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ON THE CONTINUOUS FINITE-TIME STABILIZATION OF THE DOUBLE INTEGRATOR

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Abstract. Continuous finite-time stabilization is often treated under the analytical framework of homogeneity and has been frequently illustrated in the context of the feedback control of the double integrator. For such a simple system, the simplest considered continuous finite-time controller is composed of gained (proportional) exponentially weighted *position* and *velocity* error correction terms, with the exponential weights generally less than unity and constrained to satisfy a particular relation among them under homogeneity. What happens for less-than-unity exponential weights that do not satisfy such a homogeneity-based relation? Does the finite-time stabilization hold? Through a Lyapunov function based study, we analyze and give more concrete answers to such questions than those partially provided by previous studies on the topic. We do find a more exhaustive spectrum of the exponential weights that give rise to finite-time stability of the trivial solution. Other types of stability properties are further found to take place for less-than-or-equal-to-unity exponential weights. Moreover, through complementary analysis, local or ultimate behavior of the system solutions is further characterized. The analytical findings are further illustrated through computer simulations.

Key words. Continuous finite-time control, finite-time stability/stabilization, exponential stability with respect to a homogeneous norm, double integrator

AMS subject classifications. 93D05, 93D15, 93D40, 93C10, 34H15

1. Introduction.

Stabilization achieved in finite time through continuous feedback has been a subject of increasing interest in the last decades. Ever since the early work of [9], such a topic has been often studied and/or illustrated in the context of the control of the double integrator

$$\ddot{x} = u \tag{1.1}$$

The simplest controller considered in such a context is composed by the addition of suitable nonlinear *position* and *velocity* error correction actions, namely

$$u = -k_1 \mathrm{sign}(x)|x|^{a_1} - k_2 \mathrm{sign}(\dot{x})|\dot{x}|^{a_2} \triangleq u_0(x, \dot{x}) \tag{1.2}$$

$k_i > 0$, $i = 1, 2$, which proves to render the trivial solution $x(t) \equiv 0$ globally asymptotically stable for any positive values of the control parameters k_i, a_i, $i = 1, 2$. This case was analyzed within the framework of homogeneity in [5] where, by fixing a specific relation among a_1 and a_2, namely

$$a_1 = \frac{a_2}{2 - a_2} \tag{1.3}$$

a family of dilations with respect to which the resulting closed-loop system turns out to be homogeneous of degree $a_2 - 1$ was proven to exist; the finite-time stabilization goal was thus concluded to be achieved for any

$$a_2 \in (0, 1) \tag{1.4}$$

irrespective of the (positive) control gain values k_i, $i = 1, 2$. This is so since, for a homogeneous vector field with negative degree of homogeneity, asymptotic stability of

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the origin implies finite-time convergence of every solution that it attracts (which actually concerns every solution for any initial condition). In view of its simplicity, such an analytical procedure has been applied in other related studies, such as the finite-time observer design for the output-feedback stabilization of the double integrator developed in [10], and the particular resulting finite-time-observer-based output-feedback version of (1.1)–(1.4) for which alternative and supplementary analyses were presented in [3]. It has been further extended to discontinuous vector fields in [18]. Within the analytical context of such an extension, system (1.1) controlled by (1.2)-(1.3) taking $a_2 = 0$, which gives rise to the so-called *twisting controller* [14], was proven to achieve the finite-time stabilization objective under an additional control gain condition (namely $k_1 > k_2$, which is necessary to render the trivial solution $x(t) \equiv 0$ asymptotically stable). Moreover, such a discontinuous version of (1.2) was proven in [18] to state the basis for the design of controllers that lead the closed-loop error trajectories to zero in finite time even in the presence of input-matching non-vanishing perturbations. Such a robustness property was thus further shown to be achieved by a finite-time-discontinuous-observer-based output-feedback approach of the referred discontinuous version of (1.2) (*i.e.* with $a_1 = a_2 = 0$) in [19]. Achievement of the finite-time stabilization goal has been also studied for (1.1)–(1.4) in presence of input-matching vanishing perturbations satisfying particular growth conditions in [20].

Based on (1.2)–(1.4), other (more complex) finite-time continuous stabilizers for the double integrator, that render the closed-loop system homogeneous with negative degree of homogeneity, have been presented in other works. Such is the case of [6] and [20], which proposed $u = u_0\big(\phi_1(x,\dot{x}),\dot{x}\big)$ and $u = u_0(x,\dot{x}) + \phi_2(x,\dot{x})$, respectively, with $u_0(\cdot,\cdot)$ as in (1.2), $\phi_1(x,\dot{x}) = x + \frac{1}{2-a_2}\text{sign}(\dot{x})|\dot{x}|^{2-a_2}$, $\phi_2(x,\dot{x}) = -k_3\text{sign}(\dot{x})|x|^{a_1/2}|\dot{x}|^{a_2/2}$, $k_3 > 0$, and a_i, $i = 1, 2$, as in (1.3)-(1.4), for which a family of dilations with respect to which both resulting closed-loop systems are homogeneous of degree $a_2 - 1 < 0$ proves to exist [2, Example 5.5], [20].

Beyond the attributes or benefits that might characterize or show the implementation of homogeneity-based or homogenous-closed-loop-rendering finite-time continuous control schemes, their design might happen to be restrictive in view of the fixed relation among the involved exponents; in the specific case of (1.2), this refers to the fixed relation among a_1 and a_2 stated through (1.3). What happens for values of $a_i \in (0,1)$, $i = 1, 2$, that do not satisfy such relation? Does the finite-time stabilization hold? It is well-known that finite-time stability (*i.e.* Lyapunov stability plus *finite-time attractivity* [7]) of an equilibrium implies non-uniqueness of solutions (in reverse time) which in turn implies the lack of Lipschitz-continuity of the system dynamics at the equilibrium. Hence, since with $a_i \in (0,1)$, $i = 1, 2$, (1.1)-(1.2) lacks of Lipschitz-continuity at $(x,\dot{x}) = (0,0)$, could we not expect that finite-time stability hold even if (1.3) is not satisfied? By continuous dependence (or even differentiability) of the (non-trivial) solutions of (1.1)-(1.2) on parameters [13, Chapter 3], it seems reasonable to expect that finite-time stability could hold (at least for values of a_1 that slightly differ from that fixed through (1.3), given any $a_2 \in (0,1)$). But could this be the case for any value combination of $a_i \in (0,1)$, $i = 1, 2$? Since the lack of Lipschitz-continuity is however not sufficient for non-uniqueness of solutions, having any $a_i \in (0,1)$, $i = 1, 2$, could not necessarily guarantee finite-time stability. These questions show that, beyond the simplicity and beneficial features earned by the design through (or supported by) homogeneity, we do not yet seem to have the certainty to have a complete panorama on the continuous-controller-induced finite-time stabilization (or on finite-time stability) studied through the double integrator. Getting a wider picture on finite-time stability through (1.1)-(1.2), or a broader view on the stability properties of (1.1)-(1.2) with $a_i \in (0,1)$, $i = 1,2$, is important from the control and dynamical system theories viewpoint, would generate a wider perspective for control design, and may prove to be useful to expand the capabilities accounted for closed-loop behavior/performance adjustment or refinement.

A partial answer to the questions formulated above is given in [9] where, through a particularly original analysis on (1.1)-(1.2) with $k_1 = k_2 = 1$, finite-time stability of the trivial solution $x(t) \equiv 0$ is concluded to be achieved with

$$a_2 \in (0,1) \quad , \quad a_1 > \frac{a_2}{2-a_2} \tag{1.5}$$

However, such a result from [9] turns out to lack of exhaustiveness by developing a local analysis restricted to finite-time convergent solutions that avoid non-stopping oscillations during the finite-time transient (before the definitive permanence at zero). In the own words of the author: "*If one wishes to show that a second order system is finite time, one could search for a contour that prevented trajectories from spiraling around the origin. It seems natural to search for a contour which is itself invariant. This idea lies at the core of the next two theorems.*" [9, Section 4, p. 764]. Moreover, the lack of exhaustiveness further encompasses the finite-time convergence aspect in itself, by limiting the result to conditions that permit (but do not guarantee) such type of convergence, without strictly ruling out infinite-time convergent solutions (details about the referred limitations will be given after the presentation of the main result). As a matter of fact, observe that (1.5) curiously permits values of a_1 greater than 1 (which partially contradicts the previously commented argument on the lack of Lipschitz-continuity needed to achieve the finite-time stabilization goal).

This work aims to give answers to the previously formulated questions on the finite-time stabilization of (1.1)-(1.2), and to actually achieve to give a deeper insight on the stability properties of (1.1)-(1.2) with $a_i \in (0,1]$, $i = 1,2$. Through a Lyapunov-function-based analysis, more exhaustive conditions on a_1 and a_2 that guarantee the finite-time stability of the trivial solution $x(t) \equiv 0$ are obtained without constraining the analysis or the results to a specific type of finite-time convergent solutions. Such conditions turn out to include the homogeneity related ones, namely (1.3)-(1.4) (or equivalently $a_1 \in (0,1)$ and $a_2 = a_1/(1+a_1)$), as a particular case. Furthermore, other type of stability properties are further shown to arise in the considered analytical context. The study includes a discussion section where further analysis addressed to gain insight on the contrast among the results obtained here and those from [9] is developed, and which complements the Lyapunov-function-based study with conclusions on the local or ultimate behavior of the system solutions; in particular, finite-time convergent system solutions ultimately undergoing non-stopping oscillations are confirmed to be obtainable under the found conditions, while getting solutions that do not converge in finite time is shown to be possible when the found conditions are not satisfied. A section with simulation results is further included, through which the analytical findings are illustrated.

2. Preliminaries.

Throughout this work, x_i stands for the i^{th} element of $x \in \mathbb{R}^n$. 0_n represents the origin of $\mathbb{R}^n$. $\mathbb{R}^n_+$ is the set of vectors in $\mathbb{R}^n$ whose elements are all positive, *i.e.* $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0, i = 1,\dots,n\}$. An n-dimensional closed ball and an $(n-1)$-dimensional sphere, both of radius $c > 0$, are denoted $\mathcal{B}^n_c$ and $\mathcal{S}^{n-1}_c$, respectively, *i.e.* $\mathcal{B}^n_c = \{z \in \mathbb{R}^n : \|z\| \leq c\}$ and $\mathcal{S}^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$. A fundamental fact that will be involved in this study is *Young's inequality* [4], *i.e.* for any $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and any $a, b \in \mathbb{R}_{\geq 0}$, we have that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

For a continuous scalar function V, $\dot{V}$ will represent its upper-right derivative along the trajectories of a considered system [23, 22, 7].

2.1. Finite-time stability. Consider an n-th order autonomous system $\dot{x} = f(x)$, with f being continuous on an open connected neighborhood $D \subset \mathbb{R}^n$ of the origin, where the system is considered to have an equilibrium point, *i.e.* $f(0_n) = 0_n$, and such that the system solutions $x(t; x_0)$ are unique in forward time for any initial condition $x(0; x_0) = x_0 \in D \setminus \{0_n\}$.

DEFINITION 2.1. *[8] The origin is a finite-time stable equilibrium if and only if it is Lyapunov stable and there exist an open neighborhood $\mathcal{N} \subseteq D$ of 0_n, being positively invariant under f, and a positive definite function $T : \mathcal{N} \to \mathbb{R}$, called the settling time function, such that $x(t; x_0) \neq 0_n$, $\forall t \in \big[0, T(x_0)\big)$, for every $x_0 \in \mathcal{N} \setminus \{0_n\}$, and $x(t; x_0) = 0_n$, $\forall t \geq T(x_0)$, for every $x_0 \in \mathcal{N}$. It is globally finite-time stable if it is finite-time stable with $\mathcal{N} = D = \mathbb{R}^n$.*

Remark 2.2. The origin is a globally finite-time stable equilibrium if and only if it is globally asymptotically stable and finite-time stable. Sufficiency follows from Definition 2.1 and [8, Lemma 2.2]; it has been straightforwardly stated and involved in the literature [11, Remark 1]. Necessity is a direct consequence of Definition 2.1 by the implication that global finite-time stability entails of both finite-time stability and global asymptotic stability [17].

THEOREM 2.3. *[7] Suppose there is a positive definite continuous function $V : D \to \mathbb{R}$ for which there exist real numbers $c > 0$ and $\alpha \in (0, 1)$ and an open neighborhood $\mathcal{V} \subseteq D$ of the origin such that $\dot{V}(x) \leq -cV^{\alpha}(x)$, $\forall x \in \mathcal{V} \setminus \{0_n\}$. Then the origin is a finite-time stable equilibrium. Moreover, with $\mathcal{N}$ as in Definition 2.1, the settling time function T is continuous on $\mathcal{N}$ and satisfies $T(x) \leq [V(x)]^{1-\alpha}/[c(1-\alpha)]$. If in addition $D = \mathbb{R}^n$, V is proper and $\dot{V}$ takes negative values on $\mathbb{R}^n \setminus \{0_n\}$, then the origin is globally finite-time stable.*

Since finite-time stability turns out to be a particular case of asymptotic stability (in the sense of Lyapunov's stability theory [13, Definition 4.1]), an asymptotically stable equilibrium point which is not reached in finite time by any of the trajectories that it attracts will be said to have *infinite-time attractivity* (or to be *infinite-time attractive*).

2.2. Local homogeneity. The definitions and results stated in this subsection are related to *family of dilations*, defined as $\delta_\epsilon^r(x) = (\epsilon^{r_1} x_1 \ \cdots \ \epsilon^{r_n} x_n)^T$, $\forall \epsilon > 0$, for every $x \in \mathcal{S}_1^{n-1}$, with $r = (r_1 \ \cdots \ r_n)^T$, where the dilation coefficients r_i, $i = 1, \ldots, n$, are positive scalars.

DEFINITION 2.4. *A function $V : \mathbb{R}^n \to \mathbb{R}$, resp. vector field $f : \sum_{i=1}^n f_i \frac{\partial}{\partial x}$ (with $f_i : \mathbb{R}^n \to \mathbb{R}$), is locally homogeneous of degree α with respect to δ_ϵ^r if there exists an open neighborhood of the origin $\mathcal{D}$, referred to as the domain of homogeneity, such that, for every $x \in \mathcal{D}$ and all $\epsilon \in (0, 1]$: $\delta_r^\epsilon(x) \in \mathcal{D}$ and $V\big(\delta_\epsilon^r(x)\big) = \epsilon^\alpha V(x)$, resp. $f_i\big(\delta_\epsilon^r(x)\big) = \epsilon^{\alpha + r_i} f_i(x)$ $\forall i = 1, \ldots, n$.*[1]

[1]The concept of *homogeneity in the 0-limit*, stated in [1], settles down an alternative definition

Definition 2.4 is a refined (equivalent) version of [24, Definition 2.1], stated in (and reproduced from) [25]. A function or vector field satisfying Definition 2.4 for a given $r \in \mathbb{R}^n_+$ will (for simplicity) be equivalently said to be *locally r-homogeneous of degree* α. It turns out to be homogenous (in the conventional sense) if its domain of homogeneity $\mathcal{D} = \mathbb{R}^n$. By a function, resp. vector field, referred to as (locally) homogenous of degree α, it will be meant that there is $r \in \mathbb{R}^n_+$ for which the function, resp. vector field, is (locally) r-homogeneous of degree α.

LEMMA 2.5. *[24] Suppose that, for every $i = 1, 2$, V_i is a scalar continuous function being locally r-homogeneous of degree $\alpha_i > 0$, with domain of homogeneity $\mathcal{D}_i \subset \mathbb{R}^n$. Suppose further that V_1 is positive definite on $\mathcal{D}_1$. Let $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ and $c > 0$ be such that $S_c^{n-1} \subset \mathcal{D}$. Then, for every $x \in \mathcal{D}$,*

$$c_1[V_1(x)]^{\alpha_2/\alpha_1} \le V_2(x) \le c_2[V_1(x)]^{\alpha_2/\alpha_1}$$

with $c_1 \le [\min_{z\in S_c^{n-1}} V_2(z)] \cdot [\max_{z\in S_c^{n-1}} V_1(z)]^{-\alpha_2/\alpha_1}$ and $c_2 \ge [\max_{z\in S_c^{n-1}} V_2(z)] \cdot [\min_{z\in S_c^{n-1}} V_1(z)]^{-\alpha_2/\alpha_1}$.

Remark 2.6. Observe that if V_2 happens to be positive (resp. negative) definite, then c_1 and c_2 in Lemma 2.5 are both positive (resp. negative) constants.

2.3. Exponential stability with respect to a homogeneous norm.

DEFINITION 2.7. *[15] Given $r \in \mathbb{R}^n_+$, a continuous function mapping $x \in \mathbb{R}^n$ to $\mathbb{R}$, denoted $\|x\|_r$, is called a homogeneous norm with respect to the family of dilations δ^r_ϵ if $\|x\|_r \ge 0$, $\forall x \in \mathbb{R}^n$, with $\|x\|_r = 0 \iff x = 0_n$, and $\|\delta^r_\epsilon(x)\|_r = \epsilon\|x\|_r$ for any $x \in \mathbb{R}^n$ and all $\epsilon > 0$.*

A function satisfying Definition 2.7 for a given $r \in \mathbb{R}^n_+$ will (for simplicity) be equivalently said to be an *r-homogeneous norm*. Note that it turns out be a positive definite continuous function being r-homogeneous of degree 1. By a function referred to as a homogenous norm, it will be meant that there is $r \in \mathbb{R}^n_+$ for which the function is an r-homogeneous norm. A special subset of homogenous norms is defined as follows.

DEFINITION 2.8. *[12] Given $r \in \mathbb{R}^n_+$, an r-homogeneous p-norm ($p \ge 1$) is defined as $\|x\|_{r,p} = \left[\sum_{i=1}^n |x_i|^{p/r_i}\right]^{1/p}$.*

For the sake of generality, in the rest of this subsection, definitions and results are stated under the consideration of the generalized n-th order (unforced) state equation $\dot{x} = f(t, x)$, representing both autonomous and non-autonomous systems. The vector field f is considered to be continuous in x on an open connected neighborhood $D \subset \mathbb{R}^n$ of the origin, where the system is assumed to have an equilibrium point, and such that the system solutions $x(t; t_0, x_0)$, or simply $x(t)$ whenever convenient or clear from the context, are unique in forward time for any initial state $x(t_0; t_0, x_0) = x_0 \in D \setminus \{0_n\}$ at initial time $t_0 \in [0, \infty)$. In the time-varying case, f is additionally considered to be piecewise continuous in t on $[0, \infty)$.

of local homogeneity which turns out to be more attached to the notion of *locality* generally used in control theory (that is, a function or vector field homogeneous in the 0-limit approximates a homogeneous one in a sufficiently small neighborhood of the origin). Definition 2.4 (above) is based on the idea that a function or vector field be permitted to be identical to a homogenous one in a neighborhood of the origin, which permits the statement and use of results such as Lemma 2.5. Actually, local homogeneous functions or vector fields, in the sense of Definition 2.4, are homogenous in the 0-limit (the inverse is not necessarily true).

Definition 2.9. *[15, 12] The origin is exponentially stable with respect to the r-homogeneous norm $\|\cdot\|_r$, for a given $r \in \mathbb{R}_+^n$, if there exist a neighborhood of the origin $\mathcal{U} \subseteq D$ and constants $a \geq 1$ and $b > 0$ such that $\|x(t;t_0,x_0)\|_r \leq a\|x_0\|_r e^{-b(t-t_0)}$, $\forall t \geq t_0 \geq 0$, $\forall x_0 \in \mathcal{U}$. If this is satisfied with $\mathcal{U} = D = \mathbb{R}^n$, then the origin is globally exponentially stable with respect to the r-homogeneous norm $\|\cdot\|_r$.*

For simplicity, an equilibrium point satisfying Definition 2.9 for a given $r \in \mathbb{R}_+^n$ will be equivalently said to be *r-exponentially stable*.[2]

Remark 2.10. Although *norm* is involved in the denomination stated through Definition 2.7, by noting that such a definition does not strictly define a particular type of norm (the triangle inequality is not asked to be satisfied and the considered scaling property differs to the one involved in the conventional definition of a norm; such an imprecision on the referred denomination was highlighted for Definition 2.8 in [21, Remark 5]), Definition 2.9 is corroborated to state a notion of exponential stability that differs from the conventional one, without necessarily keeping a logical relation among them (*i.e.* without necessarily one of them implying the other). In particular, if an r-homogeneous p-norm is involved, Definition 2.9 is noted to become the conventional definition of exponential stability when the elements of r are all equal to unity. For any other $r = (r_1 \ \dots \ r_n)^T \in \mathbb{R}_+^n$, it is proven in [12] that r-exponential stability does not necessarily imply exponential stability (in the conventional sense), by showing that for an r-exponentially stable equilibrium point, the (p) norm of trajectories with initial condition sufficiently close to it have an exponentially-decaying bound that depends nonlinearly on the norm of the initial state vector; more precisely $\|x(t;t_0,x_0)\| \leq a'\|x_0\|^{r_m/r_M} e^{-b' r_m (t-t_0)}$, for positive constants a' and b', with $r_m = \min_i\{r_i\}$ and $r_M = \max_i\{r_i\}$ (this is stated for $p = 2$ in [12] but the extension to any $p \geq 1$ follows from the equivalence of p-norms). Such a nonlinear dependence of the referred exponentially-decaying bound on the norm of the initial state vector is further shown in [12] (through an illustrative example) to be indispensable.

Theorem 2.11. *Let $V : [0,\infty) \times D \to \mathbb{R}$ be a continuous function such that*

$$c_1\|x\|_r^a \leq V(t,x) \leq c_2\|x\|_r^a \tag{2.2}$$

$$\dot{V}(t,x) \leq -c_3\|x\|_r^a \tag{2.3}$$

$\forall (t,x) \in [0,\infty) \times D$, where c_i, $i = 1,2,3$, and a are positive constants, and $r \in \mathbb{R}_+^n$. Then, the origin is r-exponentially stable. If the assumptions hold globally, then the origin is globally r-exponentially stable.

The proof of Theorem 2.11 follows along the lines of the proof of [13, Theorem 4.10] by simply replacing (the conventional norm) $\|\cdot\|$ by (the r-homogeneous norm) $\|\cdot\|_r$. The following corollary, generated as part of this work, will prove to be instrumental in the proof of the main result (presented in the next section).

Corollary 2.12. *Under the assumptions of Theorem 2.11, let us additionally suppose that there is a continuous function $W : [0,\infty) \times D_0 \to \mathbb{R}$ such that*

$$c_4\|x\|_r^{a_0} \leq W(t,x) \leq c_5\|x\|_r^{a_0} \tag{2.4}$$

$$\dot{W}(t,x) \geq -c_6\|x\|_r^{a_0}$$

[2] Definition 2.9 has previously adopted different (short) alternative designations, namely Δ-*exponential stability* in [12], *ρ-exponential stability* in [15], and *δ-exponential stability* in [25].

for all $t \geq 0$ and all x in an open connected neighborhood of the origin $D_0 \subseteq D$, where c_i, $i = 4, 5, 6$, and a_0 are positive constants. Then, the origin is r-exponentially stable with infinite-time attractivity. If the assumptions of Theorem 2.11 hold globally, then the origin is globally r-exponentially stable with infinite-time attractivity.

Proof. Following a procedure analogous to that of the proof of [13, Theorem 4.10], we get $\dot{W} \geq -(c_6/c_4)W$. Then, by the comparison principle [23, Theorem 4.2], we have that $W(t, x(t)) \geq W(t_0, x_0)e^{-(c_6/c_4)(t-t_0)}$, $\forall t \geq t_0$. From this and (2.4), we get

$$\|x(t)\|_r \geq \left[\frac{W(t, x(t))}{c_5}\right]^{\frac{1}{a_0}} \geq \left[\frac{W(t_0, x_0)e^{-\frac{c_6}{c_4}(t-t_0)}}{c_5}\right]^{\frac{1}{a_0}}$$
$$\geq \left[\frac{c_4\|x_0\|_r^{a_0}e^{-\frac{c_6}{c_4}(t-t_0)}}{c_5}\right]^{\frac{1}{a_0}} = \left(\frac{c_4}{c_5}\right)^{\frac{1}{a_0}} \|x_0\|_r \, e^{-\frac{c_6}{c_4 a_0}(t-t_0)} \quad \forall t \geq t_0 \tag{2.5}$$

This expression reveals that the system solution cannot reach zero in finite time, whence the r-exponential stability of the origin is concluded to be infinite-time attractive. If the assumptions of Theorem 2.11 hold globally, then there is a finite time $t_1 \geq t_0$ such that $x(t) \in D_0$, $\forall t \geq t_1$, and consequently (2.5) holds with t_0 and x_0 replaced by t_1 and $x(t_1)$, respectively, whence the r-exponential stability with infinite-time attractivity is concluded to hold for any initial condition $x_0 \in \mathbb{R}^n$ at initial time $t_0 \geq 0$. ◻

3. Main result. Consider the double integrator dynamics (1.1) in closed-loop with the control law (1.2), *i.e.*

$$\ddot{x} = -k_1 \text{sign}(x)|x|^{a_1} - k_2 \text{sign}(\dot{x})|\dot{x}|^{a_2} \tag{3.1}$$

with $k_i > 0$ and $a_i \in (0, 1]$, $\forall i \in \{1, 2\}$. Let

$$r_0 = \begin{pmatrix} \frac{2}{1+a_1} \\ 1 \end{pmatrix} \in \mathbb{R}^2_+ \tag{3.2}$$

The main result of this work is stated next.

THEOREM 3.1. *The trivial solution $x(t) \equiv 0$ of system (3.1) is*

1. *globally finite-time stable if*

$$0 < a_1 < a_2 < 1 \tag{3.3}$$

2. *globally asymptotically stable and (locally) r_0-exponentially stable with infinite-time attractivity if $0 < a_1 \leq a_2 = 1$.*

Proof. The proof is divided into four stages. The first stage shows global asymptotic stability of the trivial solution $x(t) \equiv 0$ through a non-strict Lyapunov function involving the invariance theory [16, Section 7.2]. The second stage develops a local analysis through a strict Lyapunov function that proves to be essential in the rest of the proof. Finally, based on the results obtained in the first two stages, the third and fourth stages prove items 1 and 2 of the theorem, respectively.

First stage: global asymptotic stability. Consider the following continuously differentiable positive definite radially unbounded function

$$V_0(x, \dot{x}) = \frac{k_1|x|^{1+a_1}}{1 + a_1} + \frac{\dot{x}^2}{2} \tag{3.4}$$

Its derivative along the system trajectories is obtained, after basic developments, as

$$\dot{V}_0(x,\dot{x}) = -k_2|\dot{x}|^{1+a_2} \tag{3.5}$$

whence one sees that $\dot{V}_0(x,\dot{x}) \le 0$, $\forall (x,\dot{x}) \in \mathbb{R}^2$, and $\dot{V}_0(x,\dot{x}) = 0 \iff \dot{x} = 0$. Since $\dot{x}(t) \equiv 0 \implies \ddot{x}(t) \equiv 0$ and, from (3.1), $\ddot{x}(t) \equiv \dot{x}(t) \equiv 0 \implies -k_1 \text{sign}(x(t))|x(t)|^{a_1} \equiv 0 \iff x(t) \equiv 0$ (*i.e.* $x(t) \equiv 0$ is the only system solution along which $\dot{V}_0$ remains permanently zeroed), one concludes, by the invariance theory [16, Section 7.2] (more precisely, by [16, Corollary 7.2.1]), that the trivial solution $x(t) \equiv 0$ is globally asymptotically stable (note that this intermediate conclusion holds for any $a_i > 0$, $i = 1, 2$).

Second stage: local analysis. For any $\rho > 0$, let us consider the 2-dimensional ball of radius ρ, $\mathcal{B}^2_\rho$. Observe that $(x,\dot{x}) \in \mathcal{B}^2_\rho \implies \max\{|x|,|\dot{x}|\} \le \rho$. In the rest of the proof, we shall consider that a_i, $i = 1, 2$, satisfy the following inequality

$$0 < a_1 \le a_2 \le 1 \tag{3.6}$$

Let

$$V_1(x,\dot{x}) = V_0^\beta(x,\dot{x}) + \varepsilon x\dot{x} \tag{3.7}$$

where V_0 is defined in Eq. (3.4), while β and ε are positive constants such that

$$1 \le \beta \le \beta_0 \triangleq \min\{\beta_1, \beta_2\} \le \beta_3 \tag{3.8a}$$

$$\beta_1 = \frac{a_1 + a_2}{2a_1} \quad , \quad \beta_2 = \frac{3 - a_2}{2} \quad , \quad \beta_3 = \frac{3 + a_1}{2(1 + a_1)} \tag{3.8b}$$

(one can verify that (3.6) $\implies 1 \le \beta_0$, and $\beta_0 \le \beta_3 \le \max\{\beta_1, \beta_2\}$, $\forall a_i > 0$, $i = 1, 2$) and

$$\varepsilon < \varepsilon_0 \triangleq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \tag{3.9a}$$

$$\varepsilon_1 = \left[\frac{k_1 b_1 \rho^{1+a_1-(b_1/\beta)}}{1 + a_1}\right]^\beta \quad , \quad \varepsilon_2 = \left[\frac{b_1 \rho^{2-[b_1/(\beta(b_1-1))]}}{2(b_1 - 1)}\right]^\beta$$
$$\varepsilon_3 = \frac{2^{1-\beta} b_2 \beta k_2}{\left[\frac{k_2 \rho^{b_2-1-a_1}}{k_1 b_2}\right]^{1/(b_2-1)} k_2(b_2 - 1)\rho^{a_2 b_2/(b_2-1)-2\beta+1-a_2} + b_2 \rho^{3-2\beta-a_2}} \tag{3.9b}$$

with b_1 and b_2 being positive constants such that

$$b_1 \in \left[(1 + a_1)\beta \ , \ \frac{2\beta}{2\beta - 1}\right] \tag{3.10}$$

$$b_2 \in \left[1 + a_1 \ , \ 1 + \frac{a_2}{2\beta - 1}\right] \tag{3.11}$$

(one can verify, from expressions (3.8), that $1 \le \beta \le \beta_0 \le \beta_3 \implies 1 + a_1 \le (1 + a_1)\beta \le 2\beta/(2\beta - 1)$ and $1 \le \beta \le \beta_0 \le \beta_1 \implies 1 + a_1 \le 1 + a_2/(2\beta - 1)$).

Note, on the one hand, that

$$\begin{aligned}
V_1(x,\dot{x}) &\geq V_0^\beta(x,\dot{x}) - \varepsilon\Big(|x|^{1/\beta}|\dot{x}|^{1/\beta}\Big)^\beta \\
&\geq V_0^\beta(x,\dot{x}) - \varepsilon\Big(\frac{|x|^{b_1/\beta}}{b_1} + \frac{(b_1-1)|\dot{x}|^{b_1/(\beta(b_1-1))}}{b_1}\Big)^\beta \qquad (3.12)\\
&\geq V_0^\beta(x,\dot{x}) - \frac{\varepsilon}{b_1^\beta}\Big[|x|^{(b_1/\beta)-1-a_1}|x|^{1+a_1} + (b_1-1)|\dot{x}|^{[b_1/(\beta(b_1-1))]-2}\dot{x}^2\Big]^\beta \\
&\geq V_0^\beta(x,\dot{x}) - W_0^\beta(x,\dot{x}) \triangleq W_1(x,\dot{x}) \qquad \forall (x,\dot{x}) \in \mathcal{B}_\rho^2 \qquad (3.13)
\end{aligned}$$

where

$$W_0(x,\dot{x}) = \frac{\varepsilon^{1/\beta}}{b_1}\Big[\rho^{(b_1/\beta)-1-a_1}|x|^{1+a_1} + (b_1-1)\rho^{[b_1/(\beta(b_1-1))]-2}\dot{x}^2\Big] \qquad (3.14)$$

(one can verify that (3.10) $\implies (b_1/\beta \geq 1+a_1) \wedge \big[b_1/(\beta(b_1-1)) \geq 2\big]$) and Young's inequality has been applied (taking $p = b_1$ and $q = b_1/(b_1-1)$ in (2.1)) to get (3.12). Notice further that $V_0^\beta(x,\dot{x}) - W_0^\beta(x,\dot{x}) > 0 \iff V_0^\beta(x,\dot{x}) > W_0^\beta(x,\dot{x}) \iff V_0(x,\dot{x}) > W_0(x,\dot{x}) \iff V_0(x,\dot{x}) - W_0(x,\dot{x}) > 0$. Hence, by proving that $V_0(x,\dot{x}) - W_0(x,\dot{x}) > 0$, $\forall (x,\dot{x}) \in \mathcal{B}_\rho^2 \setminus \{(0,0)\}$, positive definiteness of $W_1(x,\dot{x})$ in (3.13) —and consequently of $V_1(x,\dot{x})$ in (3.7)— (on $\mathcal{B}_\rho^2$) is concluded. In this direction, let us define

$$\begin{aligned}
\kappa_{m1} &= \frac{k_1}{1+a_1} - \frac{\rho^{(b_1/\beta)-1-a_1}}{b_1}\cdot\varepsilon^{1/\beta} \\
\kappa_{m2} &= \frac{1}{2} - \frac{(b_1-1)\rho^{[b_1/(\beta(b_1-1))]-2}}{b_1}\cdot\varepsilon^{1/\beta}
\end{aligned} \qquad (3.15)$$

and let us further note that, from expressions (3.9), one may corroborate, after basic developments, that $\varepsilon < \varepsilon_0 \leq \varepsilon_1 \implies \kappa_{m1} > 0$ and $\varepsilon < \varepsilon_0 \leq \varepsilon_2 \implies \kappa_{m2} > 0$. From this, and the expressions defining $V_0(x,\dot{x})$ and $W_0(x,\dot{x})$, we have $V_0(x,\dot{x}) - W_0(x,\dot{x}) = \kappa_{m1}|x|^{1+a_1} + \kappa_{m2}\dot{x}^2 > 0$, $\forall (x,\dot{x}) \in \mathcal{B}_\rho^2 \setminus \{(0,0)\}$, whence positive definiteness of $V_1(x,\dot{x})$ is concluded.

Note, on the other hand, that following a similar procedure we get

$$\begin{aligned}
V_1(x,\dot{x}) &\leq V_0^\beta(x,\dot{x}) + \varepsilon\Big(|x|^{1/\beta}|\dot{x}|^{1/\beta}\Big)^\beta \\
&\leq V_0^\beta(x,\dot{x}) + \varepsilon\Big(\frac{|x|^{b_1/\beta}}{b_1} + \frac{(b_1-1)|\dot{x}|^{b_1/(\beta(b_1-1))}}{b_1}\Big)^\beta \\
&\leq \Big(\frac{k_1|x|^{1+a_1}}{1+a_1} + \frac{\dot{x}^2}{2}\Big)^\beta \\
&\qquad + \varepsilon\Big(\frac{|x|^{(b_1/\beta)-1-a_1}|x|^{1+a_1}}{b_1} + \frac{(b_1-1)|\dot{x}|^{[b_1/(\beta(b_1-1))]-2}\dot{x}^2}{b_1}\Big)^\beta \\
&\leq w_2(x,\dot{x}) \leq W_2(x,\dot{x}) \qquad \forall (x,\dot{x}) \in \mathcal{B}_\rho^2 \qquad (3.16)
\end{aligned}$$

where

$$\begin{aligned}
w_2(x,\dot{x}) &= (1+\varepsilon)\Big(\kappa_{M1}|x|^{1+a_1} + \kappa'_{M2}\dot{x}^2\Big)^\beta \qquad (3.17)\\
&= (1+\varepsilon)\Big(\kappa_{M1}|x|^{1+a_1} + \kappa'_{M2}|\dot{x}|^{3-2\beta-a_2}|\dot{x}|^{2\beta-1+a_2}\Big)^\beta
\end{aligned}$$

with

$$\kappa_{M1} = \max\left\{ \frac{k_1}{1+a_1} \ , \ \frac{\rho^{(b_1/\beta)-1-a_1}}{b_1} \right\} \qquad (3.18)$$
$$\kappa'_{M2} = \max\left\{ \frac{1}{2} \ , \ \frac{(b_1-1)\rho^{[b_1/(\beta(b_1-1))]-2}}{b_1} \right\}$$

and

$$W_2(x,\dot{x}) = (1+\varepsilon)\Big(\kappa_{M1}|x|^{1+a_1} + \kappa_{M2}|\dot{x}|^{2\beta-1+a_2}\Big)^{\beta} \qquad (3.19)$$

with

$$\kappa_{M2} = \kappa'_{M2}\rho^{3-2\beta-a_2} \qquad (3.20)$$

(one can verify, from expressions (3.8), that $1 \le \beta \le \beta_0 \le \beta_2 \implies 1 + a_2 \le 2\beta - 1 + a_2 \le 2 \implies 3 - 2\beta - a_2 \ge 0$).

The derivative of V_1 along the system trajectories is obtained, after basic developments, as

$$\dot{V}_1(x,\dot{x}) = \beta V_0^{\beta-1}(x,\dot{x})\dot{V}_0(x,\dot{x}) + \varepsilon\dot{x}^2 - \varepsilon k_1|x|^{1+a_1} - \varepsilon k_2 x\,\mathrm{sign}(\dot{x})|\dot{x}|^{a_2} \qquad (3.21)$$

Under the consideration of (3.4), (3.5) and (3.8a), we further get

$$\begin{aligned}
\dot{V}_1(x,\dot{x}) &\le -\frac{\beta k_2}{2^{\beta-1}}|\dot{x}|^{2\beta-1+a_2} + \varepsilon\dot{x}^2 - \varepsilon k_1|x|^{1+a_1} \\
&\qquad + \varepsilon k_2\big(\gamma^{-(b_2-1)/b_2}|x|\big)\big(\gamma^{(b_2-1)/b_2}|\dot{x}|^{a_2}\big) \\
&\le -\frac{\beta k_2}{2^{\beta-1}}|\dot{x}|^{2\beta-1+a_2} + \varepsilon\dot{x}^2 - \varepsilon k_1|x|^{1+a_1} \qquad (3.22)\\
&\qquad + \varepsilon k_2\left(\frac{\gamma^{-(b_2-1)}|x|^{b_2}}{b_2} + \frac{(b_2-1)\gamma|\dot{x}|^{a_2b_2/(b_2-1)}}{b_2}\right) \\
&\le -\varepsilon\left(k_1 - \frac{k_2\gamma^{-(b_2-1)}|x|^{b_2-1-a_1}}{b_2}\right)|x|^{1+a_1} \\
&\qquad - \left(\frac{\beta k_2}{2^{\beta-1}} - \varepsilon|\dot{x}|^{3-2\beta-a_2}\right. \\
&\qquad\qquad \left. - \frac{\varepsilon k_2(b_2-1)\gamma|\dot{x}|^{[a_2b_2/(b_2-1)]-2\beta+1-a_2}}{b_2}\right)|\dot{x}|^{2\beta-1+a_2} \\
&\le -W_3(x,\dot{x}) \qquad \forall (x,\dot{x}) \in \mathcal{B}^2_\rho \qquad (3.23)
\end{aligned}$$

where

$$W_3(x,\dot{x}) = \varepsilon\bar{\kappa}_{m1}|x|^{1+a_1} + \bar{\kappa}_{m2}|\dot{x}|^{2\beta-1+a_2} \qquad (3.24)$$

with

$$\bar{\kappa}_{m1} = k_1 - \frac{k_2\rho^{b_2-1-a_1}}{b_2}\cdot\gamma^{-(b_2-1)} \qquad (3.25)$$
$$\bar{\kappa}_{m2} = \frac{\beta k_2}{2^{\beta-1}} - \varepsilon\rho^{3-2\beta-a_2} - \frac{\varepsilon k_2(b_2-1)\rho^{[a_2b_2/(b_2-1)]-2\beta+1-a_2}}{b_2}\cdot\gamma$$

(one can verify that (3.11) $\implies (b_2 \geq 1 + a_1) \wedge [a_2 b_2/(b_2-1) \geq 2\beta - 1 + a_2]$ and, as previously noted, that (3.8) $\implies 1 + a_2 \leq 2\beta - 1 + a_2 \leq 2 \implies 3 - 2\beta - a_2 \geq 0$), γ is a positive constant such that

$$\gamma_m \triangleq \left(\frac{k_2 \rho^{b_2-1-a_1}}{k_1 b_2}\right)^{1/(b_2-1)} < \gamma < \frac{b_2\left(\frac{\beta k_2}{2^{\beta-1}} - \varepsilon \rho^{3-2\beta-a_2}\right)}{\varepsilon k_2 (b_2-1) \rho^{[a_2 b_2/(b_2-1)]-2\beta+1-a_2}} \triangleq \gamma_M \tag{3.26}$$

(one can verify, from expressions (3.9), that $\varepsilon < \varepsilon_0 \leq \varepsilon_3 \implies \gamma_M > \gamma_m$) and Young's inequality was applied (taking $p = b_2$ and $q = b_2/(b_2-1)$ in (2.1)) to get (3.22). One can further verify, after basic developments, that (3.26) $\implies \bar{\kappa}_{mi} > 0$, $i = 1, 2$, whence $W_3(x, \dot{x})$ is corroborated to be positive definite —and consequently $\dot{V}_1(x, \dot{x})$ is concluded to be negative definite— (on $\mathcal{B}^2_\rho$). Moreover, from (3.19) and (3.24), by taking

$$r_1 = \frac{\alpha_0}{1 + a_1} \quad , \quad r_2 = \frac{\alpha_0}{2\beta - 1 + a_2} \quad , \quad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

for any $\alpha_0 > 0$, we have, for any $z = (x \quad \dot{x})^T \in \mathcal{B}^2_\rho$ and all $\epsilon \in (0, 1]$, that: $\delta^r_\epsilon(z) \in \mathcal{B}^2_\rho$ (since $\|\delta^r_\epsilon(z)\| \leq \|z\| \leq \rho$ for any $z \in \mathcal{B}^2_\rho$ and all $\epsilon \in (0, 1]$), $W_3(\epsilon^{r_1} x, \epsilon^{r_2} \dot{x}) = \epsilon^{\alpha_0} W_3(x, \dot{x})$ and $W_2(\epsilon^{r_1} x, \epsilon^{r_2} \dot{x}) = \epsilon^{\alpha_0 \beta} W_2(x, \dot{x})$, *i.e* W_2 and W_3 are locally r-homogeneous of degree $\alpha_2 = \alpha_0 \beta$ and $\alpha_3 = \alpha_0$, respectively, both with domain of homogeneity $\mathcal{B}^2_\rho$. Thus, by Lemma 2.5 and Remark 2.6 (under the consideration of the positive definiteness of W_2 and W_3), there is a positive constant c such that $W_3(x, \dot{x}) \geq c[W_2(x, \dot{x})]^{\alpha_3/\alpha_2}$, $\forall (x, \dot{x}) \in \mathcal{B}^2_\rho$, and consequently, by (3.16) and (3.23), we have that $\dot{V}_1(x, \dot{x}) \leq -W_3(x, \dot{x}) \leq -c[W_2(x, \dot{x})]^{\alpha_0/(\alpha_0 \beta)} \leq -c[V_1(x, \dot{x})]^{1/\beta}$, *i.e.*

$$\dot{V}_1(x, \dot{x}) \leq -c[V_1(x, \dot{x})]^{1/\beta} \tag{3.27}$$

$\forall (x, \dot{x}) \in \mathcal{B}^2_\rho$.

Third stage: finite-time stability. Note, from expressions (3.8), that (3.3) $\implies \beta_0 > 1$. Thus, if $0 < a_1 < a_2 < 1$ then, by taking $\beta \in (1, \beta_0)$, we have $1/\beta \in (0, 1)$, and consequently, from (3.27), we conclude, by Theorem 2.3 and Remark 2.2 (recalling the first stage), that the trivial solution $x(t) \equiv 0$ is globally finite-time stable. Item 1 of the theorem is thus proven.

Fourth stage: r_0-exponential stability with infinite-time attractivity. Let us now suppose that $0 < a_1 < a_2 = 1$. Under this assumption, we have, from expressions (3.8), that $\beta_0 = 1$. Thus, if $0 < a_1 < a_2 = 1$, then, by taking $\beta = 1$, we have $1/\beta = 1$, whence, for any $z = (x \quad \dot{x}) \in \mathcal{B}^2_\rho$ (and recalling (3.2)), we have: from (3.13)–(3.15), that

$$V_1(x, \dot{x}) \geq \kappa_{m1} |x|^{1+a_1} + \kappa_{m2} \dot{x}^2 \geq \kappa_m \|z\|^2_{r_0,2} \tag{3.28}$$

with $\kappa_m = \min\{\kappa_{m1}, \kappa_{m2}\}_{a_2=\beta=1} > 0$; from (3.16)–(3.20), that

$$V_1(x, \dot{x}) \leq (1 + \varepsilon)\big(\kappa_{M1} |x|^{1+a_1} + \kappa_{M2} \dot{x}^2\big) \leq \kappa_M \|z\|^2_{r_0,2} \tag{3.29}$$

with $\kappa_M = (1 + \varepsilon) \max\{\kappa_{M1}, \kappa_{M2}\}_{a_2=\beta=1}$; from (3.23)–(3.25), that

$$\dot{V}_1(x, \dot{x}) \leq -\varepsilon \bar{\kappa}_{m1} |x|^{1+a_1} - \bar{\kappa}_{m2} \dot{x}^2 \leq -\bar{\kappa}_m \|z\|^2_{r_0,2} \tag{3.30}$$

with $\bar{\kappa}_m = \min\{\varepsilon\bar{\kappa}_{m1}, \bar{\kappa}_{m2}\}_{a_2=\beta=1} > 0$; and from (3.21), under the consideration of (3.5) and Young's inequality (with $p = q = 2$ in (2.1)), that

$$\begin{aligned}
\dot{V}_1(x,\dot{x}) &\geq -k_2\dot{x}^2 - \varepsilon k_1|x|^{1+a_1} - \varepsilon k_2|x||\dot{x}| \\
&\geq -\varepsilon k_1|x|^{1+a_1} - k_2\dot{x}^2 - \frac{\varepsilon k_2}{2}\left(x^2+\dot{x}^2\right) \\
&\geq -\varepsilon\left(k_1 + \frac{k_2|x|^{1-a_1}}{2}\right)|x|^{1+a_1} - k_2\left(1+\frac{\varepsilon}{2}\right)\dot{x}^2 \\
&\geq -\bar{\kappa}_{M1}|x|^{1+a_1} - \bar{\kappa}_{M2}\dot{x}^2 \\
&\geq -\bar{\kappa}_M\|z\|^2_{r_0,2} \qquad (3.31)
\end{aligned}$$

with

$$\bar{\kappa}_{M1} = \varepsilon\left(k_1 + \frac{k_2\rho^{1-a_1}}{2}\right) \quad , \quad \bar{\kappa}_{M2} = k_2\left(1+\frac{\varepsilon}{2}\right)$$

and $\bar{\kappa}_M = \max\{\bar{\kappa}_{M1}, \bar{\kappa}_{M2}\}_{a_2=\beta=1}$. Thus, from these expressions, we conclude, by Theorem 2.11 and Corollary 2.12 (recalling the first stage), that the trivial solution $x(t) \equiv 0$ is globally asymptotically stable and (locally) r_0-exponentially stable with infinite-time attractivity, which proves item 2 of the theorem. ∎

Remark 3.2. From (3.2) and Remark 2.10, when $a_1 = a_2 = 1$, the stability of the trivial solution, stated through item 2 of Theorem 3.1, becomes exponential (in the conventional sense). Moreover, since with $a_1 = a_2 = 1$ system (3.1) becomes linear, the exponential stability of the trivial solution is global.

Remark 3.3. Note from (3.8a) that under (3.6), which includes all the cases of the two items of Theorem 3.1, by taking $\beta = 1$, for any $z = (x \quad \dot{x})^T \in \mathcal{B}^2_\rho$, we have: from (3.13)–(3.15), that

$$V_1(x,\dot{x}) \geq \kappa_{m1}|x|^{1+a_1} + \kappa_{m2}\dot{x}^2 \geq \kappa'_m\|z\|^2_{r_0,2}$$

with $\kappa'_m = \min\{\kappa_{m1}, \kappa_{m2}\}_{\beta=1} > 0$; from (3.16)–(3.18), that

$$V_1(x,\dot{x}) \leq (1+\varepsilon)\big(\kappa_{M1}|x|^{1+a_1} + \kappa'_{M2}\dot{x}^2\big) \leq \kappa'_M\|z\|^2_{r_0,2}$$

with $\kappa'_M = (1+\varepsilon)\max\{\kappa_{M1}, \kappa'_{M2}\}_{\beta=1}$; and from (3.23)–(3.25), that

$$\begin{aligned}
\dot{V}_1(x,\dot{x}) &\leq -\varepsilon\bar{\kappa}_{m1}|x|^{1+a_1} - \bar{\kappa}_{m2}|\dot{x}|^{1+a_2} = -\varepsilon\bar{\kappa}_{m1}|x|^{1+a_1} - \bar{\kappa}_{m2}|\dot{x}|^{a_2-1}\dot{x}^2 \\
&\leq -\varepsilon\bar{\kappa}_{m1}|x|^{1+a_1} - \bar{\kappa}_{m2}\rho^{a_2-1}\dot{x}^2 \\
&\leq -\bar{\kappa}'_m\|z\|^2_{r_0,2}
\end{aligned}$$

with $\bar{\kappa}'_m = \min\{\varepsilon\bar{\kappa}_{m1}, \bar{\kappa}_{m2}\rho^{a_2-1}\}_{\beta=1} > 0$. Thus, from these expressions, we conclude, by Theorem 2.11 (recalling the first stage), that (whatever are the values that a_i, $i = 1, 2$, take satisfying (3.6)) the trivial solution $x(t; 0_2) \equiv 0$ is globally asymptotically stable and (locally) r_0-exponentially stable, whether the (non-trivial) system solutions $x(t; z_0)$, $z_0 \in \mathbb{R}^2 \setminus \{0_2\}$, converge to the origin in finite time or not. This includes the case when $0 < a_1 = a_2 < 1$, the only one permitted by (3.6) for which the analytical context developed here has not been able to conclude on finite-time stability or infinite-time attractivity of the trivial solution. For the complementary case $0 < a_2 < a_1 \leq 1$, not encompassed by (3.6), global asymptotic stability is the best conclusion obtained here, from the first stage of the proof of Theorem 3.1.

4. Discussion. The conditions for finite-time stability of the trivial solution $x(t) \equiv 0$ of (3.1), stated through (3.3), can be alternatively expressed as $a_2 \in (0, 1)$ and $a_1 \in (0, a_2)$, or equivalently $a_1 \in (0, 1)$ and $a_2 \in (a_1, 1)$. Notice that $a_2/(2-a_2) \in (0, a_2)$, $\forall a_2 \in (0, 1)$, resp. $2a_1/(1+a_1) \in (a_1, 1)$, $\forall a_1 \in (0, 1)$, whence one corroborates that (3.3) indeed extends the conditions obtained through homogeneity. With respect to the conditions obtained in [9], more precisely stated through [9, Corollary 1] and expressed here through the expressions in (1.5), one observes that, for any $a_2 \in (0, 1)$, the choices on a_1 are significantly different, extending the lower values and limiting the upper ones. There are two reasons that explain such differences. The first of such reasons is the restriction of the (local) analysis from [9] to finite-time convergent solutions that avoid non-stopping oscillations during the finite-time transient, while no restriction to any specific type of finite-time convergent solutions is considered or formulated in the analysis developed here. Such a restriction in [9] is motivated by [9, Theorem 1] which —for a particular type of systems (that include (3.1)) with a finite-time stable equilibrium at the origin— characterizes the way in which (locally or ultimately) non-oscillating finite-time convergent solutions head towards zero. But in view of an imprecision in the proof of [9, Theorem 1] (details are given in Appendix A), the referred theorem inaccurately states that such a characterization applies to every solution that reaches the origin in finite time, thus generating the inexact idea that finite-time convergent solutions cannot reach the origin while swinging. This is counter-argued as follows. Consider (3.1) with $a_1 = a_2 = 1$ and control gains k_i, $i = 1, 2$, such that $k_2^2 - 4k_1 < 0$. The resulting differential equation corresponds to a linear system whose (non-trivial) solutions converge to zero oscillating asymptotically in time. By continuous dependence (or even differentiability) of the solutions on parameters [13, Chapter 3], a sufficiently small decrease on the values of a_i, $i = 1, 2$, resulting in the satisfaction of (3.3), would imply that the convergence of the non-trivial solutions become finite-time, but their oscillating nature could not abruptly change. On the contrary, this should be kept up to a significant change on a_i, $i = 1, 2$. Moreover, since the result from [9, Corollary 1] excludes finite-time convergent solutions that do not stop oscillating during the finite-time transient, this is the type of solutions that must take place from the extension on the choices of a_1 furnished through (3.3), or more precisely with $a_1 \in \big(0, a_2/(2 - a_2)\big)$ for any $a_2 \in (0, 1)$. This is more precisely corroborated through the following refined version of the analysis developed in [9]. From (3.1) and the fact that $\ddot{x} = d\dot{x}/dt$ and $\dot{x} = dx/dt$, we get

$$\dot{x}\frac{d\dot{x}}{dx} = -k_1 \text{sign}(x)|x|^{a_1} - k_2 \text{sign}(\dot{x})|\dot{x}|^{a_2} \tag{4.1}$$

The relations among x and $\dot{x}$ that satisfy (or are defined by) this differential equation give rise to the trajectories generated by (3.1) on the phase plane (with x and $\dot{x}$ as the system states). As precisely pointed out in [9], the trajectories that converge to the origin (locally) heading towards it, must (ultimately) approach it from the interior of a quadrant where x and $\dot{x}$ have opposite signs. This is so since the opposite signs imply that $|x|$ decreases (along the trajectories), approaching zero, while in the other quadrants, where x and $\dot{x}$ have the same sign, $|x|$ increases, moving away from zero. In such a (final) phase of the trajectories, since the *motion* of $|x|$ is monotonically kept decreasing, $\dot{x}$ keeps a functional relation with x: $\dot{x} = h(x)$, $\forall |x| \leq \bar{x}$, for a sufficiently small positive value $\bar{x}$, with $xh(x) < 0$ (or equivalently $\text{sign}\big(h(x)\big) = -\text{sign}(x)$), $\forall x \neq 0$, and $h(0) = 0$ (since the trajectories converge to the origin; note that such properties imply continuity of h at $x = 0$, thus $\lim_{x\to 0} h(x) = h(0) = 0$). Hence, under such considerations and assertions, (4.1) becomes

$$h(x)\frac{dh}{dx}(x) = -k_1\mathrm{sign}(x)|x|^{a_1} + k_2\mathrm{sign}(x)|h(x)|^{a_2} \tag{4.2}$$

which determines the existence and forms of solutions that converge to the origin (locally or ultimately) heading towards it. By further involving the following approximation:

$$\lim_{x\to 0}\frac{dh}{dx}(x) = \lim_{x\to 0}\lim_{\nu\to 0}\frac{h(x+\nu)-h(x)}{\nu} = \lim_{\nu\to 0}\lim_{x\to 0}\frac{h(x+\nu)-h(x)}{\nu}$$
$$= \lim_{\nu\to 0}\frac{h(\nu)}{\nu} = \lim_{x\to 0}\frac{h(x)}{x}$$

i.e. $(dh/dx)(x) \approx h(x)/x$ in a sufficiently small interval around $x = 0$, we get that (4.2) can be approached as

$$h^2(x) + k_1|x|^{1+a_1} = k_2|x||h(x)|^{a_2} \tag{4.3}$$

$\forall |x| \leq \bar{x}$, for a sufficiently small (positive) $\bar{x}$. Observe that functions $h(x)$ (with the above mentioned properties) that solve (4.3) shall satisfy $k_2|x||h(x)|^{a_2} \geq h^2(x)$ and $k_2|x||h(x)|^{a_2} \geq k_1|x|^{1+a_1}$ which, for any $a_2 \in (0,1]$, can be equivalently rewritten as

$$\left(\frac{k_1}{k_2}\right)^{1/a_2}|x|^{a_1/a_2} \leq |h(x)| \leq k_2^{1/(2-a_2)}|x|^{1/(2-a_2)} \tag{4.4}$$

Thus, trajectories that converge to the origin (locally or ultimately) heading towards the origin shall adopt the form of functions (with the above mentioned properties) that satisfy (4.4) in a sufficiently small region around $x = 0$. A simple analysis on the (upper and lower) bounds from (4.4) shows that, for any $a_2 \in (0,1]$, this is feasible on $\{|x| \leq \bar{x}\}$ for a sufficiently small (positive)

$$\bar{x} < \left[k_2^{1/(2-a_2)}\left(\frac{k_2}{k_1}\right)^{1/a_2}\right]^{1/[(a_1/a_2)-1/(2-a_2)]}$$

provided that $a_1 > a_2/(2-a_2)$, while if $a_1 < a_2/(2-a_2)$, there is no function $h(x)$ satisfying (4.4) in a neighborhood of $x = 0$. In other words, with $a_2 \in (0,1]$ and $a_1 > a_2/(2-a_2)$, trajectories that (locally or ultimately) head directly towards the origin do exist and they are all within the curve segments defined by the lower and upper bounds from (4.4) in a sufficiently small interval around $x = 0$, while with $a_2 \in (0,1]$ and $a_1 < a_2/(2-a_2)$, such type of trajectories cannot take place. Furthermore, in view of the invariance of the trajectories (due to the uniqueness of the non-trivial system solutions), the existence of trajectories that head directly towards the origin exclude that of trajectories that converge spiraling around it and *vice versa*. Consequently, we conclude that with $a_2 \in (0,1]$ and $a_1 > a_2/(2-a_2)$ the system trajectories converge to the origin (locally or ultimately) avoiding spiraling around it, while with $a_2 \in (0,1]$ and $a_1 < a_2/(2-a_2)$ the system solutions shall converge to zero oscillating (undergoing an infinite number of zero crossings before the definitive permanence at zero). It is worth noting that the just concluded assertions do not depend on the specific (positive) value of the control gains k_i, $i = 1, 2$. On the contrary, for any $a_2 \in (0,1]$, if $a_1 = a_2/(2-a_2)$ (the homogeneity-related case), the type of (oscillating or non-oscillating) convergence does depend on the control gains. Indeed, a simple analysis on the (upper and lower) bounds from (4.4) shows that if $a_1 = a_2/(2 - a_2)$, for any $a_2 \in (0, 1)$, then $(k_1/k_2)^{1/a_2} \leq k_2^{1/(2-a_2)}$, or equivalently $k_2^2 \geq k_1^{2-a_2}$, becomes a necessary condition for trajectories to converge to the origin avoiding spiraling around it, and consequently, $k_2^2 < k_1^{2-a_2}$ turns out to be a sufficient condition for the system solutions to converge to zero oscillating throughout the settling time; a more refined (alternative) analysis that leads to a more precise condition on the control gains k_i, $i = 1, 2$, accurately stating the dividing point among oscillating and non-oscillating solutions in the homogeneity-related case will be developed and reported in a future communication. In the more particular case when $a_1 = a_2 = 1$ (the linear system case), one corroborates directly from (4.3) that the former (non-oscillating) case takes place with $k_2^2 \geq 4k_1$, while the latter (oscillating) one arises with $k_2^2 < 4k_1$.

The second reason on the differences among the result obtained for finite-time stability in [9, Corollary 1], with respect to that presented here, is the unexhaustive search (carried out in [9]) related to the finite-time convergence in itself, leading to conditions that permit such type of convergence without strictly ruling out infinite-time convergent solutions, while the analysis developed here leads to sufficient conditions that guarantee the finite-time convergence. Indeed, as pointed out in [9], finite-time stability of the origin (in the previously referred state space) may be concluded as long as the functional relation held among x and $\dot{x}$ in the considered non-oscillating final stage of the system trajectories, $\dot{x} = h(x)$, defines a first-order differential equation with finite-time stable equilibrium at $x = 0$. With this in mind, the search for related conditions, carried out in [9], focuses on the system trajectories that (locally or ultimately) finish up by being close to the upper and lower bounds from (4.4). By forcing the exponent in the upper bound to be less than unity, the corresponding solutions were concluded to achieve the finite-time convergence, which led to conclude that such a convergence is achieved with $a_2 < 1$, omitting any further analysis on the lower bound. Through such a condition, finite-time convergence of the system trajectories is indeed made possible, but the referred omission turns out to additionally permit conditions (namely, those giving rise to an exponent in the lower bound from (4.4) being higher than unity) through which solutions that converge to zero asymptotically in time take place (for instance, those that finish up by being close to the lower bound from (4.4)). As a matter of fact, in order to guarantee the finite-time convergence, one must additionally force the exponent in the lower bound to be less than unity too. This forces all the functions $h(x)$ in the region defined through (4.4) (for sufficiently small values of $|x|$) to have the required form (in order for $\dot{x} = h(x)$ to define a first-order system with finite-time stable equilibrium at $x = 0$). Such a complementary consideration in the analysis turns out to state the supplementary condition $a_1 < a_2$. Thus, for any $a_2 \in (0, 1)$, the limitation of the upper choices on a_1 stated through the result obtained here, in relation to that from [9, Corollary 1], turns out to guarantee (and not just permit) the finite-time stability of the trivial solution $x(t) \equiv 0$, thus ruling out infinite-time convergent solutions that may take place with $a_1 \geq a_2$. The assertions concluded from the analysis and discussion developed in this section will be corroborated through simulations in the next section.

Remark 4.1. From the analysis developed in this section, one can see that in the r_0-exponential stability with infinite-time attractivity case stated through item 2 of Theorem 3.1, *i.e.* when $0 < a_1 < a_2 = 1$, the system solutions converge ultimately oscillating, since $0 < a_1 < a_2 = 1 \implies 0 < a_1 < a_2/(2 - a_2) = 1$, while in the r_0-exponential stability and asymptotic stability cases arisen with $0 < a_1 = a_2 < 1$ and $0 < a_2 < a_1 \leq 1$, respectively (recall Remark 3.3), the solutions converge ultimately

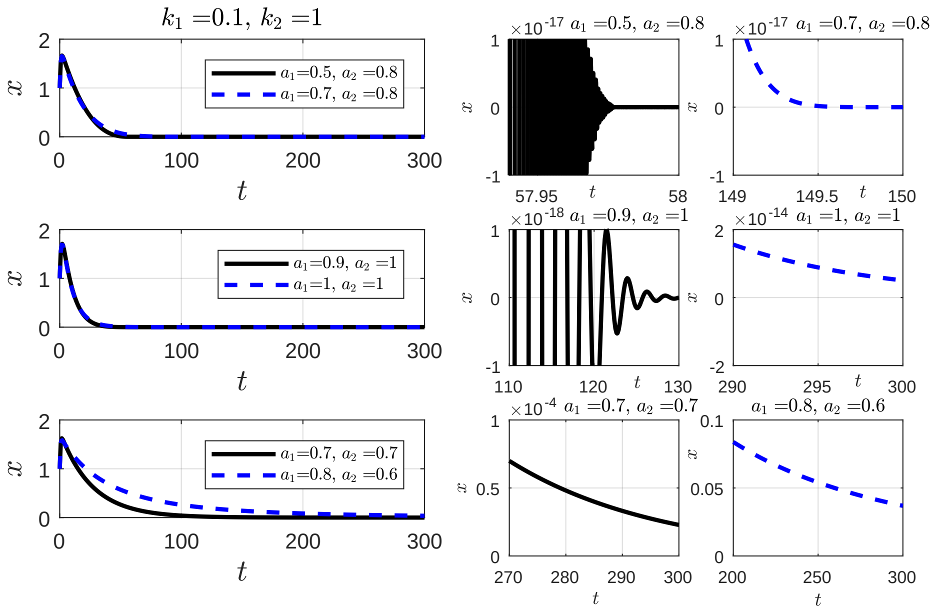

FIG. 1. *System responses taking $k_1 = 0.1$ and $k_2 = 1$. Upper graphs: $a_2 = 0.8$, $a_1 = 0.5 < 2/3 = a_1^h$ (finite-time stability with ultimate oscillation), and $a_2 = 0.8$, $a_1 = 0.7 > 2/3 = a_1^h$ (finite-time stability avoiding ultimate oscillation). Center graphs: $a_1 = 0.9$, $a_2 = 1$ ($(20/19, 1)$-exponential stability with infinite-time attractivity), and $a_1 = a_2 = 1$ (exponential stability with infinite-time attractivity). Lower graphs: $a_1 = a_2 = 0.7$ ($(20/17, 1)$-exponential stability), and $a_1 = 0.8 > 0.6 = a_2$ (asymptotic stability). Right-hand graphs: zooms of the responses included in their corresponding left-hand graph.*

avoiding oscillations, since $0 < a_1 = a_2 < 1 \implies 0 < a_2/(2 - a_2) < a_1 < 1$ and $0 < a_2 < a_1 \leq 1 \implies 0 < a_2/(2 - a_2) < a_1 \leq 1$.

5. Simulation results. In this section, we illustrate the analytical findings of Section 3 and corroborations from Section 4 through computer simulations. In this direction, it is important to keep in mind that the goal here is not to evaluate closed-loop performance from a control viewpoint, where some sort of optimization or improvement is aimed. We have rather implemented the system dynamics (3.1) with several combinations of control parameter values selected so as to make as clear as possible the referred illustrations. Subsequently, we denote a_i^h, $i \in \{1, 2\}$, the homogeneity related value of a_i for a given $a_{3-i} \in (0, 1)$, *i.e.* $a_1^h = a_2/(2 - a_2)$ for a given $a_2 \in (0, 1)$, resp. $a_2^h = 2a_1/(1 + a_1)$ for a given $a_1 \in (0, 1)$. Recall further (3.2). All the simulations were run up to 300 seconds, taking initial values $x(0) = \dot{x}(0) = 1$.

Figure 1 shows simulation results obtained taking $k_1 = 0.1$ and $k_2 = 1$ with different combinations of a_i, $i = 1, 2$; note that $k_2^2 = 1 > 0.4 = 4k_1$, satisfying the non-oscillating solution condition of the exponential stability with infinite-time attractivity case, *i.e.* with $a_1 = a_2 = 1$. More particularly, Figure 1 shows results obtained with $a_2 = 0.8$ and $a_1 = 0.5 < 2/3 = a_1^h$ (finite-time stability with ultimate oscillation), $a_2 = 0.8$ and $a_1 = 0.7 > 2/3 = a_1^h$ (finite-time stability avoiding ultimate oscillation), $a_1 = 0.9$ and $a_2 = 1$ ($(20/19, 1)$-exponential stability with infinite-time attractivity), $a_1 = a_2 = 1$ (exponential stability with infinite-time attractivity), $a_1 = a_2 = 0.7$ ($(20/17, 1)$-exponential stability) and $a_1 = 0.8 > 0.6 = a_2$ (asymptotic stability). Note that while the system response obtained with $a_2 = 0.8$ and $a_1 = 0.7 > 2/3 = a_1^h$

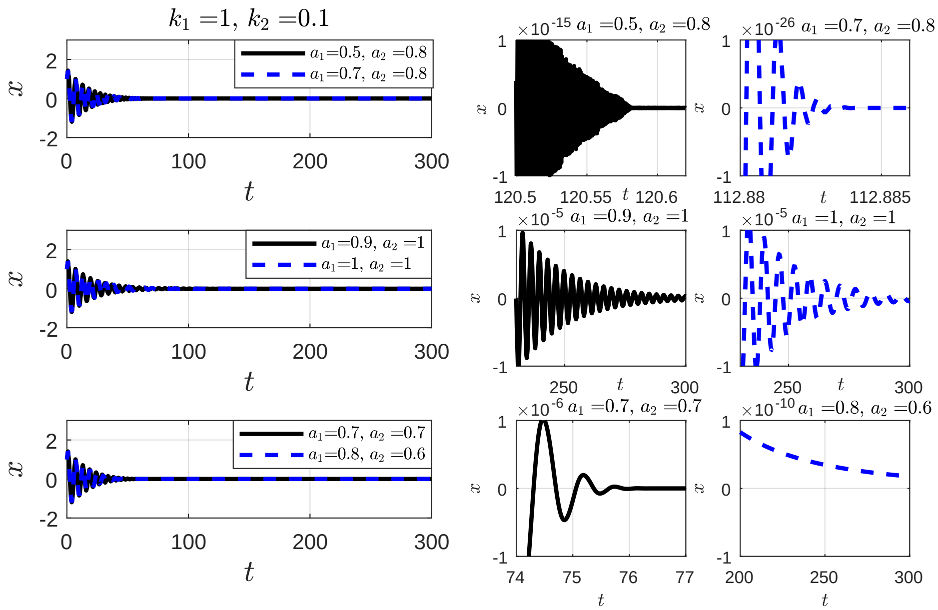

FIG. 2. *System responses taking $k_1 = 1$ and $k_2 = 0.1$. Upper graphs: $a_2 = 0.8$, $a_1 = 0.5 < 2/3 = a_1^h$ (finite-time stability with ultimate oscillation), and $a_2 = 0.8$, $a_1 = 0.7 > 2/3 = a_1^h$ (finite-time stability avoiding ultimate oscillation). Center graphs: $a_1 = 0.9$, $a_2 = 1$ ($(20/19, 1)$-exponential stability with infinite-time attractivity), and $a_1 = a_2 = 1$ (exponential stability with infinite-time attractivity). Lower graphs: $a_1 = a_2 = 0.7$ ($(20/17, 1)$-exponential stability), and $a_1 = 0.8 > 0.6 = a_2$ (asymptotic stability). Right-hand graphs: zooms of the responses included in their corresponding left-hand graph.*

converges heading directly towards the equilibrium and reaching zero at about 149.6 seconds where it remains thereafter, that gotten with $a_2 = 0.8$ and $a_1 = 0.5 < 2/3 = a_1^h$ converges ultimately experiencing non-stopping oscillations to finish up converging at around 57.975 seconds remaining at zero thereafter. Observe on the other hand that the system solution obtained with $(a_1, a_2) = (0.9, 1)$ converges quicker than that gotten with $(a_1, a_2) = (1, 1)$ and that it does converge ultimately experiencing oscillations (recall Remark 4.1). Note further that the system responses corresponding to the r_0-exponential stability and asymptotic stability cases, respectively obtained with $a_1 = a_2 = 0.7$ and $a_1 = 0.8 > 0.6 = a_2$, are both corroborated to converge avoiding oscillations (recall Remark 4.1). Moreover, these cases are observed to keep on approaching to zero by the end of the simulation time.

Figure 2 shows further simulation results obtained taking this time $k_1 = 1$ and $k_2 = 0.1$ with the same precedent combinations of a_i, $i = 1, 2$; note that in this case $k_2^2 = 0.01 < 4 = 4k_1$, satisfying the oscillating solution condition of the exponential stability with infinite-time attractivity case ($a_1 = a_2 = 1$). Note that in spite of the oscillating start of the finite-time convergent solutions involved in Figure 2 (contrarily to those involved in Figure 1), the response obtained with $a_2 = 0.8$ and $a_1 = 0.7 > 2/3 = a_1^h$ ultimately stops oscillating to head directly towards the equilibrium, reaching zero in a settling time close to 112.8835 seconds where it remains thereafter, while that gotten with $a_2 = 0.8$ and $a_1 = 0.5 < 2/3 = a_1^h$ keeps on oscillating up to its settling time at around 120.58 seconds remaining at zero thereafter. Observe on the other hand that the solutions obtained with $(a_1, a_2) = (0.9, 1)$

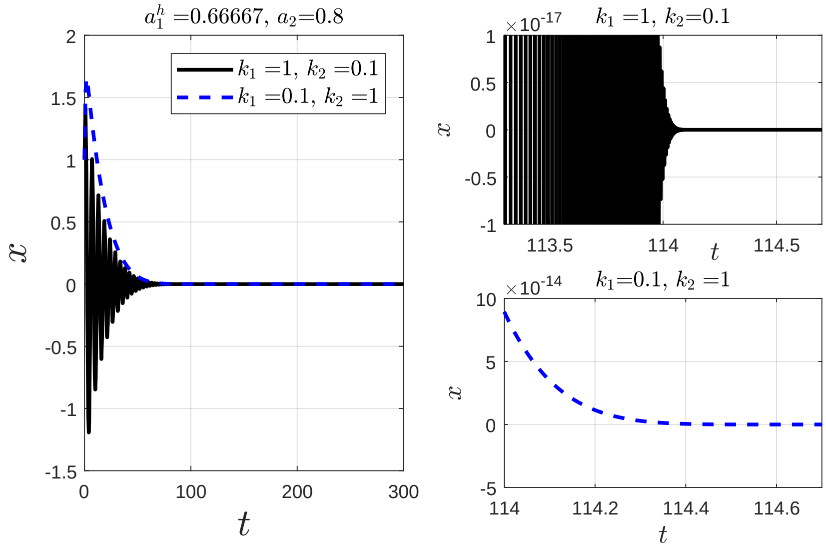

FIG. 3. *System responses taking homogeneity related values* $a_1^h = 2/3$ *and* $a_2 = 0.8$ *with:* $k_1 = 0.1$ *and* $k_2 = 1$ *(widely satisfying the homogeneity related non-oscillating response necessary condition);* $k_1 = 1$ *and* $k_2 = 0.1$ *(satisfying the homogeneity related oscillating solution sufficient condition). Right-hand graphs: zooms of the responses included in their corresponding left-hand graph.*

and $(a_1, a_2) = (1, 1)$ are corroborated to converge experiencing oscillations, while no important difference is observed among their convergence rate this time. Furthermore, one notes that the system responses corresponding to the r_0-exponential stability and asymptotic stability cases, respectively obtained with $a_1 = a_2 = 0.7$ and $a_1 = 0.8 > 0.6 = a_2$, are again both corroborated to converge avoiding oscillations. In particular, the asymptotic stability case is clearly observed to keep on approaching the equilibrium by the end of the simulation time.

Finally, Figure 3 shows further simulation results obtained taking this time the homogeneity related values $a_1 = 2/3$ $(= a_1^h)$ and $a_2 = 0.8$, with the two precedent different combinations of control gains k_i, $i = 1, 2$, namely $(k_1, k_2) = (0.1, 1)$ and $(k_1, k_2) = (1, 0.1)$; notice that in the former control gain case we have that $k_2^2 = 1 > 0.1 > k_1^{2-a_2}$, $\forall a_2 \in (0, 1)$, and in the latter one that $k_2^2 = 0.01 < 1 = k_1^{2-a_2}$, $\forall a_2 \in (0, 1)$, widely satisfying the non-oscillating response necessary condition and the oscillating solution sufficient condition of the homogeneity related case, respectively (as exposed in Section 4). One observes from the figure that with $(k_1, k_2) = (1, 0.1)$ the system response indeed converge in finite time oscillating, while with $(k_1, k_2) = (0.1, 1)$ it turns out to converge in finite time avoiding oscillations.

6. Conclusions. The double integrator fed back by an additive composition of gained (proportional) exponentially weighted *position* and *velocity* error correction terms turns out to possess multiple stability properties and give rise to multiple response behaviors. In particular, global finite-time stability of the trivial solution is proven to arise for any less-than-unity exponential weights with that related to the position error correction term, a_1, lower than that of the velocity error one, a_2, *i.e.* for any $0 < a_1 < a_2 < 1$. The homogeneity related exponential weights, namely $a_1 = a_1^h \triangleq a_2/(2 - a_2) \in (0, a_2)$ for any $a_2 \in (0, 1)$, or equivalently

$a_2 = a_2^h \triangleq 2a_1/(1+a_1) \in (a_1, 1)$ for any $a_1 \in (0,1)$, thus turn out to be a particular case over the referred richer spectrum of exponential weight values giving rise to finite-time stability of the trivial solution. Actually, such homogeneity related exponential weights happen to constitute the dividing point among finite-time convergent system solutions that ultimately keep/induce or avoid non-stopping oscillations before the definitive permanence at zero, independently of the control gain values; namely $a_2 \in (0,1)$ with: $a_1 \in (a_1^h, a_2)$ giving rise to the ultimately non-oscillating behavior and $a_1 \in (0, a_1^h)$ for the ultimately oscillating one, or equivalently $a_1 \in (0,1)$ with: $a_2 \in (a_1, a_2^h)$ for the ultimate non-oscillation case and $a_2 \in (a_2^h, 1)$ for the ultimate oscillation one. Curiously, both oscillating and non-oscillating behaviors can take place in the homogeneity related case depending on the control gain values, with $k_2^2 < k_1^{2-a_2}$ proven to be a sufficient condition for the former (oscillating) behavior and $k_2^2 \geq k_1^{2-a_2}$ a necessary condition of the latter (non-oscillating) one, when $a_2 \in (0,1)$. The conventional and a homogeneous-norm-related exponential types of stability turn out to additionally arise when $0 < a_1 \leq a_2 \leq 1$. Actually, for any such combinations of exponential weights, the trivial solution happens to have the homogeneous-norm-related exponential type of stability, becoming the conventional type when $a_1 = a_2 = 1$, with additional infinite-time attractivity in this case and when $0 < a_1 < a_2 = 1$, and sharing the finite-time stability property when $0 < a_1 < a_2 < 1$. For the complementary exponential weight condition $0 < a_2 < a_1 \leq 1$, global asymptotic stability is the best conclusion that can be drawn for the trivial solution through the analysis developed here. For this asymptotic stability case and the homogeneous-norm-related exponential stability one arisen with $0 < a_1 = a_2 < 1$, no analytical certainty about the type of convergence, among finite- and infinite-time, could be obtained. It remains to discover if the analytically obtained finite-time stability sufficient condition, $0 < a_1 < a_2 < 1$, is additionally necessary, or if there is an analytical way to know the type of convergence (among finite- or infinite-time) that does or may arise when $0 < a_1 = a_2 < 1$ and when $0 < a_2 < a_1 \leq 1$.

Appendix A. About [9, Theorem 1]. [9, Theorem 1] claims that, for systems $\dot{z} = g(z)$, $z \in \mathbb{R}^n$, with a finite-time stable equilibrium at $z = 0_n$ and g being a continuous vector field that is continuously differentiable on $\mathbb{R}^n \setminus \{0_n\}$ and has a component $g_i(z)$ that is Lipschitz-continuous at $z = 0_n$, for some $i \in \{1, \ldots, n\}$, the solutions that reach the origin in finite time do so such that $\lim_{t\to T} z_i(t)/\|z(t)\| = 0$, with T being the settling time. By denoting $z(t; p_0)$ a system solution with $z(0; p_0) = p_0$ and considering that $z(T; p_0) = 0$, the proof begins by invoking the mean value theorem, through which it is claimed that there exists $q \in [0, T]$ such that $0 = z_i(T; p_0) = z_i(0; p_0) + T g_i(z(q; p_0))$. By further considering the dependence of T and q on the initial state and denoting p a generical initial condition along the trajectory going through p_0, *i.e.* $p = z(t; p_0)$, $t \in [0, T(p_0)]$, the previous equation is more generally rewritten as

$$\frac{g_i(z(q(p); p))}{z_i(0; p)} = -\frac{1}{T(p)} \tag{A.1}$$

for any such p. At this point, the author claims that, in view of the smoothness of $z_i(t; p)$ in t and its vanishing at $t = T(p)$:

$$\lim_{p \to 0_n} \left| \frac{z_i(q(p); p)}{z_i(0; p)} \right| = 1 \tag{A.2}$$

and involves such a limit to support the rest of the proof. Nevertheless, such a limit does not hold (and does not even necessarily exist) if q is not unique. Indeed, in a general context where $z_i(t)$ can converge to zero undergoing non-stopping oscillations (giving rise to an infinite number of zero crossings) during the settling time or avoiding oscillations (for instance, depending on the value of parameters involved in the system dynamics), the limit may be valid for the latter (non-oscillating) case. But in the former (oscillating) case, there would be a multiple (actually infinite) number of mean times q satisfying (A.1) for every p, and each one of such mean times, subsequently denoted q_j, $j = 1, 2, \ldots$, would generally state different relations of $z_i(q_j(p); p)$ and $z_i(0; p)$, *i.e.* different values of $z_i(q_j(p); p)/z_i(0; p)$ for each $j = 1, 2, \ldots$; in particular, by considering that $q_{j_1}(p) > q_{j_2}(p)$ for any $j_1 > j_2$: $q_j(p) \to T(p)$ as $j \to \infty$, and consequently $\lim_{j\to\infty} z_i(q_j(p); p)/z_i(0; p) = 0$ for every p. This shows that in the oscillating case —and consequently, in the more general context where no assumption is made on the type of (oscillating or non-oscillating) convergence— the left-hand side limit in (A.2) does not have a defined value, and more particularly that (A.2) does not generally hold. Consequently, [9, Corollary 1] does not really apply to every finite-time convergent solution. It may however be considered to apply to solutions whose component z_i converge to the origin in finite time (locally or ultimately) avoiding oscillations.

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