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Research Article

Conditions to Flatten the Curvature Tensor Associated with a Symmetric Affine Connection

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In this study, we present a problem that consists in flattening the curvature associated with an affine connection on $\mathbb{R}^2$. This problem involves distributions and symmetric affine connections defined on a differential manifold M. For rank 1 constant distributions, we characterize the existence of solutions and get sufficient conditions to solve the problem. The general statement of the problem and some illustrative examples are given.

1. Introduction

The problem referred to as *flattening the curvature of an affine connection* (PFCAC) is motivated by a special case of the *curvature prescription* problem. The PFCAC arises in control theory by observing that there are conditions under which a mechanical system can be locally transformed (equivalent) to a simpler one and by realizing that the curvature plays an important role among those conditions.

Writing the equations that describe the dynamics of a mechanical system, in a specific way, sometimes facilitates solving problems that arise in control theory (controllability, stabilization, motion planning, trajectory tracking, and optimal control), as discussed by Bullo-Lewis [1]. For instance, for a subclass of *mechanical control system* $(Q, \nabla, D, \{Y_1, \ldots, Y_m\}, U)$, where Q is a smooth n-dimensional manifold (the configuration space of the system), n is the number of degrees of freedom, ∇ is an affine connection on Q, D is a regular linear velocity constraint distribution, having the property that ∇ restricts to D can be defined, $\{Y_a\colon a = 1, \ldots, m\}$ is a set of vector fields on Q (the control forces), taking values in D, and $U \subset \mathbb{R}^m$ is the control set. In this case, the equations of motion can be written as follows:

$$\ddot{q}^i = -\Gamma^i_{jk} v^j v^k + u^a Y^i_a, \quad i = \{1, \ldots, n\}. \tag{1}$$

These equations are written in a coordinate chart $(\mathcal{U}, \phi)$ of Q with coordinates $(q^1, \ldots, q^n)$. Here, one considers a collection $\mathcal{X} = \{X_1, \ldots, X_n\}$ of vector fields as a basis for T_qQ for each $q \in \mathcal{U}$, then $(v^1, \ldots, v^n)$ are the fiber coordinates for $T\mathcal{U}$ in the basis $\mathcal{X}$. For a control $u\colon I \longrightarrow \mathcal{U}$, $I \subset \mathbb{R}$ that is locally integrable, for $i, j \in \{1, \ldots, n\}$, $\nabla_{(\partial/\partial q^i)}(\partial/\partial q^j)$ is a vector field on u. Thus, $\nabla_{(\partial/\partial q^i)}(\partial/\partial q^j) = \Gamma^k_{ij}(\partial/\partial q^k)$ is a linear combination of the vector fields $\mathcal{X}$ on Q, for n^3 uniquely defined functions $\Gamma^k_{ij}\colon \mathcal{U} \longrightarrow \mathbb{R}$, $i, j, k \in \{1, \ldots, n\}$. The Γ^k_{ij} are called the *Christoffel symbols* for ∇ in the chart $(\mathcal{U}, \phi)$, and one defines the type $(1, 3)$ curvature tensor R whose components can be expressed as $R^a_{bc\,d} = (\partial\Gamma^a_{db}/\partial q^c) - (\partial\Gamma^a_{cb}/\partial q^d) + (\Gamma^a_{cm}\Gamma^m_{db} - \Gamma^a_{dm}\Gamma^m_{cb})$ for $a, b, c, d \in \{1, \ldots, n\}$. More details are provided in references [1–3].

On $T\mathcal{U}$, system (1) can be written as follows:

$$\dot{v} = S(v) + u^a Y^{\text{lift}}_a(v), \tag{2}$$

where v takes values in D, Y^{lift}_a is the vertical lift of Y_a in coordinates $Y^{\text{lift}}_a = Y^i_a(\partial/\partial v^i)$, Y^i_a are the components of Y_a, and S is the *geodesic spray* for ∇ in coordinates (q, v) on $T\mathcal{U}$, and it is expressed as follows:

$$S(v) = v^k \frac{\partial}{\partial q^k} - \Gamma^k_{ij}(v) v^i v^j \frac{\partial}{\partial v^k}. \tag{3}$$

We initially focus on mechanical systems of form (1) that are locally equivalent (or transformable) to a 3-*dimensional canonical form* with 2 inputs u^1, u^2:

$$\begin{aligned}\ddot{x}_1 &= u^1,\\ \ddot{x}_2 &= u^2,\\ \ddot{x}_3 &= u^1 x_1.\end{aligned} \tag{4}$$

This system is sometimes referred to as *the extended chained form* (ECF). An interesting point is that equation (4) is itself a model of controlled mechanical system [4], and affine control system can be expressed as follows:

$$\dot{z} = S(z) + u^1 Y_1^{\text{lift}}(z) + u^2 Y_2^{\text{lift}}(z), \tag{5}$$

where S is the *geodesic spray* for ∇ on $\mathbb{R}^3$. In coordinates (x, z) on $T\mathbb{R}^3$, it is expressed as follows:

$$S(z) = \sum_{k=4}^{6} z^k \frac{\partial}{\partial x^k}. \tag{6}$$

The problem of transforming a system of form (2) into (5) may be viewed as a problem of equivalence under the relation defined by *regular, static state-feedback* (RSSF); that is, two systems are equivalent under RSSF if one can be transformed into the other via invertible, static state feedback of the form $(v, u) \mapsto (z, w)$, where $z = \varphi(v)$ and $u = \alpha(v) + \beta(v)w$ represent, respectively, a coordinate transformation of the state variable and a change of control coordinates, with φ, a local diffeomorphism; $\beta(v)$, an invertible matrix for all $v \in T\mathbb{R}^3$; α, a smooth function. The new input components are $u_i = \alpha_i(v) + \beta_{ij}(v) w_j$, so after the transformation one gets

$$\dot{v} = S(v) + \left(\alpha_i(v) + \beta_{ij}(v) w_j\right) Y_i^{\text{lift}}(v), \tag{7}$$

and applying the change of state variables the system becomes

$$\dot{z} = S(z) + \left(\alpha_i(z) + \beta_{ij}(z) w\right) Y_i^{\text{lift}}(z). \tag{8}$$

If we label $\widehat{S}(z) = S(z) + \alpha_i(z) Y_i^{\text{lift}}(z)$ and $\widehat{Y}(z) = \beta_{ij}(z) Y_i^{\text{lift}}(z)$, then equation (8) can be written as ECF:

$$\dot{z} = \widehat{S}(z) + w \widehat{Y}(z), \tag{9}$$

where the new geodesic spray is expressed as follows:

$$\widehat{S}(z) = \dot{z}^k \frac{\partial}{\partial z^k}. \tag{10}$$

Comparing geodesic sprays (3) and (10), we note that the Γ_{ij}^k for the ECF are all zero; therefore, for the ECF, the curvature tensor is identically zero. In addition, one can observe that, after the transformation, the terms $\alpha_i(z) Y_i^{\text{lift}}(z)$ have been added to the geodesic spray $S(x)$ and those terms take values in the distribution D (space generated by the input vector fields with coefficients u) so new input coefficients are effectively defined.

Note that $\Gamma_{ij}^k + \alpha_{ij}^k$ define new functions, which we denote by $\widetilde{\Gamma}_{ij}^k$, this leads us to consider the existence of a new affine connection $\widetilde{\nabla}$, determined by $\widetilde{\Gamma}_{ij}^k$ and expressed in terms of the original ∇ and the terms $\alpha_i(z) Y_i^{\text{lift}}(z)$, which can be naturally associated with a type $(1,2)$ tensor F on D.

Curvature may be "flattened" by getting rid of the quadratic force terms related with the Γ_{ij}^k, but it is not an intrinsic fact because the curvature tensor may be zero even though the Christoffel symbols are not [4]. Then, by adding functions α_{ij}^k, one gets $\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \alpha_{ij}^k$; if we require $\widetilde{\nabla}$ to be torsionless, we need to have $\widetilde{\Gamma}_{ij}^k = \widetilde{\Gamma}_{ji}^k$, but since $\Gamma_{ij}^k = \Gamma_{ji}^k$, one gets $\alpha_{ij}^k = \alpha_{ji}^k$, that is, F must be a *symmetric tensor* with respect to its lower indices. A more "intrinsic" approach would be to determine which D-valued tensor field F yields a new connection $\widetilde{\nabla}_X Y = \nabla_X Y + F(X, Y)$ with zero curvature, which we can find out in the following proposition.

Proposition 1. *Let Σ be a simple mechanical system that can be expressed as the ECF via RSSF. Then there exists a tensor field $F \in \Gamma(D \otimes T^* Q^{\otimes 2})$ such that $\widetilde{\nabla} = \overline{\nabla} + F$ has associated a type (1,3) curvature tensor $\widetilde{R} \equiv 0$.*

Proof. Since Σ is transformable to an ECF, there is a local difeomorphism $\varphi \colon Q \longrightarrow \mathbb{R}^3$, such that, for every $q \in Q$ there is a neighborhood $\mathscr{U}$ of q such that $\varphi(\mathscr{U})$ is an open subset of $\mathbb{R}^3$ and $\varphi|_{\mathscr{U}} \colon \mathscr{U} \longrightarrow \varphi(\mathscr{U})$ is a difeomorphism. So the tangent mapping $T_q\varphi \colon T_q Q \longrightarrow T_{\varphi(q)}\mathbb{R}^3$ is a linear isomorphism for all $q \in Q$, provided that $\varphi(q) = p$. Then, for every $X \in \Gamma(TQ)$, there is a unique vector field $Y \in \Gamma(T\mathbb{R}^3)$, which is φ related to X, that is, $Y_p = T_q\varphi(X_q)$, and the inverse mapping satisfies $(T_q\varphi)^{-1}(Y_p) = X_q$.

Since (Q, g_Q) and $(\mathbb{R}^3, g_E)$ are Riemannian manifolds, then the pullback of g_E by φ, denoted by $\varphi^* g_E$, defines a Riemannian metric over Q, that is, for every $v, w \in T_q Q$:

$$\begin{aligned}(\varphi^* g_E)_q(v, w) &= (g_E)_{\varphi(q)}\left(T_q\varphi(v), T_q\varphi(w)\right)\\ &= (g_E)_{\varphi(q)}(x, y)\\ &= (g_E \circ \varphi)_q\left(\left(T_q\varphi\right)^{-1}(x), \left(T_q\varphi\right)^{-1}(y)\right)\\ &= \left(g_Q\right)_q(v, w),\end{aligned} \tag{11}$$

this means that $\varphi^* g_E = g_Q$; therefore, φ is an isometry.

Assuming that ∇ denotes Levi–Civita connection of g_Q and $\widetilde{Q}$ that of g_E and since φ is an isometry, $\varphi^*(\widetilde{\nabla}) = \nabla$. On the other hand, the difference between ∇ and $\widetilde{\nabla}$ defines a tensor, that is, $\nabla_X Y - \widetilde{\nabla}_X Y = F(X, Y)$, where F is a symmetric tensor field of the type $(1,2)$ such that $F \in \Gamma(TM \otimes T^* M^{\otimes 2})$ [2]. In particular, if F takes values in D, that is, $F \in \Gamma(D \otimes T^* M^{\otimes 2})$, then $\nabla = \widetilde{\nabla} + F$ have a curvature tensor $\widetilde{R}$, which is identically zero, of the type $(1,3)$.

Then, a problem that consists in getting rid of the curvature arises; that is, determining if there exists a type $(1,2)$ tensor $F(X, Y)$ in order to get a new connection $\widetilde{\nabla}_X Y = \nabla_X Y + F(X, Y)$ with zero type $(1,3)$ tensor curvature. All the above leads us to think that PFCAC could be written in a similar way to one kind of curvature prescription. The goal of this work is to state the PFCAC and to solve it in dimension 2 by considering constant rank 1 distributions.

The statement, details, and analysis of the PFCAC will be given in the ensuing sections of this work, and the contents of which are organized as follows: in Section 2, one recalls a number of basic concepts and notations to be used in the sequel. Section 3 gives a formulation of the PFCAC, along with some examples. The materials and methods for solving the PFAC are given in Section 4. In Section 5, some results are presented for dimension 2, along with some illustrative examples. Finally, concluding remarks and possible future lines of research are given in Section 6. □

2. Preliminary Notions

This section includes some conventions and notations that will be used in the sequel. The reader may consult references [1–3, 5–9] for more information.

We use the Einstein summation convention throughout, and also we consider all geometric objects are smooth (C^∞), unless otherwise specified.

2.1. Vector Fields and Distributions. Given a manifold M of dimension n, we denote a coordinate chart (U, φ) near a point $x \in M$, with coordinate functions $x = (x^1, \ldots, x^n)$ on U corresponding to φ. $\Gamma(TM)$ denotes the set of vector fields on M. If $X \in \Gamma(TM)$, X assigns, to each $p \in M$, a tangent vector $X_p \in T_pM$, where T_pM denotes the tangent space to M at p.

T^*M denotes the cotangent bundle and $\Gamma(T^*M)$ denotes the set of 1-forms. If $\sigma \in \Gamma(T^*M)$, then $\sigma\colon M \longrightarrow T^*M$, and a 1-form $\sigma_p \in T_p^*M$ is assigned to each $p \in M$, where T_p^*M denotes the cotangent space to M at p.

A rank k distribution D on M is a mapping D, which assigns to each $p \in M$, a k-dimensional (vector) subspace D_p of T_pM. D is called smooth if, for every point $p \in M$, there exists a neighborhood U of p and k smooth vector fields $X_1, \ldots, X_k$ on U, such that $D_x = \operatorname{span}\{X_1 \ldots, X_k\}$ for all $x \in U$. A vector field X on M is said to take values in D if $X_p \in D_p$ for all $p \in M$. Sometimes we abuse the notation and write $X \in D$.

2.2. Riemannian Metric. Let g be a *Riemannian metric* on M; g is a type $(0,2)$ symmetric, positive-definite smooth tensor field. The tensor g_p induces an inner product on T_pM for each $p \in M$. The inner product allows one to define the notions of length and angle between vectors. In coordinates, g can be expressed as follows:

$$g = g_{ij} dx^i \otimes dx^j, \tag{12}$$

where g_{ij} are the components of a matrix-valued function with the following properties:

(i) Symmetry: for $p \in M$ and X, $Y \in T_pM$, $g_p(X,Y) = g_p(Y,X)$
(ii) Positive-definite: for $p \in M$ and X, $Y \in T_pM$, $g_p(X,X) > 0$, whenever $X \neq 0$

An Euclidian metric $\tilde{g}$ on $\mathbb{R}^n$ is expressed in coordinates as follows:

$$\tilde{g}\left(\frac{\partial}{\partial r^i}, \frac{\partial}{\partial r^j}\right) = \delta_{ij}, \tag{13}$$

where $(\partial/\partial r^i)$, $(\partial/\partial r^j)$ are unit vectors on $\mathbb{R}^n$ and matrix form $\tilde{g} = Id_{n\times n}$.

2.3. Affine Connection. A Riemannian metric defines a unique Levi-Civita affine connection, but not every affine connection is the Levi-Civita connection of a metric [2].

An affine connection on a manifold M is defined as a mapping

$$\begin{aligned} \nabla\colon \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM), \\ (X,Y) &\longmapsto \nabla_X Y, \end{aligned} \tag{14}$$

where $\nabla_X Y$ is the covariant derivative of the vector field Y in the direction of the vector field X. The mapping satisfies the following properties:

(1) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$,
(2) $\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$,
(3) $\nabla_X fY = f\nabla_X Y + (Xf)Y$, for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Here $X(f)$ is the derivative of f in the direction of X.

An affine connection ∇ can be uniquely represented in terms of an ordered local frame $(\partial/\partial x^i)$, by the specification of n^3 functions Γ_{ij}^k for $1 \leq i, j, k \leq n$, referred to as the Christoffel symbols, by the following expression:

$$\nabla_{(\partial/\partial x^i)} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \quad i, j = 1, \ldots, n. \tag{15}$$

If X and $Y \in \Gamma(TM)$ are given in terms of a local frame $(\partial/\partial x^i)$, then $X = X^i(\partial/\partial x^i)$, $Y = Y^j(\partial/\partial x^j)$ and $\nabla_X Y$ can be written in coordinates as follows:

$$\nabla_X Y = \left(\frac{\partial}{\partial x^i} Y^k X^i + X^i Y^j \Gamma_{ij}^k\right) \frac{\partial}{\partial x^k}. \tag{16}$$

Given an affine connection ∇ on a manifold M, one readily defines two tensor fields M related to ∇, namely, a type $(1,2)$ tensor field $T\colon \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ called *torsion*, with local expression given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{17}$$

and a type $(1,3)$ tensor field $R\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$, called *curvature tensor*, and given by

$$R(X,Y,Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z. \tag{18}$$

An affine connection is called *torsion-free* (or *symmetric*), if $T = 0$, that is, if

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0, \quad \text{for all } X, Y \in \Gamma(TM), \tag{19}$$

where $[X,Y]$ is the Lie bracket of X and Y [2, 3, 10]. In this case, the Christoffel symbols are said to be symmetric with respect to their lower indices, that is, $\Gamma^k_{ij} = \Gamma^k_{ji}$. In the sequel, we shall consider only symmetric affine connections and, for the sake of simplicity, write ∂_i instead of $(\partial/\partial x^i)$ to denote the partial derivative with respect to a coordinate x^i when the latter is clear from the context.

3. Problem Statement

Returning to the statement of PFCAC, given a finite dimensional manifold M, a distribution D, and a symmetric affine connection ∇ on M, the main problem to solve is as follows:

(i) Determine a type $(1,2)$ symmetric tensor field $F \in \Gamma(D \otimes T^* M^{\otimes 2})$ that takes values on D and such that $\tilde{\nabla}_X Y = \nabla_X Y + F(X,Y)$ has a curvature tensor $\tilde{R}$ that is identically zero.

One easily checks that $\tilde{\nabla}$ satisfies conditions (1)–(3) of the affine connection definition.

Let $(\partial/\partial x^i)$ for $i = 1,\ldots,n$ be a basis for TM, with (dx^j) its corresponding dual base, and D a distribution such that $D_p = \text{span}\{X_l\colon l = 1,\ldots,m\}$ if F takes values in D, that is, $F \in \Gamma(D \otimes T^* M^{\otimes 2})$, then F can be expressed as a type $(1,2)$ tensor in the coordinate system (U,φ) as follows:

$$F_p = a^l_{ij} X^k_l \frac{\partial}{\partial x^k} \otimes dx^i \otimes dx^j. \tag{20}$$

Thus, the new affine connection $\tilde{\nabla}$ can be expressed in coordinates by the following equation:

$$\left(\tilde{\nabla}_{(\partial/\partial x^i)} \frac{\partial}{\partial x^j}\right)_x = \left(\nabla_{(\partial/\partial x^i)} \frac{\partial}{\partial x^j}\right)_x + F_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \tag{21}$$

where $(\nabla_{(\partial/\partial x^i)}(\partial/\partial x^j))_x = \Gamma^k_{ij}(x)(\partial/\partial x^i)$ and $F_x((\partial/\partial x^i), (\partial/\partial x^j)) = a^l_{rs} X^k_l (\partial/\partial x^k) \otimes dx^r \otimes dx^s((\partial/\partial x^i), (\partial/\partial x^j))$; since $dx^r(\partial/\partial x^k) = \delta^r_k$ one has $F_x((\partial/\partial x^i), (\partial/\partial x^j)) = a^l_{rs} X^k_l(x)(\partial/\partial x^k)\delta^r_i \delta^s_j$, equation (21) is equivalent to

$$\tilde{\nabla}_{(\partial/\partial x^i)} \frac{\partial}{\partial x^j} = \left(\Gamma^k_{ij} + a^l_{ij} X^k_l\right) \frac{\partial}{\partial x^k}. \tag{22}$$

Hence, $\tilde{\nabla}$ will be determined by the Christoffel symbols $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \alpha^l_{ij} X^k_l$ for $i,j,k = 1,\ldots,n$, where Γ^k_{ij} are the Christoffel symbols defining ∇, X^k_l is the k-th component of the vector field X_l that take values in D, and α^l_{ij} are the functions to be found. Indeed, the type $(1,3)$ curvature tensor components $\tilde{R}^a_{bc\,d}$ associated with $\tilde{\nabla}$ are as follows:

$$\begin{aligned}\tilde{R}^a_{bcd} &= R^a_{bcd} + \partial_c\left(\alpha^r_{db} X^a_r\right) - \partial_d d\alpha^r_{cb} X^a_r \\ &\quad + \Gamma^\mu_{db}\alpha^\eta_{c\mu} X^a_\eta + \alpha^r_{db} X^\mu_r \Gamma^a_{c\mu} + \alpha^r_{db}\alpha^\eta_{c\mu} X^\mu_r X^a_\eta \\ &\quad - \Gamma^\mu_{cb}\alpha^\eta_{d\mu} X^a_\eta - \alpha^r_{cb} X^\mu_r \Gamma^a_{d\mu} - \alpha^r_{cb}\alpha^\eta_{d\mu} X^\mu_r X^a_\eta,\end{aligned} \tag{23}$$

for $a,b,c,d,\mu = 1,\ldots,n$ and $r,\eta = 1,\ldots,m$. One sees that $\tilde{R}$ involves only first order partial derivatives, consequently, solving the PFCAC consists in determining conditions for the integrability of first-order of PDEs; in turn, this conditions determine the existence of the field F. The curvature tensor $\tilde{R}$ has n^4 components, of which $\tilde{R}^a_{bcd} = -\tilde{R}^a_{bdc}$ and $\tilde{R}^a_{bcd} = 0$ when $c = d$; therefore, resulting system of PDEs has only $(n^2(n^2-1)/3)$ independent equations with $(m \cdot n(n+1)/2)$ unknowns.

For example, in $\mathbb{R}^2$ there are 16 equations, but 4 are independent with 3 unknowns α_{11}, $\alpha_{12} = \alpha_{21}$ y α_{22}. In $\mathbb{R}^3$ there are 81 equations, 24 of which are independent, with 12 unknown α^l_{11}, $\alpha^l_{12} = \alpha^l_{21}$, $\alpha^l_{13} = \alpha^l_{31}$, α^l_{22}, $\alpha^l_{23} = \alpha^l_{32}$ and α^l_{33} for $l = 1,2$. Note that the resulting system of PDEs are overdetermined (in that there are more equations than unknowns). In the next section, sufficient conditions to solve the PFCAC are described.

The resulting systems of the problem are solved by algebraic manipulation of the equations involved, in analytical form. Some characteristics related to Γ^k_{ij} and R^a_{bcd} in the solutions are identified and based on these, PDE system integrability conditions are found.

4. Results and Discussions

The following result gives a sufficient condition for solving the PFCAC for an affine connection ∇ on $\mathbb{R}^n$.

Proposition 2. *Let $p \in \mathbb{R}^n$ and $U \subset \mathbb{R}$ be a neighborhood of p.*

If $C_x = span\left\{(\nabla_{(\partial/\partial x^i)}(\partial/\partial x^j))_x\colon i,j = 1,\ldots,n\right\} \subseteq D_x$ for each $x \in U$, then exists a type (1,2) tensor field $F \in \Gamma(D \otimes \mathbb{R}^{n\otimes 2})$ such that $\tilde{\nabla}_X Y = \nabla_X Y + F(X,Y)$ has a curvature tensor $\tilde{R}$ of type (1,3) identically zero.

Proof. Since U is open and $p \in U$, there exists a neighborhood V of p such that $V \subset U$ and there exist vector fields $X_1,\ldots,X_m \in \Gamma(TV)$ such that, for all $x \in V$, $D_x = \text{span}\{X_1(x),\ldots,X_m(x)\}$. Then by $C_x \subseteq D_x$, for each $Z \in C_x$, there are functions $a_{ij}(x)$ such that $Z = a^r_{ij}(x) X^k_r(x)(\partial/\partial x^k)$, for each $i,j = 1,\ldots,n$, can be written as a linear combination of $X_1,\ldots,X_m$. In coordinates, Z can be written as $(\nabla_{\partial_i}\partial_j)_x = a^l_{ij}(x) X^k_l(x)(\partial/\partial x^k)$; on the other hand, we also have $(\nabla_{(\partial/\partial x^i)}(\partial/\partial x^j))_x = \Gamma^k_{ij}(x)(\partial/\partial x^k)$, which implies that $\Gamma^k_{ij}(x)(\partial/\partial x^k) = a^l_{ij}(x) X^s_l(x)(\partial/\partial x^s)$. If $s = k$, then $\Gamma^k_{ij}(x)(\partial/\partial x^k) - a^l_{ij}(x) X^s_l(x)\ (\partial/\partial x^s) = 0$. Equivalently, $\Gamma^k_{ij}(x) - a^l_{ij}(x) X^s_l(x) = 0$ for all $x \in U$. If $F_x \triangleq F^r_{ij}(\partial/\partial x^r) \otimes dx^i \otimes dx^j$, one has $F_x((\partial/\partial x^s), (\partial/\partial x^t)) = a^l_{st} X^r_l(\partial/\partial x^r) \in D_x$; therefore, $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + a^l_{ij}(x) X^s_l = 0$. Therefore, since all $\tilde{\Gamma}^k_{ij}$ are identically null, the curvature tensor $\tilde{R}$ is identically zero.

It is possible to define the vector field F on all of U using partitions of the unity, as shown in reference [5]. Note that Proposition 2 only provides sufficient conditions, as we can see in the following example. □

Example 1. Consider the ECF described by equation (4) *with the Riemannian metric g defined on $M = \{x \in \mathbb{R}^3 \colon |x_2| < 1\}$, which determines the geodesic spray S on $\mathbb{R}^6$*:

$$[g]_x = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix},$$

$$S(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ \dfrac{x_5 x_6 - x_2 x_3 x_4}{x_2^2 - 1} \\ x_4 x_6 \\ \dfrac{x_4 x_5 - x_2 x_5 x_6}{x_2^2 - 1} \end{pmatrix}, \tag{24}$$

S is associated with the affine connection ∇, defined by the nonzero Christoffel symbols:

$$\begin{aligned} \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{23}^3 = \Gamma_{32}^3 &= \frac{x_2}{2\left(x_2^2 - 1\right)}, \\ \Gamma_{23}^1 = \Gamma_{32}^1 = \Gamma_{12}^3 = \Gamma_{21}^3 &= -\frac{x_2}{2\left(x_2^2 - 1\right)}, \\ \Gamma_{13}^2 = \Gamma_{31}^2 &= -\frac{1}{2}. \end{aligned} \tag{25}$$

Note that ∇ has a nonzero curvature tensor, because

$$\begin{aligned} R_{131}^1 = R_{123}^2 = R_{321}^2 = R_{313}^3 &= \frac{x_2}{4\left(x_2^2 - 1\right)}, \\ R_{112}^2 = R_{332}^2 = R_{113}^3 &= \frac{1}{4\left(x_2^2 - 1\right)}, \\ R_{212}^1 = R_{232}^3 &= \frac{x_2^2 + 1}{4\left(x_2^2 - 1\right)^2}, \\ R_{223}^1 = R_{221}^3 &= \frac{x_2}{2\left(x_2^2 - 1\right)^2}, \end{aligned} \tag{26}$$

and their respective skew symmetries $R_{bcd}^a = -R_{bdc}^a$. In this case, we have a system of PDEs with 24 equations and 12 unknowns. To flatten the curvature R, we add a symmetric field tensor of type $(1, 2)$ $F \in \Gamma(D \otimes \mathbb{R}^{*3\otimes 2})$ to the connection ∇, here D is a distribution on $\mathbb{R}^3$. The new connection $\widetilde{\nabla}$ is determined by the Christoffel symbols $\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \beta_{ij}^l X_l^k$, where β_{ij}^k are functions to be found and X^k are components of the vector field that takes values in the distribution D.

(1) With the distribution

$$D_x = span\left\{\left(\frac{\partial}{\partial x_1}\right)|_x, \left(\frac{\partial}{\partial x_2}\right)|_x\right\}. \tag{27}$$

One gets

$$F_x = \beta_1(x)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2(x)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{28}$$

with $\beta_1(x) = (x_5(2x_4x_2 - x_5x_3x_2 + x_5x_1 - x_6x_2^2 - x_6)/x_2^2 - 1)$ and $\beta_2(x) = (2x_6x_4 x_2^4 - 2x_6x_4 - 3x_5^2x_2 - x_5^2x_2^3/2(x_2^4 - 1))$.

(2) With the distribution

$$D_x = \text{span}\left\{\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}\right)|_x, \left(\frac{\partial}{\partial x_2}\right)|_x\right\}. \tag{29}$$

One gets

$$\begin{aligned} F_x &= \beta_1(x)\begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} + \beta_2(x)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \beta_1(x) &= \frac{x_5^2\left(3x_1 + 16x_3x_2 + 3x_1x_2^2 + 2x_3x_2^3\right) + x_5\left(6x_6x_2^4 - 6x_6\right)}{18\left(x_2^6 - x_2^2 + x_2^4 - 1\right)}, \\ \beta_2(x) &= \frac{2x_6x_4x_2^4 - 2x_6x_4 - 3x_5^2x_2 - x_5^2x_2^3}{2\left(x_2^4 - 1\right)}. \end{aligned} \tag{30}$$

(3) With the distribution

$$D_x = \text{span}\left\{\left(\frac{\partial}{\partial x_1}\right)|_x, \left(\frac{\partial}{\partial x_2}|_x + x_2\frac{\partial}{\partial x_3}\right)|_x\right\}, \tag{31}$$

it cannot be solved, since the resulting system of EDP is inconsistent.

Some questions naturally arise, such as why can the PFCAC be solved for some but not for other distributions? How much of the geodesic spray can be modified by using control vector fields that take values in the control distribution, to make the effective curvature vanish? Is there a smaller-rank distribution D such that $F \in \Gamma(D)$ and the new $\widetilde{\Gamma}_{ij}^k$ are zero? With a view towards answering these questions, we perform an analysis in dimension 2, locally, around a point $p \in \mathbb{R}^2$ with a constant rank 1 distribution D and a symmetric affine connections ∇. We refer to this problem as PFCAC2, by the Frobenius integrability theorem [6], for any constant rank 1 distribution, we can choose a vector field $X \in \Gamma(D)$ defined on a neighborhood of p such that $X(p) \neq 0$ and; therefore, there is a coordinate system (U, ϕ) around p such that $X|_U = (\partial/\partial x_1)$, and so, $D_x = \text{span}\{(\partial/\partial x_1)|_x\}$ for all $x \in U$. Henceforth, we assume that the affine connection ∇ and the distribution D are expressed in a system of local coordinates (U, x_1, x_2) on $\mathbb{R}^2$.

In general, solving the PCAC2 is equivalent to solving the following system of PDEs:

$$\begin{aligned}(\partial_2\beta_{11}-\partial_1\beta_{12})d^1 &= R^1_{112}+\beta_{12}\partial_1 d^1-\beta_{11}\partial_2 d^1+\left(\Gamma^1_{1m}\beta_{12}-\Gamma^1_{2m}\beta_{11}\right)d^m\\ &\quad+\left(\Gamma^m_{21}\beta_{1m}-\Gamma^m_{11}\beta_{2m}\right)d^1+\left(\beta_{1m}\beta_{21}-\beta_{2m}\beta_{11}\right)d^1d^m,\\ (\partial_2\beta_{11}-\partial_1\beta_{12})d^2 &= R^2_{112}+\beta_{12}\partial_1 d^2-\beta_{11}\partial_2 d^2+\left(\Gamma^2_{1m}\beta_{12}-\Gamma^2_{2m}\beta_{11}\right)d^m\\ &\quad+\left(\Gamma^m_{21}\beta_{1m}-\Gamma^m_{11}\beta_{2m}\right)d^2+\left(\beta_{1m}\beta_{21}-\beta_{2m}\beta_{11}\right)d^2d^m,\\ (\partial_2\beta_{12}-\partial_1\beta_{22})d^1 &= R^1_{212}+\beta_{22}\partial_1 d^1-\beta_{12}\partial_2 d^1+\left(\Gamma^1_{1m}\beta_{22}-\Gamma^1_{2m}\beta_{12}\right)d^m\\ &\quad+\left(\Gamma^m_{22}\beta_{1m}-\Gamma^m_{12}\beta_{2m}\right)d^1+\left(\beta_{1m}\beta_{22}-\beta_{2m}\beta_{12}\right)d^1d^m,\\ (\partial_2\beta_{12}-\partial_1\beta_{22})d^2 &= R^2_{212}+\beta_{22}\partial_1 d^2-\beta_{12}\partial_2 d^2+\left(\Gamma^2_{1m}\beta_{22}-\Gamma^2_{2m}\beta_{12}\right)d^m\\ &\quad+\left(\Gamma^m_{22}\beta_{1m}-\Gamma^m_{12}\beta_{2m}\right)d^2+\left(\beta_{1m}\beta_{22}-\beta_{2m}\beta_{12}\right)d^2d^m,\end{aligned}\tag{32}$$

where d^m is the m-th component of the vector field that generates the distribution D for $m = 1, 2$.

In order to find integrability conditions of the PDEs (32), we start with a simple case, that is, when the Christoffel symbols Γ^k_{ij} are constant for all $i, j, k = 1, 2$, we get the following result.

Proposition 3. *If ∇ is determined by Christoffel symbols Γ^k_{ij} such that in the coordinate system (U,φ) all the Γ^k_{ij} are constant, then the PFCAC2 admits a solution.*

Proof. Since Γ^2_{12} is constant for all $x \in U$, let us first consider the case where $\Gamma^2_{12} \neq 0$. Then, equation (32) is reduced to the following first-order PDE system:

$$\partial_2\beta_{11}-\partial_1\beta_{12} = R^1_{112}+\Gamma^2_{12}\beta_{12}-\Gamma^2_{11}\beta_{22},\tag{33}$$

$$\begin{aligned}\partial_2\beta_{12}-\partial_1\beta_{22} &= R^1_{212}+\beta_{11}\left(\beta_{22}+\Gamma^1_{22}\right)+\left(\Gamma^2_{22}-2\Gamma^1_{12}-\beta_{12}\right)\beta_{12}\\ &\quad+\left(\Gamma^1_{11}-\Gamma^2_{12}\right)\beta_{22},\end{aligned}\tag{34}$$

$$0 = R^2_{112}+\Gamma^2_{11}\beta_{12}-\Gamma^2_{12}\beta_{11},\tag{35}$$

$$0 = R^2_{212}+\Gamma^2_{11}\beta_{22}-\Gamma^2_{12}\beta_{12}.\tag{36}$$

Under the stated assumptions, the curvature tensor R associated with ∇ has components as follows:

$$R^a_{bcd} = \Gamma^a_{cm}\Gamma^m_{db}-\Gamma^a_{dm}\Gamma^m_{cb},\quad \text{for } a,b,c,d = 1,2.\tag{37}$$

In addition, we know that $R^a_{bcd} = 0$ when $c = d$ and $R^a_{bcd} = -R^a_{bdc}$; therefore, the nonzero components of R are as follows:

$$\begin{aligned}R^1_{112} &= \Gamma^1_{12}\Gamma^2_{21}-\Gamma^1_{22}\Gamma^2_{11},\\ R^1_{212} &= \Gamma^1_{11}\Gamma^1_{22}+\Gamma^1_{12}\Gamma^2_{22}-\Gamma^1_{21}\Gamma^1_{12}-\Gamma^1_{22}\Gamma^2_{12},\\ R^2_{112} &= \Gamma^2_{11}\Gamma^1_{21}+\Gamma^2_{12}\Gamma^2_{21}-\Gamma^2_{21}\Gamma^1_{11}-\Gamma^2_{22}\Gamma^2_{11},\\ R^2_{212} &== \Gamma^2_{11}\Gamma^1_{22}-\Gamma^2_{21}\Gamma^1_{12}.\end{aligned}\tag{38}$$

Plugging R^2_{112} and R^2_{212} into equations (35) and (36), we get, respectively, as follows:

$$\begin{aligned}0 &= \Gamma^2_{11}\Gamma^1_{21}+\Gamma^2_{12}\Gamma^2_{21}-\Gamma^2_{21}\Gamma^1_{11}-\Gamma^2_{22}\Gamma^2_{11}+\Gamma^2_{11}\beta_{12}-\Gamma^2_{12}\beta_{11},\\ 0 &= \Gamma^2_{11}\Gamma^1_{22}-\Gamma^2_{21}\Gamma^1_{12}+\Gamma^2_{11}\beta_{22}-\Gamma^2_{12}\beta_{12}.\end{aligned}\tag{39}$$

If $\beta_{12} = -\Gamma^1_{12}$ and $\beta_{22} = -\Gamma^1_{22}$, then equation (36) is satisfied, replacing β_{12} and β_{22} in equation (35), and when $\Gamma^2_{12} \neq 0$, one gets $\beta_{11} = (1/\Gamma^2_{21})(\Gamma^2_{12}(\Gamma^2_{12}-\Gamma^1_{11})-\Gamma^2_{22}\Gamma^2_{11})$; now, we just need to verify that equations (33) and (34) are satisfied. Substituting R^1_{112}, R^1_{212} and given that β_{ij} are constant functions by assumption on Γ^k_{ij}, then equations (33) and (34) are reduced to

$$\begin{aligned}0 &= \Gamma^1_{12}\Gamma^2_{21}-\Gamma^1_{22}\Gamma^2_{11}+\Gamma^2_{12}\beta_{12}-\Gamma^2_{11}\beta_{22},\\ 0 &= \Gamma^1_{11}\Gamma^1_{22}+\Gamma^1_{12}\Gamma^2_{22}-\Gamma^1_{21}\Gamma^1_{12}-\Gamma^1_{22}\Gamma^2_{12}+\beta_{11}\left(\beta_{22}+\Gamma^1_{22}\right)\\ &\quad+\left(\Gamma^2_{22}-2\Gamma^1_{12}-\beta_{12}\right)\beta_{12}+\left(\Gamma^1_{11}-\Gamma^2_{12}\right)\beta_{22},\end{aligned}\tag{40}$$

in fact, β_{11}, β_{12}; β_{21}, and β_{22} satisfy these two equations and, therefore,

$$\begin{aligned}F_x &= \frac{1}{\Gamma^2_{21}}\left(\Gamma^2_{12}\left(\Gamma^2_{12}-\Gamma^1_{11}\right)-\Gamma^2_{22}\Gamma^2_{11}\right)\begin{pmatrix}1\\0\end{pmatrix}x_3^2\\ &\quad-\Gamma^1_{12}\begin{pmatrix}1\\0\end{pmatrix}\left(x_3x_4+x_4x_3\right)-\Gamma^1_{22}\begin{pmatrix}1\\0\end{pmatrix}x_4^2.\end{aligned}\tag{41}$$

Now considering the case where $\Gamma^2_{12}|_U = 0$, system (32) is reduced to

$$\partial_2\beta_{11}-\partial_1\beta_{12} = R^1_{112}-\Gamma^2_{11}\beta_{22},\tag{42}$$

$$\begin{aligned}\partial_2\beta_{12}-\partial_1\beta_{22} &= R^1_{212}+\beta_{11}\left(\beta_{22}+\Gamma^1_{22}\right)\\ &\quad+\left(\Gamma^2_{22}-2\Gamma^1_{12}-\beta_{12}\right)\beta_{12}+\Gamma^1_{11}\beta_{22},\end{aligned}\tag{43}$$

$$0 = R^2_{112}+\Gamma^2_{11}\beta_{12},\tag{44}$$

$$0 = R^2_{212}+\Gamma^2_{11}\beta_{22},\tag{45}$$

and the nonzero components of the curvature tensor are as follows:

$$\begin{aligned} R^1_{112} &= -\Gamma^1_{22}\Gamma^2_{11}, \\ R^1_{212} &= \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^1_{12}, \\ R^2_{112} &= \Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{22}\Gamma^2_{11}, \\ R^2_{212} &= \Gamma^2_{11}\Gamma^1_{22}, \end{aligned} \quad (46)$$

by replacing R^2_{112} and R^2_{212} in equations (44) and (45), one gets

$$\begin{aligned} 0 &= \Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{22}\Gamma^2_{11} + \Gamma^2_{11}\beta_{12}, \\ 0 &= \Gamma^2_{11}\Gamma^1_{22} + \Gamma^2_{11}\beta_{22}, \end{aligned} \quad (47)$$

hence, $\beta_{12} = -\Gamma^1_{12} + \Gamma^2_{22}$ and $\beta_{22} = -\Gamma^1_{22}$. Substituting R^1_{112}, R^1_{212}, β_{12}, and β_{22}, then equation (19) is satisfied and from equation (42), one gets

$$\partial_2\beta_{11} = 0. \quad (48)$$

Therefore, $\beta_{11} = f(x_1)$ for an arbitrary function $f \in C^1(U)$. Thus, $\beta_{11} = f(x_1)$, $\beta_{12} = -\Gamma^1_{12} + \Gamma^2_{22}$, and $\beta_{22} = -\Gamma^1_{22}$ is a solution for systems (42)–(45), thus

$$\begin{aligned} F_x = f(x_1)\begin{pmatrix}1\\0\end{pmatrix}x_3^2 - \left(\Gamma^1_{12} - \Gamma^2_{22}\right)\begin{pmatrix}1\\0\end{pmatrix}(x_3x_4 + x_4x_3) \\ - \Gamma^1_{22}\begin{pmatrix}1\\0\end{pmatrix}x_4^2. \end{aligned} \quad (49)$$

In the following case, the affine connections considered are determined by their Christoffel symbols Γ^2_{ij} such that at the least one is zero and Γ^1_{ij} are all zero. In this case, sufficient conditions are obtained in the following result. □

Proposition 4. *Let an affine connection ∇ determined by Γ^2_{ij}, and suppose there is a pair $(i, j) \in \{1,2\}^2$ such that in the coordinate system (U, ϕ), $\Gamma^2_{ij}|_U = 0$, and a distribution $D_x = span\{(\partial/\partial x_1)|_x\}$ for all $x \in U$, and the following identities are satisfied:*

(1) $R^2_{212}|_U = 0$

(2) If $\Gamma^2_{12}|_U \neq 0$ and $(\partial_2 R^2_{112}/\Gamma^2_{12})|_U = 0$

(3) If $\Gamma^2_{11}|_U \neq 0$ and $(\partial_2 R^2_{112}/\Gamma^2_{11})|_U = 0$, then the PFAC2 admits a solution

Proof. Let $D_x = \text{span}\{(\partial/\partial x_1)|_x\}$ and ∇ be determined by Γ^2_{ij} in the coordinate (U, ϕ), then system (32) is reduced as follows:

$$\begin{aligned} \partial_2\beta_{11} - \partial_1\beta_{12} &= \Gamma^2_{12}\beta_{12} - \Gamma^2_{11}\beta_{22}, \\ \partial_2\beta_{12} - \partial_1\beta_{22} &= \beta_{11}\beta_{22} - \beta^2_{12} + \Gamma^2_{22}\beta_{12} - \Gamma^2_{12}\beta_{22}, \\ 0 &= R^2_{112} + \Gamma^2_{11}\beta_{12} - \Gamma^2_{12}\beta_{11}, \\ 0 &= R^2_{212} + \Gamma^2_{11}\beta_{22} - \Gamma^2_{12}\beta_{12}. \end{aligned} \quad (50)$$

(i) If $p \in \mathbb{R}^2$, then exists a neighborhood V of p with $V \in \mathscr{V}(p)$ and $V \subseteq U$ such that, for the pair $(1,1) \in \{1,2\}^2$, $\Gamma^2_{11}(x) = 0$ for all $x \in U$, then PDE system (50) is reduced to

$$\partial_2\beta_{11} - \partial_1\beta_{12} = \Gamma^2_{12}\beta_{12}, \quad (51)$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta^2_{12} + \Gamma^2_{22}\beta_{12} - \Gamma^2_{12}\beta_{22}, \quad (52)$$

$$0 = R^2_{112} - \Gamma^2_{12}\beta_{11}, \quad (53)$$

$$0 = R^2_{212} - \Gamma^2_{12}\beta_{12}, \quad (54)$$

by Assumption 1 and equations (53) and (54), one gets, respectively,

$$\begin{aligned} 0 &= \left(R^2_{112} - \Gamma^2_{12}\beta_{11}\right)|_U, \\ 0 &= \left(\Gamma^2_{12}\beta_{12}\right)|_U, \end{aligned} \quad (55)$$

from these two equations, the following cases are derived:

(a) If $\Gamma^2_{12}(x)|_U \neq 0$ for all $x \in U$, then $\beta_{12} = 0$ and $\beta_{11} = (R^2_{112}/\Gamma^2_{12})$, and for identity (2), equation (51) is satisfied and by equation (52), one gets $-\partial_1\beta_{22} = (\partial_1\Gamma^2_{12}/\Gamma^2_{12})\beta_{22}$, an ordinary equation, whose primitive is $\beta_{22} = c_0 \exp(-\int(\partial_1\Gamma^2_{12}/\Gamma^2_{12})dx_1)$, where c_0 is an integration constant, and

$$F_x = \beta_{11}\begin{pmatrix}1\\0\end{pmatrix}x_3^2 + \beta_{22}\begin{pmatrix}1\\0\end{pmatrix}x_4^2. \quad (56)$$

(b) If $\Gamma^2_{12}|_U = 0$, then $R|_U \equiv 0$; therefore, equations (51)–(54) admit a trivial solution, namely, $\beta_{ij} = 0$ for all $i, j = 1, 2$ and $F_x \equiv 0$.

(ii) Considering the pair $(2,2) \in \{1,2\}^2$, $\Gamma^2_{22}(x) = 0$ for all $x \in U$, then PDE system (50) is reduced to

$$\partial_2\beta_{11} - \partial_1\beta_{12} = \Gamma^2_{12}\beta_{12} - \Gamma^2_{11}\beta_{22}, \quad (57)$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta^2_{12} - \Gamma^2_{12}\beta_{22}, \quad (58)$$

$$0 = R^2_{112} + \Gamma^2_{11}\beta_{12} - \Gamma^2_{12}\beta_{11}, \quad (59)$$

$$0 = R^2_{212} + \Gamma^2_{11}\beta_{22} - \Gamma^2_{12}\beta_{12}, \quad (60)$$

by Assumption 1 and equations (57) and (58), one gets, respectively,

$$\begin{aligned} 0 &= \left(R^2_{112} + \Gamma^2_{11}\beta_{12} - \Gamma^2_{12}\beta_{11}\right)|_U, \\ 0 &= \left(\Gamma^2_{11}\beta_{22} - \Gamma^2_{12}\beta_{12}\right)|_U, \end{aligned} \quad (61)$$

Given these two equations, we have the following cases:

(a) If Γ^2_{12} and Γ^2_{11} are nowhere zero on U, then $\beta_{12} = \beta_{22} = 0$ and $\beta_{11} = (R^2_{112}/\Gamma^2_{12})$ satisfy equations (56)–(58) and by Assumption 2, equation (55) is satisfied and one gets

$$F_x = \frac{R^2_{112}}{\Gamma^2_{12}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3^2. \quad (62)$$

(b) If $\Gamma^2_{12}|_U = 0$ and $\Gamma^2_{11}(x)|_U \neq 0$ for all $x \in U$, then $\beta_{22} = 0$ and $\beta_{12} = -(R^2_{112}/\Gamma^2_{11})$ satisfy equations (56)–(58) and from equation (55), one gets $\partial_2 \beta_{11} = -\partial_1 (R^2_{112}/\Gamma^2_{12})$, an ordinary equation, whose primitive is $\beta_{11} = -\int (\partial_1 (R^2_{112}/\Gamma^2_{12})) dx_2 + c_1$, where c_1 is an integration constant, then

$$F_x = \beta_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3^2 - \beta_{12} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x_3 x_4 + x_4 x_3). \quad (63)$$

(c) If $\Gamma^2_{11}|_U = 0$, then $R|_U \equiv 0$, and therefore systems (55)–(58) admit a trivial solution, namely, $\beta_{ij} = 0$ for $i, j = 1, 2$ and $F_x \equiv 0$.

(iii) For the pair $(1,2) = (2,1) \in \{1,2\}^2$, $\Gamma^2_{12} \mid_U = 0$ and equation (50) is reduced to

$$\partial_2 \beta_{11} - \partial_1 \beta_{12} = -\Gamma^2_{11} \beta_{22}, \quad (64)$$

$$\partial_2 \beta_{12} - \partial_1 \beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + \Gamma^2_{22}\beta_{12}, \quad (65)$$

$$0 = R^2_{112} + \Gamma^2_{11}\beta_{12}, \quad (66)$$

$$0 = R^2_{212} + \Gamma^2_{11}\beta_{22}, \quad (67)$$

From Assumption 1 and equations (66) and (67), one gets, respectively,

$$\begin{aligned} 0 &= \left(R^2_{112} + \Gamma^2_{11}\beta_{12}\right)|_U, \\ 0 &= \left(\Gamma^2_{11}\beta_{22}\right)|_U, \end{aligned} \quad (68)$$

and the following cases are derived:

(a) If $\Gamma^2_{11}|_U \neq 0$, then with $\beta_{22} = 0$ and $\beta_{12} = -(R^2_{112}/\Gamma^2_{11})$, equations (65)–(67) are satisfied and from equation (64), one gets an ordinary equation $\partial_2 \beta_{11} = \partial_1 (R^2_{112}/\Gamma^2_{12})$ with primitive $\beta_{11} = -\int (\partial_1 (R^2_{112}/\Gamma^2_{12})) dx_2 + c$, where c is an integration constant, so

$$F_x = \beta_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3^2 - \beta_{12} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x_3 x_4 + x_4 x_3). \quad (69)$$

(b) If $\Gamma^2_{11}|_U = 0$, then $R|_U \equiv 0$; hence, systems (64)–(67) admit a trivial solution, this is, $\beta_{ij} = 0$ for $i, j = 1, 2$ and $F_x \equiv 0$.

The following case is motivated by the previous one but considering this time that the Christoffel symbols Γ^2_{ij} in the coordinate system (U, x_1, x_2) are given by smooth arbitrary functions that depend on x_1 and x_2 for all $i, j = 1, 2$. In this case, there are only some combinations of the Γ^2_{ij}, for which the PFCAC2 admits a solution, and these are mentioned in the following result: □

Proposition 5. *Let D be a distribution and ∇ be an affine connection on an open subset of $\mathbb{R}^2$, and assuming that, for a coordinate chart (U, ϕ), one has $\Gamma^2_{ij}|_U = f(x_1, x_2)$, where $f \in C^1(\mathbb{R}^2)$ and $D_x = span\{\partial/\partial x_1\}$ for all $x \in U$. Then, the PFCAC2 admits solutions in the following cases:*

C1. If $\Gamma^2_{22}|_U = constant \neq 0$, $\Gamma^2_{11}|_U = f_1(x_1)$, $\Gamma^2_{12}|_U = f_2(x_1)$ and

(i) If satisfied $R^2_{212}|_U = 0$ and $(\Gamma^2_{12}\partial_2 R^2_{112} - R^2_{112}\partial_2\Gamma^2_{21})|_U = 0$.

C2. If $\Gamma^2_{ii}|_U = \Gamma^2_{jj}|_U = x_i$ and $\Gamma^2_{ij}|_U = x_j$ for $i \neq j$ and

(ii) $R_{212}|_U = 0$ and $\partial_i \Gamma^2_{11}|_U = \partial_j \Gamma^2_{12}|_U$.

C3. If $\Gamma^2_{ij}|_U = x_k$ for $k = 1, 2$ and

(iii) $R^2_{112}|_U = R^2_{212}|_U$ and $\partial_1 (R^2_{112}/\Gamma^2_{12})|_U - \partial_2 (R^2_{112}/\Gamma^2_{12})|_U + (R^2_{112}/\Gamma^2_{12})^2|_U = 0$.

C2. If $\Gamma^2_{ii}|_U = \Gamma^2_{jj}|_U = x_i$ and $\Gamma^2_{ij}|_U = x_j$ for $i \neq j$ and

(ii) $R_{212}|_U = 0$ and $\partial_i \Gamma^2_{11}|_U = \partial_j \Gamma^2_{12}|_U$.

C3. If $\Gamma^2_{ij}|_U = x_k$ for $k = 1, 2$ and

(iii) $R^2_{112}|_U = R^2_{212}|_U$ and $\partial_1 (R^2_{112}/\Gamma^2_{12})|_U - \partial_2 (R^2_{112}/\Gamma^2_{12})|_U + (R^2_{112}/\Gamma^2_{12})^2|_U = 0$.

(ii) $R_{212}|_U = 0$ and $\partial_i \Gamma^2_{11}|_U = \partial_j \Gamma^2_{12}|_U$.

C3. If $\Gamma^2_{ij}|_U = x_k$ for $k = 1, 2$ and

(iii) $R^2_{112}|_U = R^2_{212}|_U$ and $\partial_1 (R^2_{112}/\Gamma^2_{12})|_U - \partial_2 (R^2_{112}/\Gamma^2_{12})|_U + (R^2_{112}/\Gamma^2_{12})^2|_U = 0$.

C3. If $\Gamma^2_{ij}|_U = x_k$ for $k = 1, 2$ and

(iii) $R^2_{112}|_U = R^2_{212}|_U$ and $\partial_1 (R^2_{112}/\Gamma^2_{12})|_U - \partial_2 (R^2_{112}/\Gamma^2_{12})|_U + (R^2_{112}/\Gamma^2_{12})^2|_U = 0$.

(iii) $R^2_{112}|_U = R^2_{212}|_U$ and $\partial_1 (R^2_{112}/\Gamma^2_{12})|_U - \partial_2 (R^2_{112}/\Gamma^2_{12})|_U + (R^2_{112}/\Gamma^2_{12})^2|_U = 0$.

Proof. The PDE system resulting for this case is expressed as follows:

$$\begin{aligned} \partial_2 \beta_{11} - \partial_1 \beta_{12} &= \Gamma^2_{12}\beta_{12} - \Gamma^2_{11}\beta_{22}, \\ \partial_2 \beta_{12} - \partial_1 \beta_{22} &= \beta_{11}\beta_{22} - \beta_{12}^2 + \Gamma^2_{22}\beta_{12} - \Gamma^2_{12}\beta_{22}, \\ 0 &= R^2_{112} + \Gamma^2_{11}\beta_{12} - \Gamma^2_{12}\beta_{11}, \\ 0 &= R^2_{212} + \Gamma^2_{11}\beta_{22} - \Gamma^2_{12}\beta_{12}. \end{aligned} \quad (70)$$

C1. If $\Gamma^2_{22}|_U = \text{constant} \neq 0$, $\Gamma^2_{11}|_U = f_1(x_1)$ and $\Gamma^2_{12}|_U = f_2(x_1)$, then the associated curvature tensor has components $R^1_{bcd} \equiv R^2_{212} \equiv 0$ for $b, c, d = 1, 2$, $R^2_{112} \equiv \partial_1 f_2(x_1) - f_2^2(x_1) - a f_1(x_1)$ and system (70) is reduced to

$$\partial_2 \beta_{11} - \partial_1 \beta_{12} = f_2(x_1)\beta_{12} - f_1(x_1)\beta_{22}, \quad (71)$$

$$\partial_2 \beta_{12} - \partial_1 \beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + a\beta_{12} - f_2(x_1)\beta_{22}, \quad (72)$$

$$0 = \partial_1 f_2(x_1) - f_2^2(x_1) - a f_1(x_1) + f_1(x_1)\beta_{12} - f_2(x_1)\beta_{11}, \tag{73}$$

$$0 = f_1(x_1)\beta_{22} - f_2(x_1)\beta_{12}. \tag{74}$$

From equations (73) and (74), one gets $\beta_{11} = (\partial_1 f_2(x_1) - f_2^2(x_1) - a f_1(x_1)/f_2(x_1)) + (f_1(x_1)/f_2(x_1))^2\beta_{22}$ and $\beta_{12} = (f_1(x_1)/f_2(x_1))\beta_{22}$, then if $\beta_{22} = 0$, one gets $\beta_{12} = 0$ and $\beta_{11} = (\partial_1 f_2(x_1) - f_2^2(x_1) - a f_1(x_1)/f_2(x_1))$. Note that equations (71) and (72) are also satisfied. Therefore, $\beta_{12} = \beta_{22} = 0$ and $\beta_{11} = (R_{112}^2/\Gamma_{12}^2)$ are solutions for systems (71)–(74), thus

$$F_x = \frac{R_{112}^2}{\Gamma_{12}^2}\begin{pmatrix}1\\0\end{pmatrix}x_3^2. \tag{75}$$

C2. If $\Gamma_{ii}^2|_U = \Gamma_{jj}^2|_U = x_i$ and $\Gamma_{ij}^2|_U = x_j$ for $i \neq j$, then the curvature tensor associated with ∇ has components $R_{bcd}^1 \equiv R_{212}^2 \equiv 0$ for $b, c, d = 1, 2$ and $R_{112}^2 = x_j^2 - x_i^2$. Thus, system (70) is reduced as follows:

$$\partial_2\beta_{11} - \partial_1\beta_{12} = x_j\beta_{12} - x_i\beta_{22}, \tag{76}$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + x_i\beta_{12} - x_j\beta_{22}, \tag{77}$$

$$0 = x_j^2 - x_i^2 + x_i\beta_{12} - x_j\beta_{11}, \tag{78}$$

$$0 = x_i\beta_{22} - x_j\beta_{12}. \tag{79}$$

From this, it follows that $\beta_{11} = \beta_{22} = x_j$ and $\beta_{12} = x_i$ satisfy equations (78) and (79) and by ii equations (76) and (77) also are met, thus

$$F_x = x_j\begin{pmatrix}1\\0\end{pmatrix}x_3^2 - x_i\begin{pmatrix}1\\0\end{pmatrix}(x_3x_4 + x_4x_3) - x_j\begin{pmatrix}1\\0\end{pmatrix}x_4^2. \tag{80}$$

C3. If $\Gamma_{ij}^2 \mid_U = x_k$, then ∇ has curvature tensor associated with components $R_{bc\,d}^1 \equiv 0$ and $R_{112}^2 \equiv R_{212}^2$. Thus, system (70) is reduced as follows:

$$\partial_2\beta_{11} - \partial_1\beta_{12} = x_k(\beta_{12} - \beta_{22}), \tag{81}$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + x_k(\beta_{12} - \beta_{22}), \tag{82}$$

$$0 = R_{112}^2 + x_k(\beta_{12} - \beta_{11}), \tag{83}$$

$$0 = R_{112}^2 + x_k(\beta_{22} - \beta_{12}). \tag{84}$$

If x_k is not zero on U for $k = 1, 2$, then by equations (83) and (84), one gets the following equation:

$$\beta_{11} = \frac{2R_{112}^2}{x_k} + \beta_{22}, \quad \beta_{12} = \frac{R_{112}^2}{x_k} + \beta_{22}, \tag{85}$$

replacing β_{11} and β_{12} in equations (81) and (82), one gets, respectively, as follows:

$$\partial_2\beta_{22} - \partial_1\beta_{22} = R_{112}^2 + \partial_1\left(\frac{R_{112}^2}{x_k}\right) - 2\partial_2\left(\frac{R_{112}^2}{x_k}\right), \quad \partial_2\beta_{22} - \partial_1\beta_{22} = R_{112}^2 - \left(\frac{R_{112}^2}{x_k}\right)^2 - \partial_2\left(\frac{R_{112}^2}{x_k}\right). \tag{86}$$

The system has a solution if the following equation is true:

$$R_{112}^2 + \partial_1\left(\frac{R_{112}^2}{x_k}\right) - 2\partial_2\left(\frac{R_{112}^2}{x_k}\right) = R_{112}^2 - \left(\frac{R_{112}^2}{x_k}\right)^2 - \partial_2\left(\frac{R_{112}^2}{x_k}\right), \tag{87}$$

but by identity (3)), it is satisfied; therefore, systems (81)–(84) have a solution and depend on k, that is,

(i) If $k = 1$, then $R_{112}^2 \equiv R_{212}^2 = 1$ and if $x_1 \neq 0$, then one gets

$$\partial_2\beta_{22} - \partial_1\beta_{22} = 1 - \frac{1}{x_1^2}, \tag{88}$$

from which it can be deduced that if $\beta_{22} = x_2 - (1/x_1)$, then $\beta_{11} = (1/x_1) + x_2$ and $\beta_{12} = x_2$ are solutions for the system and

$$F_x = \left(\frac{1}{x_1} + x_2\right)\begin{pmatrix}1\\0\end{pmatrix}x_3^2 + x_2\begin{pmatrix}1\\0\end{pmatrix}(x_3x_4 + x_4x_3) + \left(x_2 - \frac{1}{x_1}\right)\begin{pmatrix}1\\0\end{pmatrix}x_4^2. \tag{89}$$

(ii) If $k = 2$, then $R_{112}^2 \equiv R_{212}^2 = -1$ and if $x_2 \neq 0$, then one gets

$$\partial_2\beta_{22} - \partial_1\beta_{22} = -1 + \frac{2}{x_1^2}, \tag{90}$$

from which one can deduced that if $\beta_{22} = x_1 - (2/x_2^2)$, then $\beta_{11} = x_1$, $\beta_{12} = x_1 + (1/x_2)$ and, therefore, the system has a solution, thus

$$F_x = x_1\begin{pmatrix}1\\0\end{pmatrix}x_3^2 + \left(x_1 + \frac{1}{x_2}\right)\begin{pmatrix}1\\0\end{pmatrix}(x_3x_4 + x_4x_3) + \left(x_1 - \frac{2}{x_2^2}\right)\begin{pmatrix}1\\0\end{pmatrix}x_4^2. \tag{91}$$

Note that in Propositions 4 and 5, $\Gamma_{ij}^1|_U = 0$; however, if these are nonzero on (U, ϕ) and if we also consider the previous cases of Γ_{ij}^2, the PFCAC2 has a solution. Then, also for a connection ∇ defined by Γ_{ij}^1 arbitrary and Γ_{ij}^2 as in Propositions 4 and 5, the PFCAC2 admits a solution as follows:. □

Proposition 6. *Let D be a distribution and ∇ be an affine connection on $\mathbb{R}^2$. If in the coordinate system (U, ϕ), ∇ is determined by $\Gamma_{ij}^1|_U = f_{ij}(x_1, x_2)$ and $\Gamma_{ij}^2|_U$ (Propositions 4 and 5) for $i, j = 1, 2$, and also $D_x = span\{(\partial/\partial x_1)|_x\}$ for all $x \in U$, then, the PFCAC2 admits a solution.*

Proof. Based on the assumptions of the statement, system (32) is reduced to the following PDE system:

$$\partial_2\beta_{11} - \partial_1\beta_{12} = R_{112}^1 + \Gamma_{12}^2\beta_{12} - \Gamma_{11}^2\beta_{22}, \quad (92)$$

$$\begin{aligned}\partial_2\beta_{12} - \partial_1\beta_{22} = R_{212}^1 + \beta_{11}\left(\beta_{22} + \Gamma_{22}^1\right) + \left(\Gamma_{22}^2 - 2\Gamma_{12}^1 - \beta_{12}\right)\beta_{12} \\ + \left(\Gamma_{11}^1 - \Gamma_{12}^2\right)\beta_{22},\end{aligned} \quad (93)$$

$$0 = R_{112}^2 + \Gamma_{11}^2\beta_{12} - \Gamma_{12}^2\beta_{11}, \quad (94)$$

$$0 = R_{212}^2 + \Gamma_{11}^2\beta_{22} - \Gamma_{12}^2\beta_{12}. \quad (95)$$

From Proposition 2, $\overline{\beta}_{ij} = -\Gamma_{ij}^1$ is one solution for the case, where ∇ is determined by the only nonzero Christoffel symbols Γ_{ij}^1, with a distribution $D = \text{span}\{(\partial/\partial x_1)|_x\}$. In this case, we have Γ_{ij}^2 as in Propositions 4 and 5, then we know $\widetilde{\beta_{ij}}$ is a solution, from which one can infer that one solution for systems (92)–(95) is $\beta_{ij} = \widetilde{\beta_{ij}} - \Gamma_{ij}^1$, and this is verified by substituting in each equation. By assumption, we know that the components of the curvature tensor are as follows:

$$\begin{aligned}R_{112}^1 &= \partial_1\Gamma_{12}^1 - \partial_2\Gamma_{11}^1 + \Gamma_{12}^1\Gamma_{21}^2 - \Gamma_{22}^1\Gamma_{11}^2, \\ R_{212}^1 &= \partial_1\Gamma_{22}^1 - \partial_2\Gamma_{12}^1 + \Gamma_{11}^1\Gamma_{22}^1 + \Gamma_{12}^1\Gamma_{22}^2 - \Gamma_{21}^1\Gamma_{12}^1 - \Gamma_{22}^1\Gamma_{12}^2, \\ R_{112}^2 &= \partial_1\Gamma_{21}^2 - \partial_2\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{21}^1 + \Gamma_{12}^2\Gamma_{21}^2 - \Gamma_{21}^2\Gamma_{11}^1 - \Gamma_{22}^2\Gamma_{11}^2, \\ R_{212}^2 &= \partial_1\Gamma_{22}^2 - \partial_2\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^1 - \Gamma_{21}^2\Gamma_{12}^1,\end{aligned} \quad (96)$$

and when replacing in equations (92)–(95), one gets

$$\partial_2\widetilde{\beta}_{11} - \partial_1\widetilde{\beta}_{12} = \Gamma_{12}^2\widetilde{\beta}_{12} - \Gamma_{11}^2\widetilde{\beta}_{22}, \quad (97)$$

$$\partial_2\widetilde{\beta}_{12} - \partial_1\widetilde{\beta}_{22} = \widetilde{\beta}_{11}\widetilde{\beta}_{22} - \widetilde{\beta}_{12}^2 + \Gamma_{22}^2\widetilde{\beta}_{12} - \Gamma_{12}^2\widetilde{\beta}_{22}, \quad (98)$$

$$\begin{aligned}0 = \partial_1\Gamma_{21}^2 - \partial_2\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{21}^2 \\ - \Gamma_{22}^2\Gamma_{11}^2 + \Gamma_{11}^2\widetilde{\beta}_{12} - \Gamma_{12}^2\widetilde{\beta}_{11},\end{aligned} \quad (99)$$

$$0 = \partial_1\Gamma_{22}^2 - \partial_2\Gamma_{12}^2 + \Gamma_{11}^2\widetilde{\beta}_{22} - \Gamma_{12}^2\widetilde{\beta}_{12}. \quad (100)$$

Given that Γ_{ij}^2 are defined as in Propositions 4 and 5 and $\widetilde{\beta_{ij}}$ is one solution, thus systems (97)–(100) can be solved and one gets

$$F_x = \beta_{ij}\begin{pmatrix} 1 \\ 0 \end{pmatrix} x^i x^j. \quad (101)$$

The following examples single out some systems of PDEs that cannot be solved. This is made clear by exposing inconsistencies among the equations. □

Example 2. If ∇ is defined by $\Gamma_{22}^2|_U = x_1$ and $\Gamma_{ij}^2 \equiv 0$ for $(i, j) = (2, 2)$ on U, then the curvature tensor associated has only one nonzero component, namely, $R_{212}^2 = 1$. Then, system (32) is reduced to

$$\begin{aligned}\partial_{11} &= \partial_{12}, \\ \partial_{12} &= \partial_{22} + \beta_{11}\beta_{22} - \beta_{12}^2 + x_1\beta_{12}, \\ 0 &= 1,\end{aligned} \quad (102)$$

in this case, the inconsistency is immediately evident; therefore, the system has no solution.

Example 3. If ∇ is determined by $\Gamma_{11}^2|_U = a$, with $a \in \mathbb{R}$, $\Gamma_{22}^2|_U = x_1$ and all other Christoffel symbols identically zero on U, then the curvature tensor has nonzero components. If $a \neq 0$, $R_{112}^2|_U = -ax_1$ and $R_{212}^2|_U = 1$, but if $a = 0$, we have the previous case. Then, assuming that $a \neq 0$, the resulting system is

$$\partial_2\beta_{11} - \partial_1\beta_{12} = -a\beta_{22}, \quad (103)$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + x_1\beta_{12}, \quad (104)$$

$$0 = -ax_1 + a\beta_{12}, \quad (105)$$

$$0 = 1 + a\beta_{22}, \quad (106)$$

by equations (105) and (106), one gets $\beta_{12} = x_1, \beta_{22} = -(1/a)$ and when replacing in equation (103), we obtain $\beta_{11} = 2x_2$. Nevertheless, equation (104) is not satisfied because we have $0 = -2x_2$, so the system is inconsistent and therefore has no solution.

The following example shows a particular type of inconsistency because the solvability of the system depends on a constant.

Example 4. If ∇ is determined by $\Gamma_{11}^2|_U = a$, $\Gamma_{12}^2|_U = b$, for $a, b \in \mathbb{R}$ and $\Gamma_{22}^2|_U = x_2$, then the curvature tensor has a nonzero component $R_{112}^2 = b^2 - ax_2$ and one gets the system:

$$\partial_2\beta_{11} - \partial_1\beta_{12} = b\beta_{12} - a\beta_{22}, \quad (107)$$

$$\partial_2\beta_{12} - \partial_1\beta_{22} = \beta_{11}\beta_{22} - \beta_{12}^2 + x_2\beta_{12} - b\beta_{22}, \quad (108)$$

$$0 = b^2 - ax_2 + a\beta_{12} - b\beta_{11}, \quad (109)$$

$$0 = a\beta_{22} - b\beta_{12}, \quad (110)$$

from equations (109) and (110), it follows that if $b \neq 0$, then $\beta_{11} = (b^2 - ax_2/b) + (a^2/b^2)\beta_{22}$ and $\beta_{12} = (a/b)\beta_{22}$ but when replacing in equations (107) and (108), one gets the following equation:

$$0 = \frac{a}{b}. \quad (111)$$

If $a = 0$, systems (107)–(110) have solution $\beta_{11} = b$ and $\beta_{12} = \beta_{22} = 0$ (Proposition 4, i(a); otherwise the system is inconsistent. If $b = 0$ and $a \neq 0$, by Proposition 4, the system is inconsistent as it does not satisfy the identity (3). On the other hand, with $a = b = 0$, ∇ has associated a zero curvature tensor and therefore the system admits trivial solution $\beta_{ij} = 0$.

5. Conclusions and Further Research

Although the PFCAC is easy to state, solving it represents an interesting challenge, as we can see in the PFCAC2. In this study, we have studied the PFCAC2 and we realized that one of the difficult parts is to characterize or study the set of solutions of the resulting PDE system. In addition, PFCAC2 could only be solved for a few cases, with constant rank 1 distributions. The main results give only sufficient conditions for existence of F; at this point, we are unable to determine necessary conditions.

As further research, it is intended to make a more geometric interpretation, consider more general distributions, look for necessary conditions, and solve the flattening problem for higher dimensions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The problem was originally suggested by D.A.L. to E.C. as a research subject. E.C. derived sufficient conditions for its solvability and singled out necessary conditions for rank-one distributions in a two-dimensional setting. As the first author, E.C. wrote the original draft of the manuscript. A.C.S. contributed to checking the technical proofs and improving the English version and the overall presentation of the text.

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