\varphi, \dot{\varphi})$ which belongs to $\mathcal{C}\left([-1,0], \mathbb{R}^2\right)$, the Banach space of continuous functions mapping $[-1,0]$ to $\mathbb{R}^2$ and equipped with the supremum norm $\|(\varphi, \dot{\varphi})\| = \sup_{\theta \in [-1,0]} \|(\varphi(\theta), \dot{\varphi}(\theta))\|$.

**Theorem 8.** *The zero solution of the system (27) is locally asymptotically stable if $K_p, K_d > 0$ satisfy*

$$K_p + \frac{1}{\tau}K_d < \frac{2n^2}{(\tau c)^3}. \tag{28}$$

*The set*

$$\mathcal{U} := \left\{(\varphi, \dot{\varphi}) : \|(\varphi, \dot{\varphi})\| < \sigma \text{ and } v(\varphi, \dot{\varphi}) < \gamma_1 \sigma^2\right\}, \tag{29}$$

*where* $\gamma_1 = \min\left\{\frac{w_0^2 n K_p}{2}, 1\right\}$ *and* $\sigma = \frac{2}{w_0^2 n\left(K_p + \frac{1}{\tau}K_d\right)} - w_0 > 0$, *is an estimate of the attraction region.*

***Proof.*** Let us consider the following Lyapunov-Krasovskii functional candidate:

$$v(\varphi,\dot{\varphi}) = \frac{w_0^2 K_1}{2}\varphi^2(0) + \dot{\varphi}^2(0) + \frac{w_0^2 K_1}{2}\int_{-1}^{0}\int_{\theta}^{0}\dot{\varphi}^2(\xi)\,d\xi d\theta + \frac{w_0^2 K_2}{2}\int_{-1}^{0}\dot{\varphi}^2(\xi)\,d\xi. \quad (30)$$

Clearly, the functional (30) satisfies the following inequalities:

$$\gamma_1 \left\|(\varphi(0),\dot{\varphi}(0))\right\|^2 \le v(\varphi,\dot{\varphi}) \le \gamma_2 \left\|(\varphi,\dot{\varphi})\right\|^2,$$

with $\gamma_1 = \min\left\{\frac{w_0^2 K_1}{2}, 1\right\}$ and $\gamma_2 = \max\left\{\frac{w_0^2 K_1}{2}, 1 + \frac{w_0^2 K_1}{4} + \frac{w_0^2 K_2}{2}\right\}$. The time derivative of the functional (30) along the solutions of (27) is

$$\begin{aligned}\frac{d}{dt}v(y_t,\dot{y}_t) &= w_0^2 K_1 \dot{y}(t)\left(y(t) - y(t-1)\right) - w_0^2 K_2 \dot{y}(t)\dot{y}(t-1)\\ &\quad -\dot{y}^2(t)\left(\dot{y}(t) + 2w_0\right)\left(K_1 y(t-1) + K_2\dot{y}(t-1) + p_0\right)\\ &\quad + \frac{w_0^2}{2}\left(K_1 + K_2\right)\dot{y}^2(t) - \frac{w_0^2 K_1}{2}\int_{-1}^{0}\dot{y}^2(t+\theta)d\theta - \frac{w_0^2 K_2}{2}\dot{y}^2(t-1).\end{aligned}$$

Observing that

$$\begin{aligned}w_0^2 K_1 \dot{y}(t)\left(y(t) - y(t-1)\right) &= w_0^2 K_1 \dot{y}(t)\int_{-1}^{0}\dot{y}(t+\theta)d\theta\\ &\le \frac{w_0^2 K_1}{2}\left(\dot{y}^2(t) + \int_{-1}^{0}\dot{y}^2(t+\theta)d\theta\right)\end{aligned}$$

and

$$-w_0^2 K_2 \dot{y}(t)\dot{y}(t-1) \le \frac{w_0^2 K_2}{2}\left(\dot{y}^2(t) + \dot{y}^2(t-1)\right)$$

we get the following upper bound for the time derivative:

$$\frac{d}{dt}v(y_t,\dot{y}_t) \le w_0^2\left(K_1 + K_2\right)\dot{y}^2(t) - \dot{y}^2(t)\left(\dot{y}(t) + 2w_0\right)\left(K_1 y(t-1) + K_2\dot{y}(t-1) + p_0\right). \quad (31)$$

From $\dot{y}(t) = \dot{\tilde{q}}(t) = w(t) - w_0$ and $w(t) \ge 0$ we have $\dot{y}(t) \ge -w_0$ and, therefore,

$$\dot{y}(t) + 2w_0 \ge w_0. \quad (32)$$

The condition (28) is equivalent to $K_1 + K_2 < \frac{2}{w_0^3}$ from which follows that

$$\sigma = \varsigma - w_0 > 0, \text{ where } \varsigma = \frac{2}{w_0^2\left(K_1 + K_2\right)},$$

If for some $t \ge 0$ we have that $(y_t,\dot{y}_t) \in \mathcal{U}$, where $\mathcal{U}$ is given by (29), then $\|(y_t,\dot{y}_t)\| < \sigma$.

Since $\sigma < \varsigma$ then $\|(y_t, \dot{y}_t)\| < \sigma$ implies $\|(y(t-1), \dot{y}(t-1))\| < \varsigma$. It follows that

$$K_1 y(t-1) + K_2 \dot{y}(t-1) + p_0 > 0. \tag{33}$$

From (32) and (33) we have

$$\begin{array}{rl} & -\dot{y}^2(t)\left(\dot{y}(t) + 2w_0\right)\left(K_1 y(t-1) + K_2 \dot{y}(t-1) + p_0\right) \\ \leq & -w_0 \dot{y}^2(t)\left(K_1 y(t-1) + K_2 \dot{y}(t-1) + p_0\right) \end{array}$$

and using this in (31) we arrive at

$$\frac{d}{dt} v(y_t, \dot{y}_t) \leq -\eta\left(y_t, \dot{y}_t\right), \tag{34}$$

where

$$\eta\left(y_t, \dot{y}_t\right) = w_0 \dot{y}^2(t)\left(K_1\left(y(t-1) - w_0\right) + K_2\left(\dot{y}(t-1) - w_0\right) + p_0\right),$$

that holds when $(y_t, \dot{y}_t) \in \mathcal{U}$. Since $\|(y(t-1), \dot{y}(t-1))\| < \sigma$ then $y(t-1) - w_0 > -\varsigma$ and $\dot{y}(t-1) - w_0 > -\varsigma$ holds. From this follows

$$K_1\left(y(t-1) - w_0\right) + K_2\left(\dot{y}(t-1) - w_0\right) + p_0 > 0, \tag{35}$$

that implies $\eta\left(y_t, \dot{y}_t\right) \geq 0$. Now we will show that $\mathcal{U}$ is a positively invariant set with respect to equation (27). Let an initial function $(\varphi, \dot{\varphi}) \in \mathcal{U}$. For this initial function we have that $\|(\varphi, \dot{\varphi})\| < \sigma$ and therefore $\frac{d}{dt} v(y_t, \dot{y}_t)\big|_{t=0} \leq 0$. Then, there exists $t_0 \in (0,1)$ (maybe sufficiently small) such that

$$v(y_t, \dot{y}_t) \leq v\left(\varphi, \dot{\varphi}\right) \text{ for } t \in [0, t_0).$$

From this and the inequalities $\gamma_1 \|(y(t), \dot{y}\,(t))\|^2 \leq v(y_t, \dot{y}_t)$ and $v\left(\varphi, \dot{\varphi}\right) < \gamma_1 \sigma^2$ follow that $\|(y(t), \dot{y}\,(t))\| < \sigma$ holds for all $t \in [0, t_0)$. Observing that for $t \in [-h, 0]$ we have $\|(y(t), \dot{y}\,(t))\| = \|(\varphi(t), \dot{\varphi}\,(t))\| \leq \|(\varphi, \dot{\varphi})\| < \sigma$ then we conclude that

$$\|(y_t, \dot{y}_t)\| < \sigma \text{ for all } t \in [0, t_0),$$

which implies $(y_t, \dot{y}_t) \in \mathcal{U}$ for all $t \in [0, t_0)$. Let us suppose that $(y_t, \dot{y}_t)$ does not belong to $\mathcal{U}$ for all $t \geq 0$, then there exists a $t_1 \geq t_0$ such that $\|(y(t), \dot{y}\,(t))\| < \sigma$ for all $t \in [0, t_1)$ and $\|(y(t_1), \dot{y}(t_1))\| = \sigma$. The continuity of $\frac{d}{dt} v(y_t, \dot{y}_t)$ implies that

$$\left.\frac{d}{dt} v(y_t, \dot{y}_t)\right|_{t=t_1} \leq 0$$

and the inequalities

$$\gamma_1 \|(y(t_1), \dot{y}\,(t_1))\|^2 \leq v(y_{t_1}, \dot{y}_{t_1}) \leq v\left(\varphi, \dot{\varphi}\right) < \gamma_1 \sigma^2$$

imply that $\|(y(t_1), \dot{y}(t_1))\| < \sigma$. The contradiction shows that $(y_t, \dot{y}_t)$ belongs to $\mathcal{U}$ for all $t \geq 0$ for any initial function $(\varphi, \dot{\varphi}) \in \mathcal{U}$ and therefore the set $\mathcal{U}$ is a positively invariant set with respect to (27). For an initial function $(\varphi, \dot{\varphi}) \in \mathcal{U}$ we now show that the corresponding solution $y(t)$ asymptotically converges to zero when $t \to \infty$. Since $\|(y(t), \dot{y}(t))\| < \sigma$ for all $t \geq 0$ then $y(t)$ and $\dot{y}(t)$ are uniformly bounded for all $t \geq 0$, which implies $\eta(y_t, \dot{y}_t)$ is also uniformly bounded for all $t \geq 0$. From (34) we have

$$\int_0^t \eta(y_\xi, \dot{y}_\xi)\, d\xi \leq v(\varphi, \dot{\varphi}) - v(y_t, \dot{y}_t).$$

Since $v(y_t, \dot{y}_t)$ is not increasing and bounded from below by zero then it converges as $t \to \infty$, and therefore

$$\lim_{t\to\infty} \int_0^t \eta(y_\xi, \dot{y}_\xi)\, d\xi$$

exists and is finite. An application of the Barbalat's lemma yields at $\lim_{t\to\infty} \eta(y_t, \dot{y}_t) = 0$. The uniform continuity of the function $\eta(y_t, \dot{y}_t)$ implies that $(y_t, \dot{y}_t)$ converges to the largest invariant set of (27) where $\eta(y_t, \dot{y}_t) = 0$. Taking into account (35) follows that $\eta(y_t, \dot{y}_t) = 0$ if and only if $\dot{y}^2(t) = 0$ for all $t \geq 0$ implying $y(t) \equiv a$ for all $t \geq 0$, where $a$ is a constant. From the equation (27) follows that the only constant solution is the trivial one. Hence, the set $\mathcal{U}$ is an estimate of the attraction region. □

The estimate of the attraction region $\mathcal{U}$ can be complicated to compute since it involves the Lyapunov functional $v(\varphi, \dot{\varphi})$ given by (30). By using the lower and upper bounds for the functional $v(\varphi, \dot{\varphi})$ one obtains a computational more convenient estimate of the attraction region given by

$$\mathcal{V} = \left\{ (\varphi, \dot{\varphi}) : \|(\varphi, \dot{\varphi})\| < \sqrt{\frac{\gamma_1}{\gamma_2}}\sigma \right\} \subseteq \mathcal{U}. \tag{36}$$

**Remark 3.** *The functional (30) is generated from the one used in (Michiels et al. 2006) for the analysis of P controllers by adding an additional integral term. As a consequence, if $K_d = 0$, i.e., when one considers a P controller, then the stability condition (28) reduces to the one obtained in Theorem 2 of (Michiels et al. 2006). Thus, the Theorem 8 generalizes for PD controllers the Theorem 2 in (Michiels et al. 2006) stated for P controllers.*

## 5. Robust stability analysis

In this section, we address the robust stability analysis of PD AQM controllers under uncertain network parameters. To this aim, let us consider the nonlinear system (10) for some nominal parameters $(n_0, \tau_0, c_0)$, i.e., the system

$$\ddot{\tilde{q}}_0(t) = 1 - 0.5\left(\dot{\tilde{q}}_0(t) + w_0\right)^2 \left(K_p\left(n_0\tilde{q}_0(t-1) - q_d\right) + \frac{K_d n_0}{\tau_0}\dot{\tilde{q}}_0(t-1) + p_0\right), \tag{37}$$

where $w_0 = \frac{\tau_0 c_0}{n_0}$ and $p_0 = \frac{2}{w_0^2}$, whose unique equilibrium point is $q_0^* = \frac{q_d}{n_0}$. Let us assume that the controller's gains $K_p$ and $K_d$ are designed such that the equilibrium $q_0^*$ is locally asymptotically stable and consider that such controller is implemented in the system (10), but with new network parameters $(n_1, \tau_1, c_1)$. This leads to the following nonlinear system:

$$\ddot{\tilde{q}}_1(t) = 1 - 0.5\left(\dot{\tilde{q}}_1(t) + w_1\right)^2\left(K_p\left(n_1\tilde{q}_1(t-1) - q_d\right) + \frac{K_d n_1}{\tau_1}\dot{\tilde{q}}_1(t-1) + p_0\right), \quad (38)$$

where $w_1 = \frac{\tau_1 c_1}{n_1}$. Direct calculations show that the unique equilibrium of (38) is

$$q_1^* = \frac{1}{n_1 K_p}\left(\frac{2}{w_1^2} - \frac{2}{w_0^2}\right) + \frac{1}{n_1}q_d.$$

Note that with respect to the original system (6), in coordinate $q(t)$, the corresponding equilibrium $q_1 = n_1 q_1^*$ is

$$q_1 = \frac{1}{K_p}\left(\frac{2}{w_1^2} - \frac{2}{w_0^2}\right) + q_d. \quad (39)$$

The expression (39) shows that a PD AQM control cannot regulate to the desired value, for which is designed with nominal parameters, under uncertain network parameters, since the corresponding closed-loop system has a new equilibrium, which is unique and different to the desired one. If well is true that this characteristic of the PD control on regulating under uncertain parameters is well-know, called as the steady-state error in the classical control literature, we believe the expression (39) has not been reported in the literature of PD AQM control. In this context, it appears desirable at least to assure that the new equilibrium $q_1$ is also locally asymptotically stable for the same controller's gains $K_p$ and $K_d$ designed for local asymptotic stability of the desired value $q_d$. Based on this, let us consider that the network parameters $(n, \tau, c)$ are constants satisfying the following interval type condition:

$$n \in [n_1, n_2], \ \tau \in [\tau_1, \tau_2] \ \text{and} \ c \in [c_1, c_2]. \quad (40)$$

Given a desired value $q_d$, we aim at determining some network parameters, say $n_r, \tau_r$ and $c_r$, satisfying the condition (40) and designing the gains $K_p$ and $K_d$ for guaranteeing the local asymptotic stability of the whole family of equilibria determined for all values of the parameters $(n, \tau, c)$ satisfying (40), which is defined by

$$q_1(K_p, n, \tau, c) = \frac{1}{K_p}\left(\frac{2}{w^2} - \frac{2}{w_r^2}\right) + q_d, \quad (41)$$

where $w = \frac{\tau c}{n}$ and $w_r = \frac{\tau_r c_r}{n_r}$.

The robust stability problem formulated above has a nice solution by observing that any nonlinear system, determined by the network parameters satisfying (40), has the same form when their equilibrium points are translated to the origin. To see this, let us translate the equilibrium points $q_0^*$ and $q_1^*$ of the nonlinear systems (37) and (38) to the origin. Thus, let $y_0(t) = \tilde{q}_0(t) - q_0^*$ and $y_1(t) = \tilde{q}_1(t) - q_1^*$. Then, direct calculations derived from (37) and (38) yield at the following nonlinear systems:

$$\ddot{y}_0(t) = 1 - 0.5\left(\dot{y}_0(t) + w_0\right)^2\left(n_0 K_p y(t-1) + \frac{n_0}{\tau_0}K_d\dot{y}(t-1) + p_0\right) \quad (42)$$

and

$$\ddot{y}_1(t) = 1 - 0.5\left(\dot{y}_1(t) + w_1\right)^2 \left(n_1 K_p y(t-1) + \frac{n_1}{\tau_1} K_d \dot{y}(t-1) + p_1\right), \tag{43}$$

having trivial solution. Clearly, the nonlinear systems (42) and (43) are in fact of the same form of the nonlinear system (12), where $(n,\tau,c) = (n_0,\tau_0,c_0)$ for (42) while that $(n,\tau,c) = (n_1,\tau_1,c_1)$ for (43). Thus, by applying the nonlinear stability conditions in Theorem 8 we can derive the following robust stability result.

**Proposition 9.** *The whole family of equilibria points defined by (41), where $(n_r,\tau_r,c_r) = (n_1,\tau_2,c_2)$ and $(n,\tau,c)$ are any parameters satisfying (40), is locally asymptotically stable if $K_p, K_d > 0$ satisfy*

$$K_p + \frac{1}{\tau_1} K_d < \frac{2n_r^2}{(\tau_r c_r)}. \tag{44}$$

*The sets*

$$\mathcal{U}_r = \left\{(\varphi,\dot{\varphi}) : \|(\varphi,\dot{\varphi})\| < \sigma_r \text{ and } v(\varphi,\dot{\varphi}) < \gamma_{1r}\sigma_r^2\right\}, \tag{45}$$

*and*

$$\mathcal{V}_r = \left\{(\varphi,\dot{\varphi}) : \|(\varphi,\dot{\varphi})\| < \sqrt{\frac{\gamma_{1r}}{\gamma_{2r}}}\sigma_r\right\} \subseteq \mathcal{U}_r, \tag{46}$$

*where $\gamma_{1r} = \min\left\{\frac{(\tau_1 c_1)^2 K_p}{2n_2}, 1\right\}, \gamma_{2r} = \min\left\{\frac{w_r^2 n_r K_p}{2}, 1 + \frac{w_r^2 n_r K_p}{4} + \frac{w_r c_r K_d}{2}\right\}$ and $\sigma_r = \frac{2}{w_r^2 n_r\left(K_p + \frac{1}{\tau_1}K_d\right)} - w_r > 0$, are estimate of the robust attraction region.*

***Proof.*** Since for all $(n,\tau,c)$ satisfying (40) we have that

$$\frac{n_1^3}{(\tau_2 c_2)^3} \leq \frac{n^3}{(\tau c)^3} \text{ and } \frac{1}{\tau} K_d \leq \frac{1}{\tau_1} K_d,$$

then by selecting $n_r = n_1, \tau_r = \tau_2$ and $c_r = c_2$ the result directly follows from Theorem 8. The set $\mathcal{U}_r$ given in (45) ($\mathcal{V}_r$ given by (46)) is the one contained in the whole family of sets $\mathcal{U}$ given by (29) ($\mathcal{V}$ given by (36)), which is generated by all the parameters $(n,\tau,c)$ satisfying (40). The expressions in (45) and (46) are directly obtained by observing that $\sigma_r \leq \sigma, \gamma_{1r} \leq \gamma_1$ and $\gamma_2 \leq \gamma_{2r}$. □

By linearizing the nonlinear systems (42) and (43) one obtains systems of the form of the linear delay system (13) with parameters $(n,\tau,c) = (n_0,\tau_0,c_0)$ and $(n,\tau,c) = (n_1,\tau_1,c_1)$, respectively. Thus, by applying the linear stability conditions in Proposition 2 and the geometric property in Proposition 7 we can derive the following robust stability result.

**Proposition 10.** *The whole family of equilibria points defined by (41), where $(n_r,\tau_r,c_r) = (n_1,\tau_2,c_2)$ and $(n,\tau,c)$ are any parameters satisfying (40), is locally asymptotically stable if $(K_p,K_d) \in \Gamma_{(n_r,\tau_r,c_r)}$.*

***Proof.*** From Proposition 7 we have that $\Gamma_{(n_1,\tau_2,c_2)} \subseteq \Gamma_{(n,\tau,c)}$ for all $(n,\tau,c)$ satisfying (40). Then, the results follows directly from Proposition 2. $\square$

**Remark 4.** *Since the expression (39) depends only on the proportional gain $K_p$ and not on the derivative gain $K_d$, then the above analysis is also valid for P AQM controllers, i.e., a P AQM control also exhibits a new equilibrium point under uncertain network parameters and the robust stability results provided in the Propositions 9 and 10 are also valid for P AQM controllers.*

## 6. Simulations

In this section, we present some numerical simulations to illustrate the main results of the paper. As a numerical illustration let us consider the case when $q_d = 175$ packets and the network parameters $(n,\tau,c)$ are constants satisfying the interval type condition (40), where $n_1 = 60$ TCP flows, $n_2 = 100$ TCP flows, $\tau_1 = 0.200$ s, $\tau_2 = 0.246$ s, $c_1 = 3500$ packets/s and $c_2 = 3750$ packets/s.

According with our results under this setup we firstly determine the parameters $(n_r,\tau_r,c_r) = (n_1,\tau_2,c_2)$ and then apply Propositions 9 and 10 to obtain the PD controller's gains which locally stabilize the whole family of equilibrium points determined by (41). From the nonlinear robust stability condition (44) we get that if $K_p, K_d > 0$ and

$$K_p + \frac{1}{\tau_1}K_d = K_p + 5K_d < \frac{2n_r^2}{(\tau_r c_r)^3} = 9.1714 \times 10^{-6} \tag{47}$$

then the whole family of equilibrium points will be asymptotically stable. On the other hand, from the linear robust stability conditions in Proposition 10 we have that if $(K_p, K_d) \in \Gamma_{(n_r,\tau_r,c_r)}$ then the whole family of equilibrium points will be asymptotically stable. We compute the region $\Gamma_{(n_r,\tau_r,c_r)}$ by means of the Proposition 2 and plotted it along with the region generated by the nonlinear condition (47) in Fig. 3. As it can be seen from Fig. 3, the stability region determined by the nonlinear condition (47) is contained in the stability region $\Gamma_{(n_r,\tau_r,c_r)}$ generated from the linear stability conditions **as expected. On the other hand, with the condition (47), the nonlinearities of the system are considered, and estimates of the attraction region can be computed, as we will show below.** Now, let us select two pairs of controllers gains: $PD1 = (K_{p1}, K_{d1}) = \left(5 \times 10^{-6}, 0.5 \times 10^{-6}\right)$ inside of the region generated by the nonlinear stability condition (47) and $PD2 = (K_{p2}, K_{d2}) = \left(8 \times 10^{-5}, 3 \times 10^{-5}\right)$ inside the stability region $\Gamma_{(n_r,\tau_r,c_r)}$, see Fig. 3.

We illustrate the robust stability properties of these two PD controllers via Matlab/Simulink simulations implemented on the nonlinear model. The scenario for the simulations is the following: For $t \in [0,30]$ the network parameters are $(n,\tau,c) = (60, 0.246, 3750)$, for $t \in (30,60]$ the network parameters are $(n,\tau,c) = (100, 0.200, 3500)$ and for $t \in (60,90]$ the network parameters are $(n,\tau,c) = (80, 0.220, 3650)$. Under this scenario follows from (41) that for each PD controller ($PD1$ and $PD2$) we have three equilibrium points which depend on the proportional gains, see Table 1. As it can be seen in Table 1 the equilibirum points corresponding to $PD2$ are smaller than the equilibirum points associated to $PD1$ since $K_{p2} > K_{p1}$. The responses of the controllers $PD1$ and $PD2$ are given in Figs 4 and 5 respectively. As it can be seen, both controllers assure the asymptotic stability of the corresponding

| $(n,\tau,c)$ | $K_{p1}$ | $K_{p2}$ |
|---|---|---|
| (60,0.246,3750) | 175 | 175 |
| (100,0.200,3500) | 6646.2 | 579.4 |
| (80,0.220,3650) | 2453.1 | 317.4 |

Table 1.: Equilibirum points for PD1 and PD2

three equilibrium points as expected from the theoretical results.

Finally, let us compute the estimates of the attraction region associated to the controller $PD1$. Direct calculations lead to the following estimates of the attraction region:

$$\mathcal{U}_r = \{(\varphi,\dot{\varphi}) : \|(\varphi,\dot{\varphi})\| < 18.80127 \text{ and } v(\varphi,\dot{\varphi}) < 4.3302\}$$

and

$$\mathcal{V}_r = \{(\varphi,\dot{\varphi}) : \|(\varphi,\dot{\varphi})\| < 1.97413\} \subseteq \mathcal{U}_r.$$

## 7. Conclusions

In this paper, we investigated the local stability of some classes of TCP/AQM delay models by using a PD controller as AQM strategy. We showed that the corresponding closed-loop delay system is of neutral type and provided a formal justification for transforming it to a retarded delay one. The complete characterization of the set of all stabilizing PD controllers for the linearization of the models is provided in counterpart with the existing works that only give an estimate of this set. **This now allows the designers to select the controller gains for achieving some performance specifications based on the exact stability region and not in an estimate of it as occurring in the existing works.** A nonlinear stability analysis is performed by means of the Lyapunov-Krasovskii functional approach and simply-to-check sufficient conditions for asymptotic stability of the closed-loop equilibrium point along with estimates of the attraction region are obtained. **The robustness issue, of importance in the practical implementation of the controllers due to the highly varying network parameters in real scenarios, is also addressed**. We showed that a new equilibrium point is generated when a PD stabilizing controller is implemented in the system, but with new different network parameters and gave an explicit formula for it. Robust **linear and nonlinear** stability conditions to design the controller's gains for assuring the local stability of the whole family of equilibria points generated from uncertain network parameters of interval type are derived. **Since in the real application it is reasonable to have lower and upper bounds of the network parameters, based on measurements, then our robust stability contribution now allows the designers to choose the PD controller gains to maintain the desired stability despite varying network conditions and thus achieve a better performance. Finally, there are several directions for extending this work. With the knowledge of the robust set of all stabilizing controllers and robust nonlinear conditions, the controller design for satisfying some performance objectives it is an important issue to be ad-**

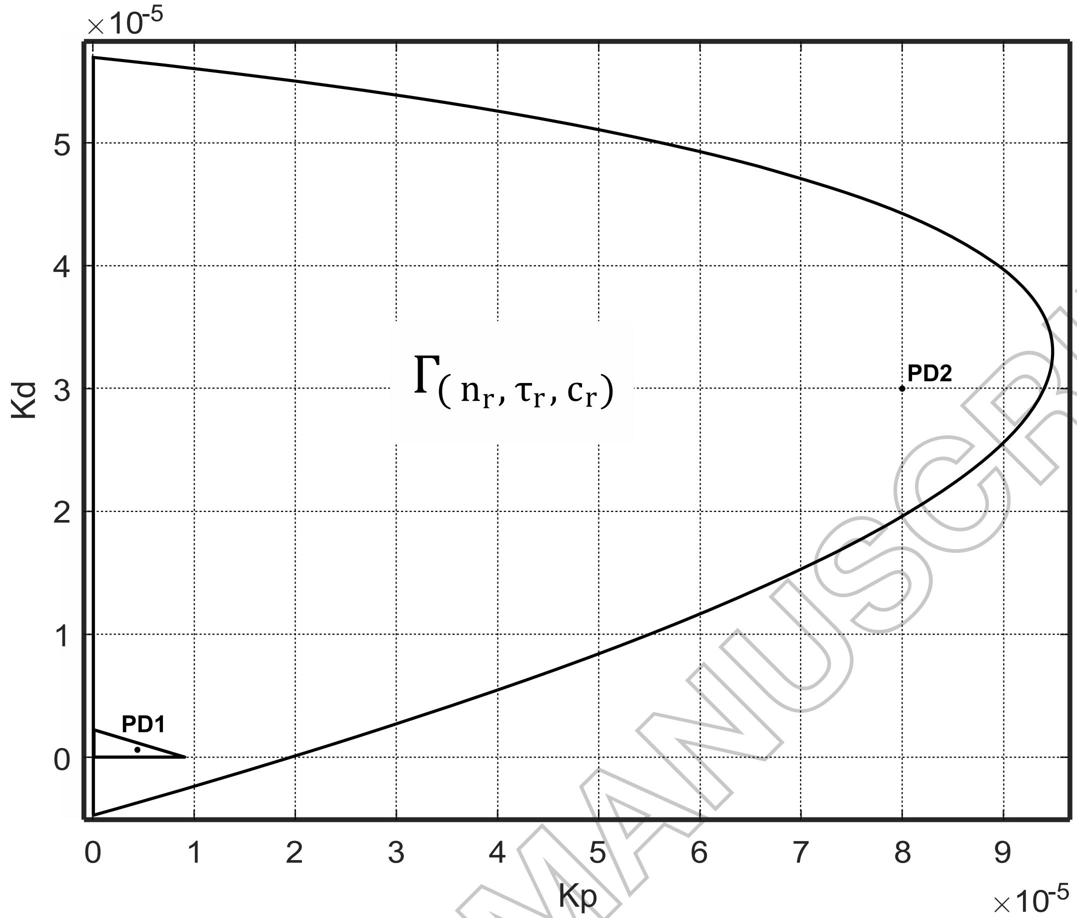


Figure 3.: Stability regions $\Gamma_{(n_r,\tau_r,c_r)}$ and the nonlinear one determined by the (47).

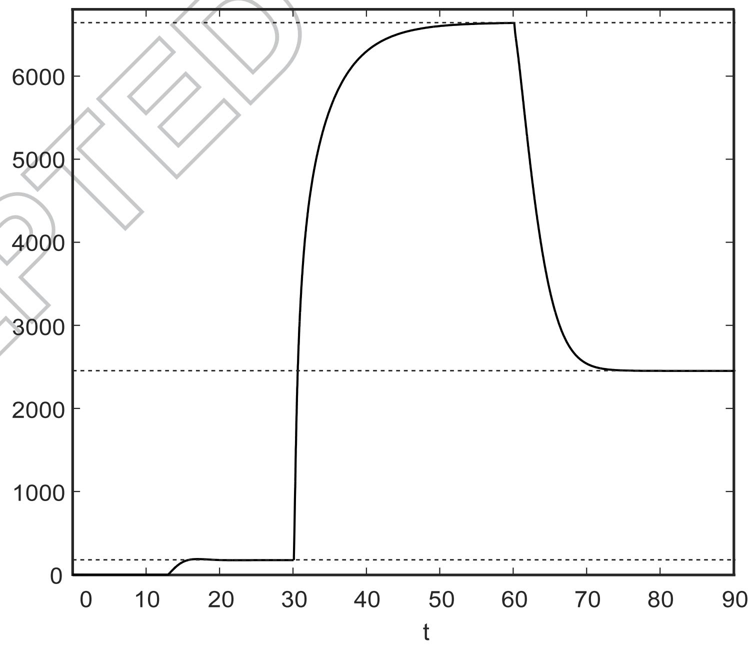


Figure 4.: Response of $q(t)$ for the $PD1$ controller.

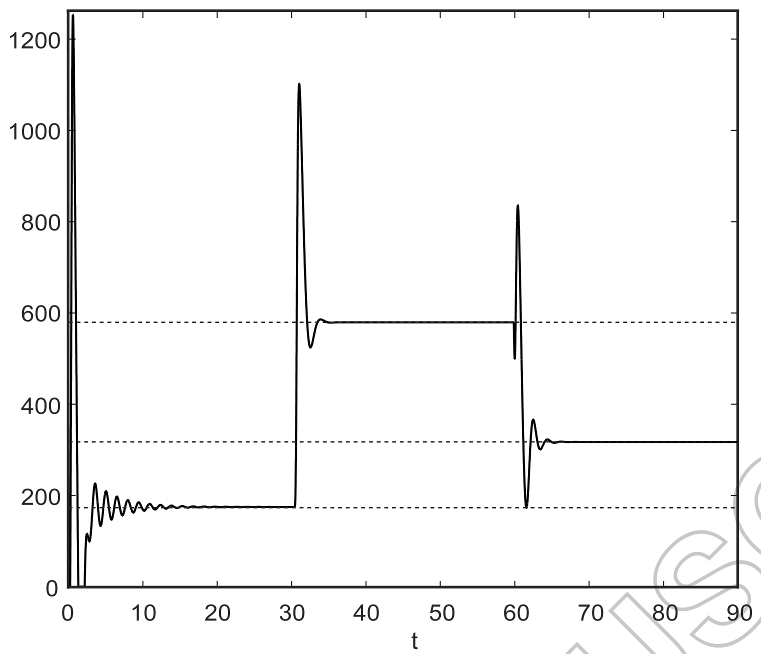


Figure 5.: Response of $q(t)$ for the $PD2$ controller.

**dressed. Extensions of this work to networks with multiple bottleneck links and heterogeneous round-trip times also deserve further study.**

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